A DIFFERENT SHORT PROOF OF BROOKS’ THEOREM

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Abstract. Lovász gave a short proof of Brooks’ theorem by coloring greedily in a good order. We give a different short proof by reducing to the cubic case.

In [5] Lovász gave a short proof of Brooks’ theorem by coloring greedily in a good order. Here we give a different short proof by reducing to the cubic case. One interesting feature of the proof is that it doesn’t use any connectivity concepts. Our notation follows Diestel [2].

Theorem 1 (Brooks [1] 1941). Every graph $G$ with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K_{\Delta(G) + 1}$.

Proof. Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, $G$ is $\Delta$-regular.

First, suppose $\Delta \geq 4$. Pick $v \in V(G)$ and let $w_1, \ldots, w_\Delta$ be $v$’s neighbors. Since $K_{\Delta + 1} \not\subseteq G$, by symmetry we may assume that $w_2$ and $w_3$ are not adjacent. Choose a $(\Delta + 1)$-coloring $\{\{v\}, C_1, \ldots, C_\Delta\}$ of $G$ where $w_i \in C_i$ so as to maximize $|C_1|$. Then $C_1$ is a maximal independent set in $G$ and in particular, with $H := G - C_1$, we have $\chi(H) = \chi(G) - 1 = \Delta = \Delta(H) + 1 \geq 4$. By minimality of $|G|$, we get $K_{\Delta} \subseteq H$. But $\{\{v\}, C_2, \ldots, C_\Delta\}$ is a $\Delta$-coloring of $H$, so any $K_{\Delta}$ in $H$ must contain $v$ and hence $w_2$ and $w_3$, a contradiction.

Therefore $G$ is 3-regular. Since $G$ is not a forest it contains an induced cycle $C$. Put $T := N(C)$. Then $|T| \geq 2$ since $K_4 \not\subseteq G$. Take different $x, y \in T$ and put $H_{xy} := G - C$ if $x$ is adjacent to $y$ and $H_{xy} := (G - C) + xy$ otherwise. Then, by minimality of $|G|$, either $H_{xy}$ is 3-colorable or adding $xy$ created a $K_4$ in $H_{xy}$.

Suppose the former happens. Then we have a 3-coloring of $G - C$ where $x$ and $y$ receive different colors. We can easily extend this partial coloring to all of $G$ since each vertex of $C$ has a set of two available colors and some pair of vertices in $C$ get different sets.

Whence adding $xy$ created a $K_4$, call it $A$, in $H_{xy}$. We conclude that $T$ is independent and each vertex in $T$ has exactly one neighbor in $C$. Hence $|T| \geq |C| \geq 3$. Pick $z \in T - \{x, y\}$. Then $x$ is contained in a $K_4$, call it $B$, in $H_{xz}$. Since $d(x) = 3$, we must have $A - \{x, y\} = B - \{x, z\}$. But then any $w \in A - \{x, y\}$ has degree at least 4, a contradiction.

We note that the reduction to the cubic case is an immediate consequence of more general lemmas on hitting all maximum cliques with an independent set (see [4], [6] and [3]). H. Tverberg pointed out that this reduction was also demonstrated in his paper [7].

References