Coloring graphs from almost maximum degree sized palettes, a prospectus

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1 Introduction

The federal prison system is collapsing, due in part to inefficient resource management. You have recently been hired as an efficiency expert by one of the large private firms heading the privatization of the prison system. Your first task is to design new, cheap prisons that will minimize inmate fighting. Cell walls are expensive, so you decide to replace the many two-person cells with a few giant cells able to handle arbitrarily many inmates. But you also need to minimize inmate fighting and the only data you have is which inmates have fought with each other in the past. Without further information, the best you can do is to not put inmates who have fought before into the same cell. So how many of these massive cells are you going to need?

As a first pass, you imagine lining the inmates up and having them enter the first cell they can without encountering one of their former rivals. You realize that the worst case that could happen is if a given inmate was behind all of his rivals and all of them went into different cells. So, if Δ is the maximum number of inmates a given inmate has fought with, then you will need at most $\Delta + 1$ cells. But cells cost money, can you do better? Is there always a way to put the inmates into Δ cells? You quickly see that this won't be possible if there is a group of $\Delta + 1$ inmates who have all fought one another or if Δ is two and there is an odd number of inmates that can be arranged in a circle so that each has fought both and only his neighbors.

After a few days of hard thought, you are able to prove that these are the only obstructions to using Δ cells (Brooks 1941 [8]). Usually Δ is at least three and there is no such *clique* of $\Delta + 1$ inmates who have all fought one another, so using Δ cells trades a little possibility of fighting for a large savings in cost.

*The author of the prospectus and prospectus itself are, of course, imaginary. Nevertheless, it is clear that such persons as the writer of this prospectus not only may but positively must, exist—given the necessity of mathematical truth.

And yet you aren't satisfied, how much better can you do? What obstructions are there to using $\Delta - 1$ cells and do they occur infrequently? After working on this for a long time, you find obstructions other than large cliques for all Δ up to eight (see Figure 1 for an example) but your construction techniques all fail for Δ at least nine, so you conjecture that the only obstructions in this case are the Δ cliques (Borodin and Kostochka 1977 [7]).

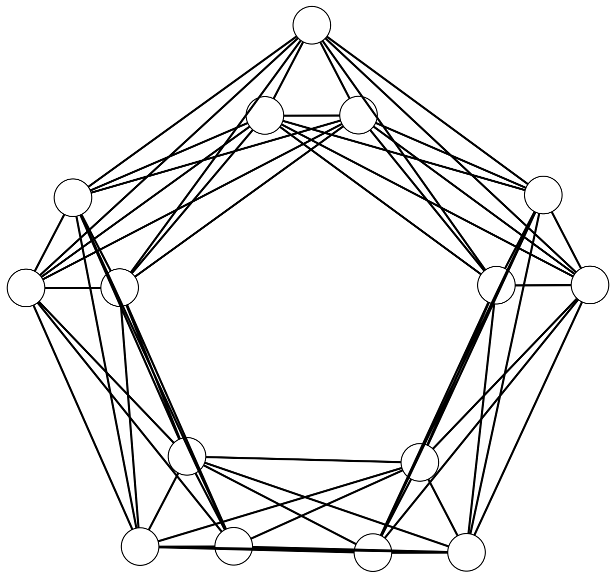

Figure 1: A group of prisoners that cannot be put into $\Delta - 1$ cells.

Even though you suspect the conjecture to be true, you can't be sure there isn't some often occurring arrangement of prisoners with Δ at least nine that you haven't considered. However, there is a little more structure in your data—you notice that, in all cases, the most experienced fighters (the ones who have fought with Δ others) have not fought with each other. And behold, assuming that none of the most experienced fighters have fought each other, you are able to prove that for Δ at least seven the only obstruction to using $\Delta - 1$ cells is a clique of Δ inmates (Kierstead and Kostochka 2009 [27]).

Throughout the course of your work you discover that there is a whole field, called Graph Coloring, which does research into such problems. In fact, in your latest proof you used a few high-tech tools due to Gallai [19], Stiebitz [58] and Alon-Tarsi [3]. The proof technique naturally failed for Δ equal to six, but you conjecture to yourself that it must hold (Kierstead and Kostochka 2009 [27]), so you need different tools. After much searching, you find the needed tool in an obscure Russian text (Mozhan 1983 [41]) and give a technical algorithmic proof for the $\Delta = 6$ case (Rabern 2010 [51]). Actually, you prove a bit more—as long as there aren't cliques of size around $\frac{\Delta}{2}$ among the most experienced fighters, you can use $\Delta - 1$ cells.

What about when $\Delta = 5$? By refining the algorithmic method you are able to prove that, in this case, there is only one obstruction (see Figure 2),

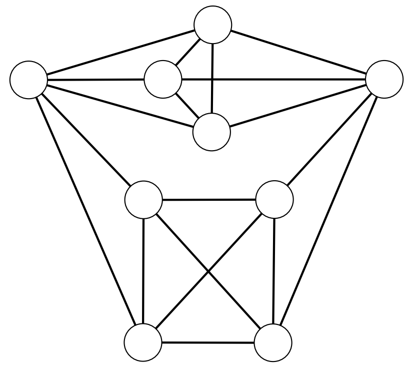

Figure 2: A group of prisoners that cannot be put into $\Delta - 1$ cells.

besides a Δ clique, to using $\Delta - 1$ cells (Kostochka, Rabern and Stiebitz 2011 [35]).

Eventually, you become so obsessed with this problem that you decide to quit your job and learn all you can about this field of Graph Coloring. Moreover, you decide to adopt that field's parlance in place of all your cell and prisoner talk and write this prospectus using Graph Theory terminology and notation.

2 A short history

Here we collect statements of the results and conjectures that have bearing on this inquiry woven together with some historical remarks and our improvements. The first non-trivial result about coloring graphs with around Δ colors is Brooks' theorem from 1941.

Theorem 2.1 (Brooks [8]). *Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.*

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings. Counterexamples exist showing that the $\Delta \geq 9$ condition is tight.

Conjecture 2.2 (Borodin and Kostochka [7]). *Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.*

Note that another way of stating this is that for $\Delta \geq 9$, the only obstruction to $(\Delta - 1)$-coloring is a K_Δ. In the same paper, Borodin and Kostochka prove the following weaker statement.

Theorem 2.3 (Borodin and Kostochka [7]). *Every graph satisfying $\chi \geq \Delta \geq 7$ contains a $K_{\left\lfloor \frac{\Delta+1}{2} \right\rfloor}$.*

The proof is quite simple once you have a decomposition lemma of Lovász from the 1960's [38].

Lemma 2.4 (Lovász [38]). *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.*

Proof. For a partition $P := (V_1, \dots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(\|G[V_i]\| - r_i |V_i|\right).$$

Let $P := (V_1, \dots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$. Suppose there is $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) > r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$, there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$ and thus moving x from V_i to V_j gives a new partition violating minimality of $f(P)$. Hence $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. $\square$

Now to prove Borodin and Kostochka's result, let G be a graph with $\chi \geq \Delta \geq 7$ and use $r_1 := \left\lceil \frac{\Delta-1}{2} \right\rceil$ and $r_2 := \left\lfloor \frac{\Delta-1}{2} \right\rfloor$ in Lovász's lemma to get a partition (V_1, V_2) of $V(G)$ with $\Delta(G[V_i]) \leq r_i$ for each $i \in [2]$. Since $r_1 + r_2 = \Delta - 1$ and $\chi \geq \Delta$, it must be that $\chi(G[V_i]) \geq r_i + 1$ for some $i \in [2]$. But $\Delta \geq 7$, so $r_i \geq 3$ and hence by Brooks' theorem $G[V_i]$ contains a $K_{\left\lfloor \frac{\Delta+1}{2} \right\rfloor}$.

