A SIMILAR SHORTER PROOF OF BROOKS’ THEOREM

This is the same, but just excluded diamonds first instead.

**Theorem 1** (Brooks 1941). *Every graph satisfies $\chi \leq \max\{3, \omega, \Delta\}$.*

*Proof.* Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, $G$ is $\Delta$-regular.

First, suppose $G$ is 3-regular. If $G$ contains a diamond $D$, then we may 3-color $G - D$ and easily extend the coloring to $D$ by first coloring the nonadjacent vertices in $D$ the same. So, $G$ doesn’t contain diamonds. Since $G$ is not a forest it contains an induced cycle $C$. Since $K_4 \not\subseteq G$ we have $|N(C)| \geq 2$. So, we may take different $x, y \in N(C)$ and put $H := G - C$ if $x$ is adjacent to $y$ and $H := (G - C) + xy$ otherwise. Then, $H$ doesn’t contain $K_4$ as $G$ doesn’t contain diamonds. By minimality of $|G|$, $H$ is 3-colorable. That is, we have a 3-coloring of $G - C$ where $x$ and $y$ receive different colors. We can easily extend this partial coloring to all of $G$ since each vertex of $C$ has a set of two available colors and some pair of vertices in $C$ get different sets.

Hence we must have $\Delta \geq 4$. Consider a $\Delta$-coloring of $G - v$ for some $v \in V(G)$. Each color must be used on every $K_\Delta$ in $G - v$ and hence some color must be used on every $K_\Delta$ in $G$. Let $M$ be such a color class expanded to a maximal independent set. Then $\chi(G - M) = \chi(G) - 1 = \Delta > \max\{3, \omega(G - M), \Delta(G - M)\}$, a contradiction. $\square$