graph theory notes*

Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [4]. For a graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the vertices of degree $\delta(G)$ and let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree larger than $\delta(G)$.

Theorem (Stiebitz)**.** *If G is a color-critical graph, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$.*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately from Lemma 3 using $X = V(\mathcal{L}(G))$. The main induction step in the proof requires a non-trivial fact about bipartite graphs, we save the proof of this fact until later but state it here. A bipartite graph G with parts A and B has *positive surplus* with respect to A if $|N(S)| > |S|$ for all $\emptyset \neq S \subseteq A$. Note that A could be empty here, in which case the graph has positive surplus vacuously.

Lemma 1 (Stiebitz)**.** *Let G be a bipartite graph with parts $A \neq \emptyset$ and B such that G has positive surplus with respect to A. Then there is $x \in A$ such that for any different $y, z \in N(x)$, the bipartite graph G' formed by contracting $\{y, z\}$ and removing x has positive surplus with respect to $A \setminus \{x\}$.*

Lemma 1 is used in Claim 2 of the proof of Lemma 2. Note that Lemma 2 is trivially true in the $X = \emptyset$ case, we allow this to avoid having to handle a base case where $G[X]$ has one component separately.

Lemma 2 (Stiebitz)**.** *Let G be a graph and $X \subseteq V(G)$ such that*

- *$d_G(x) \leq k - 1$ for all $x \in X$; and*
- *$\chi(G - X) \leq k - 1$; and*
- *for each component C of $G[X]$, we have $\chi(G - V(C)) \leq k - 1$.*

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Suppose $G - X$ is the disjoint union of (possibly not connected) graphs $M_1, \dots, M_{\ell+1}$ such that the bipartite graph $\mathcal{B}$ formed by contracting each M_i and each component of $G[X]$ has positive surplus with respect to the $G[X]$ side. If f_i is a $(k-1)$-coloring of M_i for each $i \in [\ell+1]$, then there are permutations $\pi_1, \dots, \pi_{\ell+1}$ of $[k-1]$ such that the $(k-1)$-coloring of $G - X$ given by $(\pi_1 \circ f_1) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k-1)$-coloring of G.

Proof. Suppose the lemma is false and choose a counterexample G and nonempty $X \subseteq V(G)$ so that $|X|$ is as small as possible. So, $G - X$ is the disjoint union of graphs $M_1, \dots, M_{\ell+1}$ and we have $(k-1)$-colorings f_i of M_i for each $i \in [\ell+1]$ so that no permutations allow us to extend to a $(k-1)$-coloring of G.

Claim 1. *Each non-separating vertex in $G[X]$ has neighbors in at least two of the M_i.* Suppose to the contrary that we have a component C of $G[X]$ and $x \in V(C)$ a non-separating vertex that has neighbors in at most one of the M_i. Let $X' = X \setminus \{x\}$. The hypotheses of the lemma are satisfied with X' in place of X and $M_i' = G[V(M_i) \cup \{x\}]$ in place of M_i since $\mathcal{B}$ remains the same and hence still has positive surplus. Since $\mathcal{B}$ has positive surplus, we must have $|C| \geq 2$ and hence x has at most $k-2$ neighbors in $G - X$, so we can greedily complete the given $(k-1)$-coloring of $G - X$ to $G - X'$. Applying minimality of $|X|$ to this coloring of $G - X'$, we get permutations that allow us to extend to a $(k-1)$-coloring of G. But these same permutations also allow us to extend the given $(k-1)$-coloring of $G - X$ to G, a contradiction.

Claim 2. *There is a component C of $G[X]$ and $i \neq j$ such that some non-separating $x \in V(C)$ has neighbors in M_i and M_j and the bipartite graph formed from $\mathcal{B}$ by contracting $\{M_i, M_j\}$ and removing C has positive surplus.*

By Lemma 1, $G[X]$ has a component C such that for any neighbors M_i, M_j of C in $\mathcal{B}$, the bipartite graph formed from $\mathcal{B}$ by contracting $\{M_i, M_j\}$ and removing C has positive surplus. Let x be a non-separating vertex in C. By Claim 1, there are different M_i, M_j in which x has neighbors, so C with x and i, j works.

