graph theory notes*

The combinatorial nullstellensatz and Schauz's coefficient formula

In [2], Alon and Tarsi introduced a beautiful algebraic technique for proving the existence of list colorings. Alon [1] further developed this technique into the *Combinatorial Nullstellensatz*. Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1,\dots,k_n}$ for the coefficient of $x_1^{k_1}\cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1,\dots,x_n]$.

Combinatorial Nullstellensatz (Alon)**.** *Suppose $f \in \mathbb{F}[x_1,\dots,x_n]$ and $k_1,\dots,k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. If $f_{k_1,\dots,k_n} \neq 0$, then for any $A_1,\dots,A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$, there exists $(a_1,\dots,a_n) \in A_1 \times \cdots \times A_n$ with $f(a_1,\dots,a_n) \neq 0$.*

Michałek [5] gave a very short proof of the Combinatorial Nullstellensatz just using long division. Schauz [6] sharpened the Combinatorial Nullstellensatz by proving the following coefficient formula. Versions of this result were also proved by Hefetz [3] and Lasoń [4]. Our presentation is similar to Lasoń's.

Coefficient Formula (Schauz)**.** *Suppose $f \in \mathbb{F}[x_1,\dots,x_n]$ and $k_1,\dots,k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. For any $A_1,\dots,A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$, we have*

$$f_{k_1,\dots,k_n} = \sum_{(a_1,\dots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1,\dots,a_n)}{N(a_1,\dots,a_n)},$$

where

$$N(a_1,\dots,a_n) := \prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (a_i - b).$$

We first give Michałek's proof of the Combinatorial Nullstellensatz and use this to derive the coefficient formula.

Proof of Combinatorial Nullstellensatz. Suppose the result is false and choose $f \in \mathbb{F}[x_1,\dots,x_n]$ for which it fails minimizing $\deg(f)$. Then $\deg(f) \geq 2$ and we have $k_1,\dots,k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$ and $A_1,\dots,A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$ such that $f(a_1,\dots,a_n) = 0$ for all $(a_1,\dots,a_n) \in A_1 \times \cdots \times A_n$. By symmetry, we may assume that $k_1 > 0$. Fix $a \in A_1$ and divide f by $x_1 - a$ to get $f = (x_1 - a)Q + R$ where the degree of x_1 in R is zero. Then the

*clarifications, errors, simplifications ⇒ `landon.rabern@gmail.com` coefficient of $x_1^{k_1-1}x_2^{k_2}\cdots x_n^{k_n}$ in Q must be non-zero and $\deg(Q) < \deg(f)$. So, by minimality of $\deg(f)$ there is $(a_1,\ldots,a_n) \in (A_1 \setminus \{a\}) \times \cdots \times A_n$ such that $Q(a_1,\ldots,a_n) \neq 0$. Since $0 = f(a_1,\ldots,a_n) = (a_1 - a)Q(a_1,\ldots,a_n) + R(a_1,\ldots,a_n)$ we must have $R(a_1,\ldots,a_n) \neq 0$. But x_1 has degree zero in R, so $R(a,\ldots,a_n) = R(a_1,\ldots,a_n) \neq 0$. Finally, this means that $f(a,\ldots,a_n) = (a-a)Q(a,\ldots,a_n) + R(a,\ldots,a_n) \neq 0$, a contradiction. □

Proof of Coefficient Formula. Let $f \in \mathbb{F}[x_1,\ldots,x_n]$ and $k_1,\ldots,k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. Also, let $A_1,\ldots,A_n \subseteq \mathbb{F}$ with $|A_i| = k_i+1$. For each $(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n$, let $\chi_{(a_1,\ldots,a_n)}$ be the characteristic function of the set $\{(a_1,\ldots,a_n)\}$; that is $\chi_{(a_1,\ldots,a_n)} \colon A_1 \times \cdots \times A_n \to \mathbb{F}$ with $\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = 1$ when $(x_1,\ldots,x_n) = (a_1,\ldots,a_n)$ and $\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = 0$ otherwise. Consider the function

$$F = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\chi_{(a_1,\ldots,a_n)}.$$

Then F agrees with f on all of $A_1 \times \cdots \times A_n$ and hence $f - F$ is zero on $A_1 \times \cdots \times A_n$. We will apply the Combinatorial Nullstellensatz to $f - F$ to conclude that $(f - F)_{k_1,\ldots,k_n} = 0$ and hence $f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n}$ where $F_{k_1,\ldots,k_n}$ will turn out to be our desired sum. To apply the Combinatorial Nullstellensatz, we need to represent F as a polynomial, we can do so by representing each $\chi_{(a_1,\ldots,a_n)}$ as a polynomial as follows. For $(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n$, let

$$N(a_1,\ldots,a_n) := \prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (a_i - b).$$

Then it is readily verified that

$$\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = \frac{\prod_{i\in[n]}\prod_{b\in A_i\setminus\{a_i\}}(x_i - b)}{N(a_1,\ldots,a_n)}.$$

Using this to define F we get

$$F(x_1,\ldots,x_n) = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\frac{\prod_{i\in[n]}\prod_{b\in A_i\setminus\{a_i\}}(x_i - b)}{N(a_1,\ldots,a_n)}.$$

