A DIFFERENT SHORT PROOF OF BROOKS' THEOREM

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Abstract. Lovász gave a short proof of Brooks' theorem by coloring greedily in a good order. We give a different short proof by reducing to the cubic case.

In [5] Lovász gave a short proof of Brooks' theorem by coloring greedily in a good order. Here we give a different short proof by reducing to the cubic case. One interesting feature of the proof is that it doesn't use any connectivity concepts. Our notation follows Diestel [2].

Theorem 1 (Brooks [1] 1941)**.** *Every graph G with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K^{\Delta(G)+1}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ-regular.

First, suppose $\Delta \geq 4$. Pick $v \in V(G)$ and let $w_1, \ldots, w_\Delta$ be v's neighbors. Since $K^{\Delta+1} \not\subseteq G$, by symmetry we may assume that w_2 and w_3 are not adjacent. Choose a $(\Delta+1)$-coloring $\{\{v\}, C_1, \ldots, C_\Delta\}$ of G where $w_i \in C_i$ so as to maximize $|C_1|$. Then C_1 is a maximal independent set in G and in particular, with $H := G - C_1$, we have $\chi(H) = \chi(G) - 1 = \Delta = \Delta(H) + 1 \geq 4$. By minimality of $|G|$, we get $K^\Delta \subseteq H$. But $\{\{v\}, C_2, \ldots, C_\Delta\}$ is a Δ-coloring of H, so any K^Δ in H must contain v and hence w_2 and w_3, a contradiction.

Therefore G is 3-regular. Since G is not a forest it contains an induced cycle C. Put $T := N(C)$. Then $|T| \geq 2$ since $K^4 \not\subseteq G$. Take different $x, y \in T$ and put $H_{xy} := G - C$ if x is adjacent to y and $H_{xy} := (G - C) + xy$ otherwise. Then, by minimality of $|G|$, either H_{xy} is 3-colorable or adding xy created a K^4 in H_{xy}.

Suppose the former happens. Then we have a 3-coloring of $G - C$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of C has a set of two available colors and some pair of vertices in C get different sets.

Whence adding xy created a K^4, call it A, in H_{xy}. We conclude that T is independent and each vertex in T has exactly one neighbor in C. Hence $|T| \geq |C| \geq 3$. Pick $z \in T - \{x, y\}$. Then x is contained in a K^4, call it B, in H_{xz}. Since $d(x) = 3$, we must have $A - \{x, y\} = B - \{x, z\}$. But then any $w \in A - \{x, y\}$ has degree at least 4, a contradiction. □

We note that the reduction to the cubic case is an immediate consequence of more general lemmas on hitting all maximum cliques with an independent set (see [4], [6] and [3]). H. Tverberg pointed out that this reduction was also demonstrated in his paper [7].

References

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