A decade later, Catlin [10] showed that bumping the $\Delta(G)+1$ to $\Delta(G)+2$ allowed for shuffling vertices from one partition set to another and thereby proving stronger decomposition results. A few years later Kostochka [34] modified Catlin's algorithm to show that every triangle-free graph G can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. In [50], we generalized Kostochka's modification to prove the following.

Lemma 4.7 (Rabern [50]). *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G)+2-k$. Then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no incomplete r_i-regular components for each $i \in [k]$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's triangle-free coloring result.

Corollary 4.8 (Rabern [50]). *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2 \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}\Delta(G)+2$, just use $r_k = 0$ when $\Delta \equiv 2 (\text{mod } 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 4.9 (Rabern [50]). *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r.*

In fact, we proved a lemma stronger than Lemma 4.7 allowing us to forbid a larger class of components coming from any so-called *r-permissible collection*. In section 4.2 we will explore a result that both simplifies and generalizes this latter result.

Also in the 1980's, Kostochka proved the following using a complicated recoloring argument together with a technique for reducing Δ in a counterexample based on hitting every maximum clique with an independent set.

Theorem 2.5 (Kostochka [33]). *Every graph satisfying $\chi \geq \Delta$ contains a $K_{\Delta-28}$.*

Kostochka [33] proved the following result which shows that graphs having clique number sufficiently close to their maximum degree contain an independent set hitting every maximum clique. In [49] we improved the antecedent to $\omega \geq \frac{3}{4}(\Delta+1)$. Finally, King [30] made the result tight.

Lemma 5.1 (Kostochka [33]). *If G is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

Lemma 5.3 (Rabern [49]). *If G is a graph satisfying $\omega \geq \frac{3}{4}(\Delta+1)$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

Lemma 5.5 (King [30]). *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta+1)$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

If G is a vertex critical graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ and we expand the independent set I produced by Lemma 5.5 to a maximal independent set M and remove M from G, we see that $\Delta(G-M) \leq \Delta(G)-1$, $\chi(G-M) = \chi(G)-1$ and $\omega(G-M) = \omega(G)-1$. Using this, the proof of many coloring results can be reduced to the case of the smallest Δ for which they work. In Section 5, we give three such applications.

A little after Kostochka proved his bound, Mozhan [41] used a function minimization and vertex shuffling procedure different than, but related to Catlin's, to prove the following.

Theorem 2.6 (Mozhan [41]). *Every graph satisfying $\chi \geq \Delta \geq 10$ contains a $K_{\lfloor \frac{2\Delta+1}{3} \rfloor}$.*

Finally, in his dissertation Mozhan proved the following. We don't know the method of proof as we were unable to obtain a copy of his dissertation. However, we suspect the method is a more complicated version of the above proof.

Theorem 2.7 (Mozhan). *Every graph satisfying $\chi \geq \Delta \geq 31$ contains a $K_{\Delta-3}$.*

In [51], we used part of Mozhan's method to prove the following result. For a graph G let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree at least $\chi(G)$.

Theorem 2.8 (Rabern [51])**.** *$K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.*

Setting $\omega(\mathcal{H}(G)) = 1$ proved a conjecture of Kierstead and Kostochka [27].

Corollary 2.9 (Rabern [51])**.** *$K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.*

In joint work with Kostochka and Stiebitz [35], we generalized and improved this result, again using Mozhan's technique. In section 4.1, we will improve these results further and simplify the proofs by using Catlin's vertex shuffling algorithm in place of Mozhan's.

In 1999, Reed used probabilistic methods to prove that the Borodin-Kostochka conjecture holds for graphs with very large maximum degree.

Theorem 2.10 (Reed [55])**.** *Every graph satisfying $\chi \geq \Delta \geq 10^{14}$ contains a K_Δ.*

A lemma from Reed's proof of the above theorem is generally useful.

Lemma 2.11 (Reed [55])**.** *Let G be a critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most 4 neighbors in H. In particular, the $K_{\Delta-1}$'s in G are pairwise disjoint.*

In section 8, we improve this lemma by showing that under the same hypotheses, any vertex in $G - H$ has at most 1 neighbor in H. Moreover, we lift the result out of the context of a minimal counterexample to graphs satisfying a certain criticality condition—we refer to such graphs as mules. This allows meaningful results to be proved for values of Δ less than 9. Also in section 8, we prove that the following, a priori weaker, conjecture is equivalent to the Borodin-Kostochka conjecture.

Conjecture 2.12 (Rabern 2010)**.** *If G is a graph with $\chi = \Delta = 9$, then $K_3 * \overline{K}_6 \subseteq G$.*

At the core of these results are the list coloring lemmas proved in section 6. There we classify graphs of the form $A * B$ which are not f-choosable where $f(v) := d(v) - 1$ for each vertex v. In section 7 we use these list coloring results together with Chudnovsky and Seymour's decomposition theorem for claw-free graphs [14] and our proof in [48] of the Borodin-Kostochka conjecture for line graphs of multigraphs to prove the conjecture for claw-free graphs.

Theorem 2.13 (Rabern 2011)**.** *Every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.*

3 Overview

Our approach to coloring a graph with around Δ colors has two main steps.

1. Construct a large dense subgraph H.
2. Coloring the rest of the graph inductively leaves a list assignment on H. Try to color H from these lists.

For step (1), we have used two different constructions, both involve minimizing edges in some partition and then shuffling vertices around. Originally (in [51] and [43]), we used the construction of Mozhan [41]; however, in this inquiry we will replace this with a construction that can be attributed to Catlin [10]. This will be done in section 4.

For step (2), we do a lot of work in section 6 to prove results on coloring graphs with almost degree sized lists. Additionally, in section 8 we use minimality of a counterexample to show that the resulting list coloring has special properties allowing us to finish the coloring.

Another important tool, developed in section 5, allows us to reduce Δ in minimum counterexamples to small, sometimes manageable values.

4 Doing the vertex shuffle

Let $\mathcal{G}$ be the collection of all finite simple connected graphs. For a graph G, $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$. Let $\mathcal{C}_G$ be the components of G and $c(G) := |\mathcal{C}_G|$. If $h \colon \mathcal{G} \to \mathbb{N}$, we define h for any graph as $h(G) := \sum_{D \in \mathcal{C}_G} h(D)$. An *ordered partition* of G is a sequence $(V_1, V_2, \ldots, V_k)$ where the V_i are pairwise disjoint and cover $V(G)$. Note that we allow the V_i to be empty. When there is no possibility of ambiguity, we call such a sequence a *partition*.

4.1 Coloring when the high vertex subgraph has small cliques

In this section we generalize all of the results on coloring graphs with restrictions on the high vertex subgraph from [51] and [35]. Moreover, our proofs here are simpler and much easier to visualize thanks to using the vertex shuffling procedure of Catlin [10] in place of that of Mozhan [41]. The proof technique can be viewed as a generalization of that of Bollobás and Manvel [5]. We give a non-standard proof of Brooks' theorem to illustrate the technique.

4.1.1 Brooks' theorem

Let G be a graph. A partition $P := (V_0, V_1)$ of $V(G)$ will be called *normal* if it achieves the minimum value of $(\Delta(G) - 1)\left\|V_0\right\| + \left\|V_1\right\|$. Note that if P is a normal partition, then $\Delta(G[V_0]) \leq 1$ and $\Delta(G[V_1]) \leq \Delta(G) - 1$. The *$P$-components* of G are the components of $G[V_i]$ for $i \in [2]$. A P-component is called an *obstruction* if it is a K_2 in $G[V_0]$ or a $K_{\Delta(G)}$ in $G[V_1]$ or an odd cycle in $G[V_1]$ when $\Delta(G) = 3$. A path $x_1x_2 \cdots x_k$ is called *P-acceptable* if x_1 is contained in an obstruction and for different $i, j \in [k]$, x_i and x_j are in different P-components. For a subgraph H of G and $x \in V(G)$, we put $N_H(x) := N(x) \cap V(H)$.