Now $\deg(F) = \sum_{i\in[n]}(|A_i| - 1) = \sum_{i\in[n]} k_i = \deg(f)$. Since $f - F$ is zero on $A_1 \times \cdots \times A_n$, applying the Combinatorial Nullstellensatz to $f - F$ with $k_1,\ldots,k_n$ and sets $A_1,\ldots,A_n$ gives $(f - F)_{k_1,\ldots,k_n} = 0$ and hence

$$f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1,\ldots,a_n)}{N(a_1,\ldots,a_n)}.$$

□

1 Applications to graph coloring

Let G be a loopless multigraph with vertex set $V := \{x_1,\ldots,x_n\}$ and edge multiset E. The *graph polynomial* of G is

$$p_G(x_1,\ldots,x_n) := \prod_{\substack{\{x_i,x_j\}\in E\\ i<j}} (x_i - x_j).$$

To each orientation $\vec{G}$ of G, there is a corresponding monomial $m_{\vec{G}}(x_1, \ldots, x_n)$ given by choosing either x_i or $-x_j$ from each factor $(x_i - x_j)$ according to $\vec{G}$. Precisely, given an orientation $\vec{G}$ of G with edge multiset $\vec{E}$, put

$$m_{\vec{G}}(x_1, \ldots, x_n) := \left(\prod_{\substack{(x_i,x_j)\in\vec{E} \\ i<j}} x_i \right) \left(\prod_{\substack{(x_j,x_i)\in\vec{E} \\ i<j}} -x_j \right).$$

Then $p_G(x_1, \ldots, x_n) = \sum_{\vec{G}} m_{\vec{G}}(x_1, \ldots, x_n)$, where the sum is over all orientations $\vec{G}$ of G.

Each $m_{\vec{G}}(x_1, \ldots, x_n)$ has coefficient either 1 or -1. We are interested in collecting up all monomials of the form $x_1^{k_1} \cdots x_n^{k_n}$. Let $DE_{k_1,\ldots,k_n}(G)$ be the orientations of G where $m_{\vec{G}}(x_1, \ldots, x_n) = x_1^{k_1} \cdots x_n^{k_n}$ and $DO_{k_1,\ldots,k_n}(G)$ the orientations of G where $m_{\vec{G}}(x_1, \ldots, x_n) = -x_1^{k_1} \cdots x_n^{k_n}$. Write $p_{k_1,\ldots,k_n}(G)$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in $p_G(x_1, \ldots, x_n)$. Then we have

$$p_{k_1,\ldots,k_n}(G) = |DE_{k_1,\ldots,k_n}(G)| - |DO_{k_1,\ldots,k_n}(G)|.$$

This gives a combinatorial interpretation of the coefficients of p_G, but unfortunately it quantifies over all orientations of G. For applying the Combinatorial Nullstellensatz, it is useful to have a single orientation of G as a certificate that $p_{k_1,\ldots,k_n}(G) \neq 0$. This can be achieved in terms of Eulerian subgraphs. A digraph is Eulerian if the in-degree and out-degree are equal at every vertex. Let $EE(\vec{G})$ be the spanning Eulerian subgraphs of $\vec{G}$ with an even number of edges and let $EO(\vec{G})$ be the spanning Eulerian subgraphs of $\vec{G}$ with an odd number of edges.

Eulerian Reduction. *If $\vec{G}$ is an orientation of G, then we have*

$$\left| |EE(\vec{G})| - |EO(\vec{G})| \right| = \left| |DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| - |DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| \right|.$$

Proof. For $D \in DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G) \cup DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$, let $\vec{G} \oplus D$ be the spanning subgraph of $\vec{G}$ with edge set

$$\left\{ (x_i, x_j) \in E(\vec{G}) \mid (x_j, x_i) \in E(D) \right\}.$$

Then $\vec{G} \oplus D$ is Eulerian since all vertices have the same out-degree in $\vec{G}$ and D. If $\vec{G}$ is even, this gives bijections between $DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EE(\vec{G})$ as well as between $DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EO(\vec{G})$. When $\vec{G}$ is odd, the bijections are between $DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EO(\vec{G})$ as well as between $DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)$ and $EE(\vec{G})$. In either case, we have

$$\left| |EE(\vec{G})| - |EO(\vec{G})| \right| = \left| |DE_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| - |DO_{d^+_{\vec{G}}(x_1),\ldots,d^+_{\vec{G}}(x_n)}(G)| \right|.$$

□

Therefore, if $\vec{G}$ is an orientation of G with $d^+_{\vec{G}}(x_i) = k_i$ for all $i \in [n]$, then

$$|p_{k_1,\ldots,k_n}(G)| = \left| |EE(\vec{G})| - |EO(\vec{G})| \right|.$$

References

[1] N. Alon, *Combinatorial nullstellensatz*, Combinatorics Probability and Computing **8** (1999), no. 1–2, 7–29.

[2] N. Alon and M. Tarsi, *Colorings and orientations of graphs*, Combinatorica **12** (1992), no. 2, 125–134.

[3] Dan Hefetz, *On two generalizations of the Alon–Tarsi polynomial method*, Journal of Combinatorial Theory, Series B **101** (2011), no. 6, 403–414.

[4] Michał Lasoń, *A generalization of combinatorial nullstellensatz*, The Electronic Journal of Combinatorics **17** (2010), no. 1, N32.

[5] Mateusz Michałek, *A short proof of combinatorial nullstellensatz*, The American Mathematical Monthly **117** (2010), no. 9, 821–823.

[6] Uwe Schauz, *Algebraically solvable problems: describing polynomials as equivalent to explicit solutions*, Electron. J. Combin **15** (2008), no. 1.