graph theory notes*

The union of a forest and a star forest is 3-colorable

Norbert Sauer conjectured the following in 1993 [4] and Michael Stiebitz proved it in 1994 [5]. A *star forest* is a forest where each component has a dominating vertex called the *root*. It is easy to see that for two forests F_1 and F_2 we have $\chi(F_1 \cup F_2) \leq 4$. We can do better when one of the forests is a star forest.

Theorem (Stiebitz)**.** *If F_1 is a star forest and F_2 is a forest, then $\chi(F_1 \cup F_2) \leq 3$.*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately by applying Lemma with $k = 3$, $F = F_2$ and H the subgraph of G induced on the set of roots of F_1. The following proof and picture are from the paper *Brooks' Theorem and Beyond* with Dan Cranston [3].

Lemma (Stiebitz)**.** *Let H be an induced subgraph of a graph G with $\chi(H) \leq k$ for some $k \geq 3$. Then $\chi(G) \leq k$ if G has a spanning forest F where*

1. *for each component C of H, $F[V(C)]$ is a tree; and*
2. *$d_G(v) \leq d_F(v) + k - 2$ for every $v \in V(G - H)$.*

Proof. For any graphs U and W, we write $U - W$ for the subgraph of U induced by $V(U) \setminus V(W)$. If $uv \in E(F)$, then u is an F*-neighbor* of v, and u and v are F*-adjacent*. Suppose the lemma is false and choose a counterexample pair G, H minimizing $|G - H|$. Note that each vertex v in $G - H$ must have a neighbor in H, since otherwise we can add v to H. Thus $|H| \geq 1$.

Claim 1. *If there exists $v \in V(G - H)$ adjacent to components $A_1, \dots, A_s$ of H with $d_G(v) \leq s + k - 2$, then there exist i and j, with $i \neq j$, and a path in $F - v$ from A_i to A_j.* Suppose not and choose such a $v \in V(G - H)$. We will find a k-coloring of G. For each $i \in [s]$, let z_i be a neighbor of v in A_i. Form G', F', H' from G, F, H (repectively) by deleting v and identifying all z_i as a single new vertex z. Now $\chi(H') \leq k$, since by permuting colors in each component we can get a k-coloring of H where all the z_i use the same color. Also, F' is a spanning forest in G' since we are assuming there is no path in $F - v$ from A_i to A_j whenever $i \neq j$. It is easy to check that Conditions (1) and (2) hold for G', F', H'. Now $|G' - H'| < |G - H|$, so by minimality of $|G - H|$, we have a k-coloring of G'. This gives a k-coloring of $G - v$ where $z_1, \dots, z_s$ all get the same color. So v has at most $d_G(v) - (s - 1) \leq k - 1$ colors used on its neighborhood, leaving a color free to finish the k-coloring on G, a contradiction.

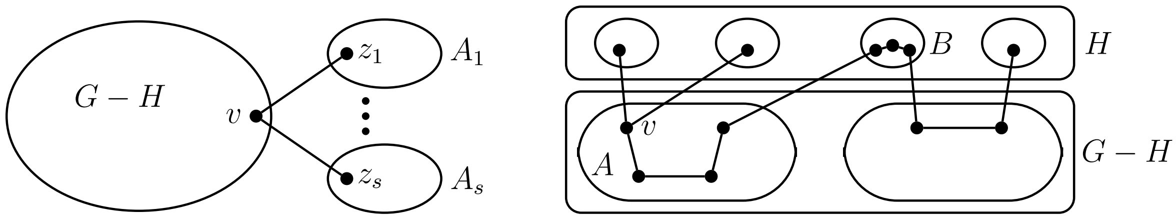

Figure 1: The left figure shows Claim 1. The right figure shows Claim 3.

Claim 2. *Every leaf of F is in H and every vertex not in H has an F-neighbor not in H.* We can rewrite this formally: $d_F(v) \geq 2$ and $d_{F-H}(v) \geq 1$ for all $v \in V(G-H)$. Applying Claim 1 with $s = 1$ implies $d_G(v) \geq k$. Now Condition (2) gives $d_F(v) \geq d_G(v) + 2 - k \geq 2$. Suppose $d_{F-H}(v) = 0$ for some $v \in V(G-H)$. Since F is a forest, Condition (1) implies that all F-neighbors of v must be in different components of H. Moreover there can be no path between two of these components in $F - v$. Condition (2) gives $d_G(v) \leq d_F(v) + k - 2$, so applying Claim 1 with $s = d_F(v)$ gives a contradiction. Thus $d_{F-H}(v) \geq 1$ for all $v \in V(G-H)$.

Claim 3. *There exists v in $G-H$ with $d_{F-H}(v) = 1$ such that every component of H that is F-adjacent to v is not F-adjacent to any other vertex in $G-H$.* Form a bipartite graph F' from F by contracting each component of H and each component of $F-H$ to a single vertex. Since F is a forest, Condition (1) implies that F' is also a forest. So some vertex contracted from a component A of $F-H$ has at most one neighbor of degree at least 2; say this neighbor is contracted from B, where $B \subseteq (F \cap H)$. (If not, then we can walk between components of H and $F-H$ until we get a cycle in F.) Let v be a leaf of A that is not F-adjacent to B; this gives $d_{F-H}(v) = d_A(v) \leq 1$. Claim 2 gives $d_{F-H}(v) \geq 1$, so in fact $d_{F-H}(v) = 1$ as desired.

Claim 4. *If the v in Claim 3 is adjacent to a component of H, then it is F-adjacent to that component.* Let $A_1, \ldots, A_r$ be the components of H that are F-adjacent to v, where $r = d_F(v) - 1$. Suppose there is another component A_{r+1} of H that is adjacent to v. Since no vertex of $G-H$ besides v is F-adjacent to any of $A_1, \ldots, A_r$, there can be no F-path in $F - v$ between any pair among $A_1, \ldots, A_r, A_{r+1}$. Now the contrapositive of Claim 1 implies that $d_G(v) > (r+1) + k - 2 = d_F(v) + k - 2$; this inequality contradicts Condition (2).

Claim 5. *The lemma holds.* Let $H' := G[V(H) \cup \{v\}]$, with v as in Claims 3 and 4. By Claim 4, Condition (1) of the hypotheses holds for H'. Condition (2) clearly holds and F is still a forest. Also, by permuting colors in the components we can get a k-coloring of H where all F-neighbors of v get the same color. Hence v has at most $d_H(v) - (d_F(v) - 2) \leq d_G(v) - 1 - (d_F(v) - 2) = d_G(v) - d_F(v) + 1 \leq k - 1$ colors on its neighborhood. Hence H' is k-colorable. But then, by minimality of $|G-H|$, G is k-colorable, a contradiction. □

Combined with a result on the existence of spanning trees with pairwise non-adjacent leaves [1], Lemma yields Brooks' theorem [2]. See [3] for details.

Question. Are there other applications of Lemma?

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References

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