Lemma 4.1. *Let G be a graph with $\Delta(G) \geq 3$. If G doesn't contain $K_{\Delta(G)+1}$, then $V(G)$ has an obstruction-free normal partition.*

Proof. Suppose the lemma is false. Among the normal partitions having the minimum number of obstructions, choose $P := (V_0, V_1)$ and a maximal P-acceptable path $x_1x_2 \cdots x_k$ so as to minimize k.

Let A and B be the P-components containing x_1 and x_k respectively. Put $X := N_A(x_k)$. First, suppose $|X| = 0$. Then moving x_1 to the other part of P creates another normal partition P' having the minimum number of obstructions. But $x_2x_3 \cdots x_k$ is a maximal P'-acceptable path, violating the minimality of k. Hence $|X| \geq 1$.

Pick $z \in X$. Moving z to the other part of P destroys the obstruction A, so it must create an obstruction containing x_k and hence B. Since obstructions are complete graphs or odd cycles, the only possibility is that $\{z\} \cup V(B)$ induces an obstruction. Put $Y := N_B(z)$. Then, since obstructions are regular, $N_B(x) = Y$ for all $x \in X$ and $|Y| = \delta(B) + 1$. In particular, X is joined to Y in G.

Suppose $|X| \geq 2$. Then, similarly to above, switching z and x_k in P shows that $\{x_k\} \cup V(A - z)$ induces an obstruction. Since obstructions are regular, we must have $|N_{A-z}(x_k)| = \Delta(A)$ and hence $|X| \geq \Delta(A) + 1$. Thus $|X \cup Y| = \Delta(A) + \delta(B) + 2 = \Delta(G) + 1$. Suppose X is not a clique and pick nonadjacent $v_1, v_2 \in X$. It is easily seen that moving v_1, v_2 and then x_k to their respective other parts violates normality of P. Hence X is a clique. Suppose Y is not a clique and pick nonadjacent $w_1, w_2 \in Y$. Pick $z' \in X - \{z\}$. Now moving z and then w_1, w_2 and then z' to their respective other parts again violates normality of P. Hence Y is a clique. But X is joined to Y, so $X \cup Y$ induces a $K_{\Delta(G)+1}$ in G, a contradiction.

Hence we must have $|X| = 1$. Suppose $X \neq \{x_1\}$. First, suppose A is K_2. Then moving z to the other part of P creates another normal partition Q having the minimum number of obstructions. In Q, $x_kx_{k-1} \cdots x_1$ is a maximal Q-acceptable path since the Q-components containing x_2 and x_k contain all of x_1's neighbors in that part. Running through the above argument using Q gets us to the same point with A not K_2. Hence we may assume A is not K_2.

Move each of $x_1, x_2, \ldots, x_k$ in turn to their respective other parts of P. Then the obstruction A was destroyed by moving x_1 and for $1 \leq i < k$, the obstruction created by moving x_i was destroyed by moving x_{i+1}. Thus, after the moves, x_k is contained in an obstruction. By minimality of k, it must be that $\{x_k\} \cup V(A - x_1)$ induces an obstruction and hence $|X| \geq 2$, a contradiction.

Therefore $X = \{x_1\}$. But then moving x_1 to the other part of P creates an obstruction containing both x_2 and x_k. Hence $k = 2$. Since x_1x_2 is maximal, x_2 can have no neighbor in the other part besides x_1. But now switching x_1 and x_2 in P creates a partition violating the normality of P. □

Theorem 4.2 (Brooks 1941)**.** *If a connected graph G is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Suppose not and choose a counterexample G minimizing $\Delta(G)$. Plainly, $\Delta(G) \geq 3$. By Lemma 4.1, $V(G)$ has an obstruction-free normal partition (V_0, V_1). Since $G[V_0]$ has maximum degree at most one and contains no K_2's, we see that V_0 is independent. Since $G[V_1]$ is obstruction-free, applying minimality of $\Delta(G)$ gives $\chi(G[V_1]) \leq \Delta(G[V_1]) < \Delta(G)$. Hence $\chi(G) \leq \Delta(G)$, a contradiction. □

4.1.2 The generalizations

For a vector $\mathbf{r} \in \mathbb{N}^k$ we take the coordinate labeling $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ as convention. Define the *weight* of a vector $\mathbf{r} \in \mathbb{N}^k$ as $w(\mathbf{r}) := \sum_{i\in[k]} r_i$. Let G be a graph. An $\mathbf{r}$-*partition* of G is a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ minimizing

$$f(P) := \sum_{i\in[k]} \left(\|G[V_i]\| - r_i\,|V_i|\right).$$

It is a fundamental result of Lovász [38] that if $P := (V_1, \ldots, V_k)$ is an $\mathbf{r}$-partition of G with $w(\mathbf{r}) \geq \Delta(G) + 1 - k$, then $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. The proof is simple: if there is a vertex in a part violating the condition, then there is some part it can be moved to that decreases $f(P)$. As Catlin [10] showed, with the stronger condition $w(\mathbf{r}) \geq \Delta(G) + 2 - k$, a vertex of degree r_i in $G[V_i]$ can always be moved to some other part while maintaining $f(P)$. Since G is finite, a well-chosen sequence of such moves must always wrap back on itself. Both Catlin [10], and independently Bollobás and Manvel [5] used such a technique to prove coloring results. We generalize these techniques by taking into account the degree in G of the vertex to be moved—a vertex of degree less than the maximum needs a weaker condition on $w(\mathbf{r})$ to be moved.

For an induced subgraph H of G, define $\delta_G(H) := \min_{v\in V(H)} d_G(v)$. We also need the following notion of a movable subgraph.

Definition 4.1. Let G be a graph and H an induced subgraph of G. For $d \in \mathbb{N}$, the *d-movable subgraph* of H with respect to G is the subgraph H^d of G induced on

$$\{v \in V(H) \mid d_G(v) = d \text{ and } H - v \text{ is connected}\}.$$

We prove two partition lemmas of similar form. All of our coloring results will follow from the first lemma, the second lemma is a degeneracy result from which the main lemma of Bollobás and Manvel [5] follows. For unification purposes, define a *t-obstruction* as an odd cycle when $t = 2$ and a K_{t+1} when $t \geq 3$.

Theorem 4.3 (Rabern [53]). *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}^k_{\geq 2}$. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

1. *there exists an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G such that if C is an r_i-obstruction in $G[V_i]$, then $\delta_G(C) \geq d$ and C^d is edgeless.*

2. *$w(\mathbf{r}) = d$ and G contains an induced subgraph Q with $|Q| = d + 1$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*

   (a) $|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i \geq 2$,

   (b) $\left|F_1^d\right| \geq 2$, $\left|F_i^d\right| \geq 1$ for $i \geq 2$,

   (c) for $i \in [k]$, each $v \in V(F_i^d)$ is universal in Q;

Theorem 4.4 (Rabern [53]). *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}^k_{\geq 1}$ where at most one of the r_i is one. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

1. *there exists an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G such that if C is an r_i-regular component of $G[V_i]$, then $\delta_G(C) \geq d$ and there is at most one $x \in V(C^d)$ with $d_{C^d}(x) \geq r_i - 1$. Moreover, P can be chosen so that either:*

   (a) for all $i \in [k]$ and r_i-regular component C of $G[V_i]$, we have $\left|C^d\right| \leq 1$; or,

   (b) for some $i \in [k]$ and some r_i-regular component C of $G[V_i]$, there is $x \in V(C^d)$ such that $\{y \in N_C(x) \mid d_G(y) = d\}$ is a clique.

