A farmer has 3000 bananas. He needs to take them to the market 1000 miles away to sell them; however, he only has the strength to carry 1000 bananas at a time. Also, he needs to eat 1 banana to have the energy to travel 1 mile. What is the maximum number of bananas he can get to the market?

Definition. For any integer $0 \leq n \leq 1000$, let $B(n)$ be the maximum number of bananas the farmer can transport n miles.

Note that $B(0) = 3000$ and $B(1000)$ is the quantity we are trying to determine. We assume that bananas are discrete entities.

Lemma. *Let* $1 \leq n \leq 1000$. *Then* $B(n-1) \geq B(n) + 2\left\lceil \frac{B(n)}{999} \right\rceil - 1$.

Proof. Pick a strategy that transports $B(n)$ bananas n miles. Since the farmer can carry only 1000 bananas at a time, one trip can move at most 999 bananas from mile $n-1$ to mile n. Hence, he must make at least $\left\lceil \frac{B(n)}{999} \right\rceil$ trips from mile $n-1$ to mile n. Modify the strategy by removing all trips from mile $n-1$ to mile n. Since all trips except the final one cost the farmer 2 bananas and the final trip costs him 1 banana, he saves $2\left\lceil \frac{B(n)}{999} \right\rceil - 1$ bananas by only transporting them to mile $n-1$. Hence $B(n-1) \geq B(n) + 2\left\lceil \frac{B(n)}{999} \right\rceil - 1$. □

Theorem. *The maximum number of bananas the farmer can transport to the market is* 533.

Proof. We first show that $B(1000) \geq 533$. The naive strategy of moving as much as possible one mile at a time takes care of this. In detail, the farmer moves 1000 bananas to mile 1, takes one banana to get back to take 1000 more bananas to mile 1, and again. He used 5 bananas in the back and forth, so now he has 2995 bananas at mile 1. Repeat this shuffle until he has $3000 - 5(200) = 2000$ bananas at mile 200. Since he can bring 2000 bananas in two loads, he only needs to use 3 bananas to move everything one mile. He does this for 333 miles, ending up with $2000 - 3(333) = 1001$ bananas at mile 533. Now he just loads up 1000 of them and marches to the market, getting there with $1000 - (1000 - 533) = 533$ bananas.

Now we show that $B(1000) \leq 533$. Assume, to get a contradiction, that $B(1000) \geq 534$. Then, by repeatedly applying the Lemma, we have $B(534) \geq 1000$. Now $2\left\lceil \frac{1000}{999} \right\rceil - 1 = 3$, so we may again repeatedly apply the Lemma to yield $B(200) \geq 2002$. Since $2\left\lceil \frac{2002}{999} \right\rceil - 1 = 5$, using the Lemma once again gives $B(0) \geq 3002$. This contradicts the fact that $B(0) = 3000$. Hence our assumption that $B(1000) \geq 534$ was false. Whence $B(1000) \leq 533$. Putting this together with the above yields $B(1000) = 533$.

□