Claim 3. *The lemma is true.* Let C, x, i, j be as in Claim 2, by symmetry we may assume $i = 1$ and $j = 2$. Let $G' = G - V(C)$ and $X' = X \setminus V(C)$. Then $G' - X'$ is the disjoint union of the ℓ graphs $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$. Pick $y_1 \in N(x) \cap V(M_1)$ and $y_2 \in N(x) \cap V(M_2)$. We permute the colors in the coloring of f_2 so that y_1 and y_2 get the same color. This will save one color for x so that we can greedily color C, ending at x. Formally, let τ be a permutation of $[k-1]$ such that $(\tau \circ f_2)(y_2) = f_1(y_1)$ and let $f_* = f_1 \cup (\tau \circ f_2)$. By Claim 2 and minimality of $|X|$, we can apply the lemma to G' with $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$ and colorings $f_*, f_3, \dots, f_{\ell+1}$ to get permutations $\pi_*, \pi_3, \dots, \pi_{\ell+1}$ such that the $(k-1)$-coloring of $G' - X'$ given by $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k-1)$-coloring of G'. But this is the same as the $(k-1)$-coloring $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$, so using the permutations $\pi_*, \pi_* \circ \tau, \pi_3, \dots, \pi_{\ell+1}$ we get a coloring of $G - X$ that extends to $G - V(C)$. In these colorings, y_1 and y_2 receive the same color. This means that x has $k - 1 - (d_G(x) - d_C(x)) + 1 \geq d_C(x) + 1$ colors available and each other vertex v in C has $k - 1 - (d_G(v) - d_C(v)) + 1 \geq d_C(v) \geq d_C(v)$ colors available. So, coloring C greedily in order of decreasing distance from x gives an extension to a $(k-1)$-coloring of G, a contradiction. □

Note that, like the previous one, the following lemma is trivially true in the $X = \emptyset$ case, again we allow this to avoid having to handle a base case where $G[X]$ is connected separately.

Lemma 3 (Stiebitz)**.** *Let G be a connected graph and $X \subseteq V(G)$ such that*

- $d_G(x) \le k-1$ *for all* $x \in X$*; and*
- $\chi(G-X) \le k-1$*; and*
- *for each component C of $G[X]$, we have $\chi(G - V(C)) \le k-1$; and*
- *$G[X]$ has ℓ components and $G-X$ has at least $\ell+1$ components.*

Then G is $(k-1)$-colorable.

Proof. Suppose not and choose a counterexample minimizing $|X|$. If $|X| = 0$, the lemma is trivially true, so we must have $|X| \ge 1$. Let $\mathcal{B}$ be the bipartite graph formed by contracting each component of $G-X$ and each component of $G[X]$. If $\mathcal{B}$ has positive surplus with respect to the $G[X]$ side, then applying Lemma 2 to any $(k-1)$-coloring of $G-X$ gives a $(k-1)$-coloring of G. So, we may assume that $\mathcal{B}$ does not have positive surplus. That is, for some t, $G[X]$ has a set of t components $\{C_1, \dots, C_t\}$ which together have neighbors in at most t components of $G-X$. But then the other $\ell - t$ components of $G[X]$ together have neighbors in at least $\ell + 1 - t$ components of $G-X$ since G is connected. Let $X' = X \setminus \bigcup_{i \in [t]} V(C_i)$. Then the hypotheses of the lemma are satisfied with X' in place of X, so minimality of $|X|$ shows that G is $(k-1)$-colorable, a contradiction. □

Bipartite graphs

A bipartite graph G with parts A and B is a *2-forest* with respect to A if G is a forest where each vertex in A has degree 2 in G. Plainly, any 2-forest has positive surplus. We first need a few lemmas about bipartite graphs with positive surplus. It is well-known that the edge-minimal positive surplus bipartite graphs are exactly the 2-forests (see [3]). More precisely,

Lemma 4. *Let G be a bipartite graph with parts $A \ne \emptyset$ and B such that G has positive surplus with respect to A. If $G-e$ does not have positive surplus for each $e \in E(G)$, then G is a 2-forest with respect to A.*

The next lemma says that a positive surplus bipartite graph always has a special vertex in A such that removing most of its incident edges leaves a positive surplus graph.