2. *$w(\mathbf{r}) = d$ and G contains a $K_t * E_{d+1-t}$ where $t \geq d + 1 - k$, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{d+1-t})$ we have $d_G(v) > d$; or,*

From Theorem 4.3, we get the following two coloring results. For a vertex critical graph G, call $v \in V(G)$ *low* if $d(v) = \chi(G) - 1$ and *high* otherwise. Let $\mathcal{H}(G)$ be the subgraph of G induced on the high vertices of G.

Corollary 4.5 (Rabern [53]). *Let G be a vertex critical graph with $\chi(G) = \Delta(G) + 2 - k$ for some $k \geq 2$. If $k \leq \frac{\chi(G)-1}{\omega(\mathcal{H}(G))+1}$, then G contains an induced subgraph Q with $|Q| = \chi(G)$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*

1. *$|F_1| = \chi(G) - (k-1)(\omega(\mathcal{H}(G)) + 1)$, $|F_i| = \omega(\mathcal{H}(G)) + 1$ for $i \geq 2$;*

2. *for each $i \in [k]$, F_i contains at least $|F_i| - \omega(\mathcal{H}(G))$ low vertices which are all universal in Q.*

Corollary 4.6 (Rabern [53]). *Let G be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.*

4.2 Destroying incomplete components in vertex partitions

In [34] Kostochka modified an algorithm of Catlin [10] to show that every triangle-free graph G can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. In fact, his modification proves that the vertex set of any triangle-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of paths. In [50] we generalized this as follows.

Lemma 4.7 (Rabern [50])**.** *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G)+2-k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no incomplete r_i-regular components for each $i \in [k]$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's theorem.

Corollary 4.8 (Rabern [50])**.** *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}\Delta(G)+2$, just use $r_k = 0$ when $\Delta \equiv 2 (\mathrm{mod}\ 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 4.9 (Rabern [50])**.** *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r.*

For the purposes of coloring it is more economical to split off $\Delta + 2 - (r+1)\left\lfloor \frac{\Delta+2}{r+1} \right\rfloor$ parts with $r_j = 0$.

Corollary 4.10 (Rabern [50])**.** *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r and $\Delta(G) + 2 - (r+1)\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ independent sets. In particular, $\chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$.*

For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks' Theorem. Lemma 4.7 is of the same form as Lovász's Lemma 2.4, but it gives a more restrictive partition at the cost of replacing $\Delta(G)+1$ with $\Delta(G)+2$. For $r \geq 3$, combining Lovász's Lemma 2.4 with Brooks' theorem gives the following better bound for a K_{r+1}-free graph G (first proved in [7], [11] and [37]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G)+1}{r+1} \right\rfloor .$$

4.2.1 A generalization

Here we prove a generalization of Lemma 4.7.

Definition 4.2. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a vertex $x \in V(G)$ is called *h-critical* in G if $G - x \in \mathcal{G}$ and $h(G - x) < h(G)$.

Definition 4.3. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a pair of vertices $\{x, y\} \subseteq V(G)$ is called an *h-critical pair* in G if $G - \{x, y\} \in \mathcal{G}$ and x is h-critical in $G - y$ and y is h-critical in $G - x$.

Definition 4.4. For $r \in \mathbb{N}$ a function $h\colon \mathcal{G} \to \mathbb{N}$ is called an *r-height function* if it has each of the following properties:

1. if $h(G) > 0$, then G contains an h-critical vertex x with $d(x) \geq r$;
2. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then $h(G - x) = h(G) - 1$;
3. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then G contains an h-critical vertex $y \notin \{x\} \cup N(x)$ with $d(y) \geq r$;
4. if $G \in \mathcal{G}$ and $\{x, y\} \subseteq V(G)$ is an h-critical pair in G with $d_{G-y}(x) \geq r$ and $d_{G-x}(y) \geq r$, then there exists $z \in N(x) \cap N(y)$ with $d(z) \geq r + 1$.

Lemma 4.11. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. If h_i is an r_i-height function for each $i \in [k]$, then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that for each $i \in [k]$, $\Delta(G[V_i]) \leq r_i$ and $h_i(D) = 0$ for each component D of $G[V_i]$.*

For each $r \in \mathbb{N}$, it is easy to see that the function $h_r\colon \mathcal{G} \to \mathbb{N}$ defined as follows is an r-height function:

$$h_r(G) := \begin{cases} 1 & G \text{ is incomplete and } r\text{-regular;} \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 4.11 with these height functions proves Lemma 4.7. Other height functions exist, but we don't yet have a sense of their ubiquity or lack thereof.

Proof of Lemma 4.11. For a partition $P := (V_1, \dots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(\|G[V_i]\| - r_i\,|V_i|\right),$$

$$c(P) := \sum_{i=1}^{k} c(G[V_i]),$$

$$h(P) := \sum_{i=1}^{k} h_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$, and subject to that $c(P)$, and subject to that $h(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G)+2-k$ there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_{V_i}(x) > r_i$ we would have $f(P^*) < f(P)$ contradicting the minimality of P. This proves that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.

Now suppose that for some i_1 there is a component A_1 of $G[V_{i_1}]$ with $h_{i_1}(A_1) > 0$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. By property 1 of height functions, we have an h_{i_1}-critical vertex $x_1 \in V(A_1)$ with $d_{A_1}(x_1) \geq r_{i_1}$. By the above we have $i_2 \neq i_1$ such that moving x_1 from V_{1,i_1} to V_{1,i_2} gives a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{1,i_2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since x_1 is h_{i_1}-critical, by the minimality of $h(P_1)$, it must be that $h_{i_2}(A_2) > h_{i_2}(C_2)$. By property 2 of height functions we must have $h_{i_2}(A_2) = h_{i_2}(C_2) + 1$. Hence $h(P_2)$ is still minimum. Now, by property 3 of height functions, we have an h_{i_2}-critical vertex $x_2 \in V(A_2) - (\{x_1\} \cup N_{A_2}(x_1))$ with $d_{A_2}(x_2) \geq r_{i_2}$.

Continue on this way to construct sequences $i_1, i_2, \ldots$, $A_1, A_2, \ldots$, $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. In particular, $\{x_s, x_{t+1}\}$ is an h_{i_s}-critical pair in $Q := G\left[\{x_{t+1}\} \cup V(A_s)\right]$ where $d_{Q-x_{t+1}}(x_s) \geq r_{i_s}$ and $d_{Q-x_s}(x_{t+1}) \geq r_{i_s}$. Thus, by property 4 of height functions, we have $z \in N_Q(x_s) \cap N_Q(x_{t+1})$ with $d_Q(z) \geq r_{i_s} + 1$.

We now modify P_s to contradict the minimality of $f(P)$. At step $t+1$, x_t was adjacent to exactly r_{i_s} vertices in V_{t+1,i_s}. This is what allowed us to move x_t into V_{t+1,i_s}. Our goal is to modify P_s so that we can move x_t into the i_s part without moving x_s out. Since z is adjacent to both x_s and x_t, moving z out of the i_s part will then give us our desired contradiction.

