A SIMILAR SHORTER PROOF OF BROOKS' THEOREM

This is the same, but just excluded diamonds first instead.

Theorem 1 (Brooks 1941)**.** *Every graph satisfies* $\chi \leq \max\{3, \omega, \Delta\}$.

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ-regular.

First, suppose G is 3-regular. If G contains a diamond D, then we may 3-color $G - D$ and easily extend the coloring to D by first coloring the nonadjacent vertices in D the same. So, G doesn't contain diamonds. Since G is not a forest it contains an induced cycle C. Since $K_4 \not\subseteq G$ we have $|N(C)| \geq 2$. So, we may take different $x, y \in N(C)$ and put $H := G - C$ if x is adjacent to y and $H := (G - C) + xy$ otherwise. Then, H doesn't contain K_4 as G doesn't contain diamonds. By minimality of $|G|$, H is 3-colorable. That is, we have a 3-coloring of $G - C$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of C has a set of two available colors and some pair of vertices in C get different sets.

Hence we must have $\Delta \geq 4$. Consider a Δ-coloring of $G - v$ for some $v \in V(G)$. Each color must be used on every K_Δ in $G - v$ and hence some color must be used on every K_Δ in G. Let M be such a color class expanded to a maximal independent set. Then $\chi(G - M) = \chi(G) - 1 = \Delta > \max\{3, \omega(G - M), \Delta(G - M)\}$, a contradiction. □