Lemma 5 (Stiebitz)**.** *Let G be a bipartite graph with parts $A \ne \emptyset$ and B such that G has positive surplus with respect to A. Then there is $x \in A$ such that removing any set of $d_G(x) - 2$ edges incident to x leaves a bipartite graph having positive surplus with respect to A.*

Proof. Suppose not and choose a counterexample G minimizing $||G||$. First, suppose there is $\emptyset \ne A' \subsetneq A$ such that $|N(A')| \le |A'| + 1$. Let $G' = G[A' \cup N(A')]$. Then G' has positive surplus with respect to A' and hence by minimality of $||G||$, there is $x \in A'$ such that removing any set of $d_G(x) - 2$ edges incident to x leaves a bipartite graph having positive surplus with respect to A'. If $W \subseteq A \setminus A'$, then $|W| + |A'| + 1 \le |N_G(W) \cup N_G(A')| \le$

$|N_G(W) \setminus N_G(A')| + |N_G(A')| \leq |N_G(W) \setminus N_G(A')| + |A'| + 1$ since G has positive surplus. Therefore $|N_G(W) \setminus N_G(A')| \geq |W|$ for each $W \subseteq A \setminus A'$. Since G is a counterexample, we have $y, z \in N(x)$ such that the graph H made by removing all edges incident to x except xy, xz does not have positive surplus with respect to A. That is, we have $S \subseteq A$ such that $|N_H(S)| \leq |S|$. Let $S' = S \cap A'$ and $W = S \setminus S'$. Since $H[A' \cup N(A')]$ has positive surplus with respect to A', we have $|N_H(S')| \geq |S'| + 1$. But also, $|N_H(W) \setminus N_H(A')| = |N_G(W) \setminus N_G(A')| \geq |W|$, so $|N_H(S)| = |N_H(S')| + |N_H(W) \setminus N_H(A')| \geq |S'| + 1 + |W| = |S| + 1$, a contradiction.

So, for every $\emptyset \neq A' \subsetneq A$ we must have $|N(A')| \geq |A'| + 2$. Pick $x \in A$ arbitrarily. Since G is a counterexample, we have $y, z \in N(x)$ such that the graph H made by removing all edges incident to x except xy, xz does not have positive surplus with respect to A. Since $d_G(x) \geq 3$, there is $w \in N_G(x) \setminus \{y, z\}$. Suppose the bipartite graph $G - xw$ has positive surplus with respect to A. Then, by minimality of $||G||$, there is $u \in A$ such that removing any set of $d_{G-xw}(u) - 2$ edges incident to u from $G - xw$ leaves a bipartite graph having positive surplus with respect to A. If $u \neq x$, then $d_{G-xw}(u) - 2 = d_G(u) - 2$, so u works as the special vertex for G, a contradiction. So, $u = x$ and thus H has positive surplus, a contradiction.

Therefore, there is $\emptyset \neq S \subseteq A$ such that $|N_{G-xw}(S)| \leq |S|$. But $|N_{G-xw}(S)| \geq |N_G(S)| - 1$, so if $S \neq A$, then $|N_{G-xw}(S)| \geq |S| + 2 - 1$. So, we must have $S = A$ and $|N(A)| = |A| + 1$ as well as $|N_{G-xw}(A)| = |N_G(A)| - 1$ and hence $d_G(w) = 1$. This was for arbitrary $x \in A$, so we conclude that every $x \in A$ has a neighbor in B of degree 1 and hence all but at most one vertex in B has degree 1 in G. But then every vertex in A has degree 2 in G and hence will work for the special vertex, giving the final contradiction. □

Proof of Lemma 1. By Lemma 5, there is $x \in A$ such that for any different $y, z \in N(x)$, the bipartite graph H formed from G by removing all edges incident to x except xz, yz has positive surplus with respect to A. But then, by Lemma 4, H contains a 2-forest F with respect to A containing the edges xy and xz. Consider the graph F' formed from F by contracting $\{y, z\}$ and removing x. Then F' is a 2-forest with respect to A since the degree of vertices in A don't change and x is a separating vertex in F. Now G' contains F' and hence has positive surplus, so we are done. □

References

[1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).

[3] László Lovász and M.D. Plummer, *Matching Theory*, vol. 367, American Mathematical Soc., 2009.

[4] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.