So, consider the set X of vertices that could have been moved out of V_{s,i_s} between step s and step $t+1$; that is, $X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,i_s}$. For $x_j \in X$, since $d_{A_j}(x_j) \geq r_{i_s}$ and x_j is not adjacent to x_{j-1} we see that $d_{V_{s,i_s}}(x_j) \geq r_{i_s}$. Similarly, $d_{V_{s,i_t}}(x_t) \geq r_{i_t}$. Also, by the minimality of t, X is an independent set in G. Thus we may move all elements of X out of V_{s,i_s} to get a new partition $P^* := (V_{*,1}, \ldots, V_{*,k})$ with $f(P^*) = f(P)$.

Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and $t+1$ are the elements of X, we see that $d_{V_{*,i_s}}(x_t) = r_{i_s}$. Since $d_{V_{*,i_t}}(x_t) \geq r_{i_t}$ we can move x_t from V_{*,i_t} to V_{*,i_s} to get a new partition $P^{**} := (V_{**,1}, \ldots, V_{**,k})$ with $f(P^{**}) = f(P^*)$. Now, recall that $z \in V_{**,i_s}$. Since z is adjacent to x_t we have $d_{V_{**,i_s}}(z) \geq r_{i_s}+1$. Thus we may move z out of V_{**,i_s} to get a new partition P^{***} with $f(P^{***}) < f(P^{**}) = f(P)$. This contradicts the minimality of $f(P)$. $\square$

5 Reducing maximum degree

5.1 Hitting all maximum cliques

As part of his proof that every graph with $\chi \geq \Delta$ contains a $K_{\Delta-28}$, Kostochka proved the following lemma.

Lemma 5.1 (Kostochka [33])**.** *If G is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

To talk about the proof we first need a definition.

Clique Graph. Let G be a graph. For a collection of cliques $\mathcal{Q}$ in G, let $X_{\mathcal{Q}}$ be the intersection graph of $\mathcal{Q}$. That is, the vertex set of $X_{\mathcal{Q}}$ is $\mathcal{Q}$ and there is an edge between $Q_1 \neq Q_2 \in \mathcal{Q}$ iff Q_1 and Q_2 intersect.

Kostochka's proof proceeded in two stages. First show that the vertices in each component of the clique graph have a large intersection. Then find an independent transversal of these intersections. Such a transversal is an independent set hitting every maximum clique. Kostochka used a custom method to find a transversal. In [49], we applied the following lemma of Haxell [23] (proved long after Kostochka's paper) to find the independent transversal.

Lemma 5.2. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. Then H has an independent set $\{v_1, \ldots, v_n\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Finding the independent transversal using this lemma gives the following.

Lemma 5.3 (Rabern [49])**.** *If G is a graph satisfying $\omega \geq \frac{3}{4}(\Delta + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

Aharoni, Berger and Ziv [1] showed that Haxell's proof actually gets more than Lemma 5.2. From their extension, King [30] proved the following lopsided version of Haxell's lemma.

Lemma 5.4 (King [30])**.** *Let G be a graph partitioned into r cliques $V_1, \ldots, V_r$. If there exists $k \geq 1$ such that for each i every $v \in V_i$ has at most $\min\{k, |V_i| - k\}$ neighbors outside V_i, then G contains an independent set with r vertices.*

Using this gives the best possible form of the lemma.

Lemma 5.5 (King [30])**.** *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$, then G contains an independent set I such that $\omega(G - I) < \omega(G)$.*

5.1.1 A simple proof of Kostochka's first stage

The proofs for Kostochka's first stage can be made much simpler than the originals and we do so here.

Lemma 5.6 (Hajnal [22])**.** *Let G be a graph and $\mathcal{Q}$ a collection of maximum cliques in G. Then*

$$\left|\bigcup \mathcal{Q}\right| + \left|\bigcap \mathcal{Q}\right| \geq 2\omega(G).$$

Proof. Suppose the lemma is false and let $\mathcal{Q}$ be a counterexample with $|\mathcal{Q}|$ minimal. Put $r := |\mathcal{Q}|$ and say $\mathcal{Q} = \{Q_1, \ldots, Q_r\}$. Consider the set $W := (Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i$. Plainly, W is a clique. Thus we may derive a contradiction as follows.

$$\begin{aligned}
\omega(G) &\geq |W| \\
&= \left|(Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i\right| \\
&= \left|Q_1 \cap \bigcup_{i=2}^r Q_i\right| + \left|\bigcap_{i=2}^r Q_i\right| - \left|\bigcap_{i=1}^r Q_i \cap \bigcup_{i=2}^r Q_i\right| \\
&= |Q_1| + \left|\bigcup_{i=2}^r Q_i\right| - \left|\bigcup_{i=1}^r Q_i\right| + \left|\bigcap_{i=2}^r Q_i\right| - \left|\bigcap_{i=1}^r Q_i\right| \\
&= \omega(G) + \left|\bigcup_{i=2}^r Q_i\right| + \left|\bigcap_{i=2}^r Q_i\right| - \left|\bigcup_{i=1}^r Q_i\right| - \left|\bigcap_{i=1}^r Q_i\right| \\
&\geq \omega(G) + 2\omega(G) - \left(\left|\bigcup_{i=1}^r Q_i\right| + \left|\bigcap_{i=1}^r Q_i\right|\right) \\
&> \omega(G).
\end{aligned}$$

□

Lemma 5.7 (Kostochka [33])**.** *If $\mathcal{Q}$ is a collection of maximum cliques in a graph G with $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ such that $X_{\mathcal{Q}}$ is connected, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose not and choose a counterexample $\mathcal{Q} := \{Q_1, \ldots, Q_r\}$ minimizing r. Plainly, $r \geq 3$. Let A be a noncutvertex in $X_{\mathcal{Q}}$ and B a neighbor of A. Put $\mathcal{Z} := Q - \{A\}$. Then $X_{\mathcal{Z}}$ is connected and hence by minimality of r, $\cap\mathcal{Z} \neq \emptyset$. In particular, $|\cup\mathcal{Z}| \leq \Delta(G) + 1$. Hence $|\cup\mathcal{Q}| \leq |\cup\mathcal{Z}| + |A - B| \leq 2(\Delta(G) + 1) - \omega(G) < 2\omega(G)$. This contradicts Hajnal's lemma. □

With a little more work we can prove the following generalization of Kostochka's lemma which has a Helly feel. We won't use this result here, but it has some independent interest.

Lemma 5.8. *Fix $k \geq 2$. Let G be a graph satisfying $\omega > \frac{k+1}{2k+1}(\Delta + 1)$. If $\mathcal{Q}$ is a collection of maximum cliques in G such that any k elements of $\mathcal{Q}$ have common intersection, then $\cap\mathcal{Q} \neq \emptyset$.*

Proof. Suppose not and choose a counterexample $\mathcal{Q} := \{Q_1, \ldots, Q_r\}$ minimizing r. Plainly, $r \geq k+1$. Put $\mathcal{Z}_i := \mathcal{Q} - \{Q_i\}$. Then any k elements of $\mathcal{Z}_i$ have common intersection and hence by minimality $\cap\mathcal{Z}_i \neq \emptyset$. In particular $\cup\mathcal{Z}_i$ contains a universal vertex and thus $|\cup\mathcal{Z}_i| \leq \Delta(G)+1$. Now, by Hajnal's Lemma, $|\cap\mathcal{Z}_i| \geq 2\omega(G) - (\Delta(G)+1) > 2\omega(G) - \frac{2k+1}{k+1}\omega(G) = \frac{1}{k+1}\omega(G)$.

Put $m := \min_i |Q_i - \cup\mathcal{Z}_i|$. Note that the $\cap Z_i$ are pairwise disjoint since $\cap\mathcal{Q} = \emptyset$. Thus $\cup\mathcal{Q}$ contains the disjoint union of the $\cap Z_i$ as well as at least m vertices in each clique outside the rest. In particular,

$$|\cup\mathcal{Q}| \geq \frac{1}{k+1}\omega(G)r + mr \geq \omega(G) + (k+1)m.$$

In addition,

$$|\cup\mathcal{Q}| \leq m + \Delta(G) + 1.$$

Hence,

$$m \leq \frac{\Delta(G) + 1 - \omega(G)}{k} < \frac{1}{k+1}\omega(G).$$

Finally,

$$|\cup\mathcal{Q}| \leq m + \Delta(G) + 1 < \frac{1}{k+1}\omega(G) + \frac{2k+1}{k+1}\omega(G) = 2\omega(G).$$

Applying Hajnal's Lemma gives a contradiction. □

5.2 The quintessential reduction example

Reed [54] has conjectured that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$

If we could always find an independent set whose removal decreased both ω and Δ, then the conjecture would follow by simple induction since we can give the independent set a single color and use at most $\left\lceil \frac{\omega+\Delta+1}{2} \right\rceil - 1$ colors on what remains. Expanding the independent set given by Lemma 5.5 to a maximal one shows that this sort of argument goes through when $\omega > \frac{2}{3}(\Delta+1)$. Thus a minimum counterexample to Reed's conjecture satisfies $\omega \leq \frac{2}{3}(\Delta+1)$.

5.3 Reducing for Brooks

We can use the facts on hitting maximum cliques to reduce Brooks' theorem down to the $\Delta = 3$ case as follows. Let G be a counterexample to Brooks' theorem minimizing $\Delta(G)$. Suppose $\Delta(G) \geq 4$. We may assume G is critical. If $\omega(G) < \Delta(G)$, then removing any maximal independent set from G decreases $\chi(G)$ and $\Delta(G)$ both by one giving a counterexample with smaller Δ. Hence

$\omega(G) \geq \Delta(G)$. But then $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ and Lemma 5.5 gives us an independent set I such that $\omega(G - I) < \omega(G)$. Let M be a maximal independent set containing I. Then $G - M$ is a counterexample with smaller Δ.

There is also a simple direct proof (not using the hitting maximum cliques lemmas) that Brooks' theorem can be reduced to the $\Delta = 3$ case.

Question. Is there a simple proof (not just specializing a general proof) of Brooks' theorem for $\Delta = 3$?

5.4 Reducing for Borodin-Kostochka

More generally, we can use the facts on hitting maximum cliques to prove the following reduction lemma.

Definition 5.1. For $k, j \in \mathbb{N}$, let $\mathcal{C}_{k,j}$ be the collection of all vertex critical graphs satisfying $\chi = \Delta = k$ and $\omega < k - j$. Put $\mathcal{C}_k := \mathcal{C}_{k,0}$. Note that $\mathcal{C}_{k,j} \subseteq \mathcal{C}_{k,i}$ for $j \geq i$.

Lemma 5.9. *Fix $k, j \in \mathbb{N}$ with $k \geq 3j + 6$. If $G \in \mathcal{C}_{k,j}$, then there exists $H \in \mathcal{C}_{k-1,j}$ such that $H \lhd G$.*

Proof. Let $G \in \mathcal{C}_{k,j}$. We first show that there exists a maximal independent set M such that $\omega(G - M) < k - (j + 1)$. If $\omega(G) < k - (j + 1)$, then any maximal independent set will do for M. Otherwise, $\omega(G) = k - (j+1)$. Since $k \geq 3j + 6$, we have $\omega(G) = k - (j + 1) > \frac{2}{3}(k + 1) = \frac{2}{3}(\Delta(G) + 1)$. Thus by Lemma 5.5, we have an independent set I such that $\omega(G - I) < \omega(G)$. Expand I to a maximal independent set to get M.

Now $\chi(G - M) = k - 1 = \Delta(G - M)$, where the last equality follows from Brooks' theorem and $\omega(G - M) < k - (j + 1) \leq k - 1$. Since $\omega(G - M) < k - (j + 1)$, for any $(k - 1)$-critical induced subgraph $H \unlhd G - M$ we have $H \in \mathcal{C}_{k-1,j}$. $\square$

As a consequence we get the result of Kostochka that the Borodin-Kostochka conjecture can be reduced to the $k = 9$ case.

Lemma 5.10. *Let $\mathcal{H}$ be a hereditary graph property. For $k \geq 5$, if $\mathcal{H} \cap \mathcal{C}_k = \emptyset$, then $\mathcal{H} \cap \mathcal{C}_{k+1} = \emptyset$. In particular, to prove the Borodin-Kostochka conjecture it is enough to show that $\mathcal{C}_9 = \emptyset$.*

6 List coloring with almost degree sized lists

Let G be a graph. A *list assignment* to the vertices of G is a function from $V(G)$ to the finite subsets of $\mathbb{N}$. A list assignment L to G is *good* if G has a coloring c where $c(v) \in L(v)$ for each $v \in V(G)$. It is *bad* otherwise. We call the collection of all colors that appear in L, the *pot* of L. That is $Pot(L) := \bigcup_{v \in V(G)} L(v)$. For a subgraph H of G we write $Pot_H(L) := \bigcup_{v \in V(H)} L(v)$.

For $S \subseteq Pot(L)$, let G_S be the graph $G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]$. We also write G_c for $G_{\{c\}}$. We let $\mathcal{B}(L)$ be the bipartite graph that has parts $V(G)$ and $Pot(L)$ and an edge from $v \in V(G)$ to $c \in Pot(L)$ iff $c \in L(v)$. For $f: V(G) \to \mathbb{N}$, an f-assignment on G is an assignment L of lists to the vertices of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is f-choosable if every f-assignment on G is good.

6.1 Shrinking the pot

In this section we prove a lemma about bad list assignments with minimum pot size. Some form of this lemma has appeared independetly in at least two places we know of—Kierstead [26] and Reed and Sudakov [56]. We will use this lemma repeatedly in the arguments that follow.

Given a graph G and $f: V(G) \to \mathbb{N}$, we have a partial order on the f-assignments to G given by $L < L'$ iff $|Pot(L)| < |Pot(L')|$. When we talk of *minimal* f-assignments, we mean minimal with respect to this partial order.

Lemma 6.1. *Let G be a graph and $f: V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq Pot(L)$, any coloring of G_S from L uses some color not in S.*

Proof. Suppose not and let $\emptyset \neq S \subseteq Pot(L)$ be such that G_S has a coloring ϕ from L using only colors in S. For $v \in V(G)$, let $h(v)$ be the smallest element of $Pot(L) - L(v)$ (this is well defined by assumption). Pick some $c \in S$ and construct a new list assignment L' as follows.

$$L'(v) = \begin{cases} L(v) & \text{if } v \in V(G) - V(G_S) \\ L(v) & \text{if } v \in V(G_S) \text{ and } c \notin L(v) \\ (L(v) - \{c\}) \cup \{h(v)\} & \text{if } v \in V(G_S) \text{ and } c \in L(v) \end{cases}$$

Note that L' is an f-assignment and $Pot(L') = Pot(L) - \{c\}$. Thus, by minimality of L, we can properly color G from L'. In particular, we have a coloring of $V(G) - V(G_S)$ from L using no color from S. We can complete this to a coloring of G from L using ϕ. This contradicts the fact that L is bad. □

Definition 6.1. A bipartite graph with parts A and B has *positive surplus* (with respect to A) if $|N(X)| > |X|$ for all $\emptyset \neq X \subseteq A$.

Lemma 6.2. *Let G be a graph and $f: V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then $\mathcal{B}(L)$ has positive surplus (with respect to $Pot(L)$).*

Proof. Suppose not and choose $\emptyset \neq X \subseteq Pot(L)$ such that $|N(X)| \leq |X|$ minimizing $|X|$. If $|X| = 1$, then G_X can be colored from X contradicting Lemma 6.1. Hence $|X| \geq 2$. By minimality of $|X|$, for any $Y \subset X$, $|N(Y)| \geq$

$|Y|+1$. Hence, for any $x \in X$, we have $|N(X)| \geq |N(X - \{x\})| \geq |X - \{x\}| + 1 = |X|$. Thus, by Hall's theorem, we have a matching of X into $N(X)$, but $|N(X)| \leq |X|$ so this gives a coloring of G_X from X contradicting Lemma 6.1. □

Small Pot Lemma. *Let G be a graph and $f : V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If G is not f-choosable, then G has a minimal bad f-assignment L such that $|Pot(L)| < |G|$.*

Proof. Suppose not and let L be a minimal bad f-assignment. For each $v \in V(G)$ we have $|L(v)| = f(v) < |G| \leq |Pot(L)|$ and hence $L(v) \neq Pot(L)$. Thus by Lemma 6.2 we have the contradiction $|G| \geq |N(Pot(L))| > |Pot(L)|$. □

6.2 Degree choosability

Definition 6.2. Let G be a graph and $r \in \mathbb{Z}$. Then G is *d_r-choosable* if G is f-choosable where $f(v) = d(v) - r$.

Note that a vertex critical with $\chi = \Delta + 1 - r$ contains no induced d_r-choosable graph. For $r = 0$, we have the following well known generalization of Brooks' Theorem (see [6], [17], [36] and [24]).

6.2.1 Degree-choosable graphs

Definition 6.3. A *Gallai tree* is a graph all of whose blocks are complete graphs or odd cycles.

Classification of d_0-choosable graphs. *For any connected graph G, the following are equivalent.*

- *G is d_0-choosable.*
- *G is not a Gallai tree.*
- *G contains an induced even cycle with at most one chord.*

We give some lemmas about d_0-assignments that will be useful in the later study of general d_k-assignments. Combined with the following structural result, these lemmas give a quick proof of the classification of d_0-choosable graphs. See [17] and [16] for alternate proofs of the classification.

Lemma 6.3. *Any 2-connected graph is complete, an odd cycle or contains an induced even cycle with at most one chord.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Since G is 2-connected and not complete, it contains an induced cycle C of length at least four. Then C is an induced odd cycle and thus $G - C$ is not empty. Since G is 2-connected, we may choose a shortest C-path in G with distinct ends in C—call it R. Since $G[V(C) \cup V(R)]$ is 2-connected, by minimality of $|G|$, $V(G) = V(C) \cup V(R)$.

First suppose R has length at least 3. Then since R is shortest, $G = C \cup R$ and thus one of the small cycles in $C \cup R$ is an even induced cycle or the large cycle is an even induced cycle with at most one chord, giving a contradiction.

Thus R has length 2. Let z be the vertex on R in $G - C$. If z has only two neighbors in C, then we get a contradiction as in the previous paragraph. Thus z has at least three neighbors $a, b, c \in V(C)$. Now $|C| \geq 4$ since G is not complete. Thus, without loss of generality, the vertices between a and b on C in cyclic order are $w_1, \ldots, w_k$ with $k \geq 1$. But $G - \{w_1, \ldots, w_k\}$ is 2-connected, not complete, and not an odd cycle. Hence, by minimality of $|G|$, $G - \{w_1, \ldots, w_k\}$ contains an induced even cycle with at most one chord. This final contradiction completes the proof. □

Lemma 6.4. *A connected graph is d_0-choosable iff it contains a d_0-choosable induced subgraph.*

Proof. The forward direction is plain. For the reverse, let $H \trianglelefteq G$ be d_0-choosable. Since G is connected, we can order $V(G)$ such that each vertex in $V(G - H)$ has a neighbor after it and $V(H)$ comes last. Coloring $V(G - H)$ greedily from the lists in this order leaves a d_0-assignment on H which we can complete by assumption. □

Lemma 6.5. *Let L be a bad d_0-assignment on a connected graph G and $x \in V(G)$ a noncutvertex. Then $L(x) \subseteq L(y)$ for each $y \in N(x)$.*

Proof. Suppose otherwise that we have $c \in L(x) - L(y)$ for some $y \in N(x)$. Coloring x with c leaves at worst a d_0-assignment L' on the connected $H := G - x$ where $|L'(y)| > d_H(y)$. But then we can complete the coloring, a contradiction. □

Lemma 6.6. *Any even subdivision of a bridgeless d_0-choosable graph is d_0-choosable.*

Proof. Since subdividing an edge cannot create a bridge, it suffices to show that subdividing an edge with two vertices preserves d_0-choosability. Let G be a bridgeless d_0-choosable graph. Suppose there exists $xy \in E(G)$ such that subdividing xy with vertices w and z creates a graph H which is not d_0-choosable. Let L be a bad d_0-assignment on H. Since G is bridgeless, w and z are not cutvertices of H. By Lemma 6.5, $L(w) = L(z)$. But L restricted to G is a d_0-assignment, so we have a coloring π of $H - \{w, z\}$ from L such that $\pi(x) \neq \pi(y)$. Now $L(w) - \{\pi(x)\} \neq L(z) - \{\pi(y)\}$ so we can complete the coloring to all of H, a contradiction. □

Using the Small Pot Lemma it is easy to prove that C_4 and K_4^- are d_0-choosable which combined with Lemma 6.6 shows that every even cycle with at most one chord is d_0-choosable. It turns out that the conclusion of the Small

Pot Lemma holds for general bad d_0-assignments, not just minimal ones. We will use the following lemma often in proofs when we end up with a bad d_0-assignment that may not be minimal.

Lemma 6.7. *If L a bad d_0-assignment on a connected graph G, $|Pot(L)| < |G|$.*

Proof. Suppose that the lemma is false and choose a connected graph G together with a bad d_0-assignment L where $|Pot(L)| \geq |G|$ minimizing $|G|$. Plainly, $|G| \geq 2$. Let $x \in G$ be a noncutvertex (any end block has at least one). By Lemma 6.5, $L(x) \subseteq L(y)$ for each $y \in N(x)$. Thus coloring x decreases the pot by at most one, giving a smaller counterexample. This contradiction completes the proof. □

Proof of the classification of d_0-choosable graphs. It is easy to construct a bad d_0-assignment on a Gallai tree—hence (1) implies (2). Now if a graph is not a Gallai tree, then some block is neither complete nor an odd cycle. But then, by Lemma 6.3, that block contains an induced even cycle with at most one chord. Hence (2) implies (3).

Now we prove that C_4 and K_4^- are d_0-choosable. If not, then we have a bad d_0-assignment L on C_4 or K_4^-. By Lemma 6.7, $|Pot(L)| \leq 3$. Hence some nonadjacent pair can be colored the same leaving a d_{-1}-assignment on the components which can be easily completed.

Thus, by Lemma 6.6, any even cycle with at most one chord is d_0-choosable. Combining this with Lemma 6.4 proves that (3) implies (1). □

6.2.2 What can we say for general r?

In this section we prove some lemmas about d_r-choosable graphs of the form $A * B$. We leave them out of this prospectus for brevity.

6.2.3 The case $r = 1$

In this section we will classify the d_1-choosable graphs of the form $A * B$ with $|A| \geq |B| \geq 1$. Currently the proof of this classification has around fifty lemmas and we leave out all of the details and even the statement for brevity's sake. To give the flavor, we give the classification of d_1-choosable graphs of the form $E_2 * B$ and those of the form $K_3 * B$.

Lemma 6.8. *$E_2 * B$ is not d_1-choosable iff B is the disjoint union of complete subgraphs and at most one P_3.*

Definition 6.4. A graph G is *almost complete* if $\omega(G) \geq |G| - 1$.

Lemma 6.9. *If $K_3 * B$ is not d_1-choosable, then B is $E_3 * K_{|B|-3}$, almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$ or $E_3 + K_{|B|-3}$.*

We note that the classification of the B's for which $K_1 * B$ is d_1-choosable severely restricts the possible neighborhoods in a vertex critical graph with $\chi = \Delta$.

7 The Borodin-Kostochka conjecture for claw-free graphs

In [15], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of *simple* graphs defined by excluding the claw, $K_5 - e$ and another graph D as induced subgraphs. Kierstead and Schmerl [28] improved this by removing the need to exclude D. The goal of this section is to remove the need to exclude $K_5 - e$.

We will apply a structure theorem for claw-free graphs of Chudnovsky and Seymour [14]. To do so, we first need to handle the two base classes of the structure theorem: line graphs and circular interval graphs. The first two sections prove Borodin-Kostochka for these classes. Then we use the structure theorem to prove the conjecture for quasi-line graphs. Finally, we prove that no neighborhood in a claw-free counterexample can contain a 5-cycle and the conjecture for claw-free graphs follows.

7.1 Line graphs of multigraphs

In this section we prove the Borodin-Kostochka Conjecture for line graphs of *multigraphs*. Moreoever, we prove a strengthening of Brooks' theorem for line graphs of multigraphs and conjecture the best possible such bound.

Lemma 7.1. *Fix $k \geq 0$. Let H be a multigraph and put $G = L(H)$. Suppose $\chi(G) = \Delta(G) + 1 - k$. If $xy \in E(H)$ is critical and $\mu(xy) \geq 2k + 2$, then xy is contained in a $\chi(G)$-clique in G.*

Proof. Let $xy \in E(H)$ be a critical edge with $\mu(xy) \geq 2k + 2$. Let A be the set of all edges incident with both x and y. Let B be the set of edges incident with either x or y but not both. Then, in G, A is a clique joined to B and B is the complement of a bipartite graph. Put $F = G[A \cup B]$. Since xy is critical, we have a $\chi(G) - 1$ coloring of $G - F$. Viewed as a partial $\chi(G) - 1$ coloring of G this leaves a list assignment L on F with $|L(v)| = \chi(G) - 1 - (d_G(v) - d_F(v)) = d_F(v) - k + \Delta(G) - d_G(v)$ for each $v \in V(F)$. Put $j = k + d_G(xy) - \Delta(G)$.

Let M be a maximum matching in the complement of B. First suppose $|M| \leq j$. Then, since B is perfect, $\omega(B) = \chi(B)$ and we have

$$\begin{aligned} \omega(F) &= \omega(A) + \omega(B) = |A| + \chi(B) \\ &\geq |A| + |B| - j = d_G(xy) + 1 - j \\ &= \Delta(G) + 1 - k = \chi(G). \end{aligned}$$

Thus xy is contained in a $\chi(G)$-clique in G.

Hence we may assume that $|M| \geq j + 1$. Let $\{\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}\}$ be a matching in the complement of B. Then, for each $1 \leq i \leq j + 1$ we have

$$
\begin{aligned}
|L(x_i)| + |L(y_i)| &\geq d_F(x_i) + d_F(y_i) - 2k \\
&\geq |B| - 2 + 2|A| - 2k \\
&= d_G(xy) + |A| - 2k - 1 \\
&\geq d_G(xy) + 1.
\end{aligned}
$$

Here the second inequality follows since $\alpha(B) \leq 2$ and the last since $|A| = \mu(xy) \geq 2k+2$. Since the lists together contain at most $\chi(G) - 1 = \Delta(G) - k$ colors we see that for each i,

$$
\begin{aligned}
|L(x_i) \cap L(y_i)| &\geq |L(x_i)| + |L(y_i)| - (\Delta(G) - k) \\
&\geq d_G(xy) + 1 - \Delta(G) + k \\
&= j + 1.
\end{aligned}
$$

Thus we may color the vertices in the pairs $\{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\}$ from L using one color for each pair. Since $|A| \geq k+1$ we can extend this to a coloring of B from L by coloring greedily. But each vertex in A has $j+1$ colors used twice on its neighborhood, thus each vertex in A is left with a list of size at least $d_A(v) - k + \Delta(G) - d_G(v) + j + 1 = d_A(v) + 1$. Hence we can complete the $(\chi(G)-1)$-coloring to all of F by coloring greedily. This contradiction completes the proof. □

Theorem 7.2. *If G is the line graph of a multigraph H and G is vertex critical, then*

$$
\chi(G) \leq \max\left\{\omega(G), \Delta(G) + 1 - \frac{\mu(H) - 1}{2}\right\}.
$$

Proof. Let G be the line graph of a multigraph H such that G is vertex critical. Say $\chi(G) = \Delta(G) + 1 - k$. Suppose $\chi(G) > \omega(G)$. Since G is vertex critical, every edge in H is critical. Hence, by Lemma 7.1, $\mu(H) \leq 2k+1$. That is, $\mu(H) \leq 2(\Delta(G) + 1 - \chi(G)) + 1$. The theorem follows. □

This upper bound is tight. To see this, let $H_t = t \cdot C_5$ (i.e. C_5 where each edge has multiplicity t) and put $G_t = L(H_t)$. As Catlin [12] showed, for odd t we have $\chi(G_t) = \frac{5t+1}{2}$, $\Delta(G_t) = 3t - 1$, and $\omega(G_t) = 2t$. Since $\mu(H_t) = t$, the upper bound is achieved.
We need the following lemma which is a consequence of the fan equation (see [4, 9, 18, 20]).

Lemma 7.3. *Let G be the line graph of a multigraph H. Suppose G is vertex critical with $\chi(G) > \Delta(H)$. Then, for any $x \in V(H)$ there exist $z_1, z_2 \in N_H(x)$ such that $z_1 \neq z_2$ and*

- $\chi(G) \leq d_H(z_1) + \mu(xz_1)$,

- $2\chi(G) \leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2)$.

Lemma 7.4. *Let G be the line graph of a multigraph H. If G is vertex critical with $\chi(G) > \Delta(H)$, then*

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Proof. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. By Lemma 7.3 we have $z \in N_H(x)$ such that $\chi(G) \leq d_H(z) + \mu(xz)$. Hence

$$\Delta(G) + 1 \geq d_H(x) + d_H(z) - \mu(xz) \geq d_H(x) + \chi(G) - 2\mu(xz).$$

Which gives

$$\chi(G) \leq \Delta(G) + 1 - \Delta(H) + 2\mu(H).$$

Adding Vizing's inequality $\chi(G) \leq \Delta(H) + \mu(H)$ gives the desired result. □

Combining this with Theorem 7.2 we get the following upper bound.

Theorem 7.5. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{7\Delta(G) + 10}{8}\right\}.$$

Proof. Suppose not and choose a counterexample G with the minimum number of vertices. Say $G = L(H)$. Plainly, G is vertex critical. Suppose $\chi(G) > \omega(G)$. By Theorem 7.2 we have

$$\chi(G) \leq \Delta(G) + 1 - \frac{\mu(H) - 1}{2}.$$

By Lemma 7.4 we have

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$

Adding three times the first inequality to the second gives

$$4\chi(G) \leq \frac{7}{2}(\Delta(G) + 1) + \frac{3}{2}.$$

The theorem follows. □

Corollary 7.6. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 11$, then G contains a $K_{\Delta(G)}$.*

With a little more care we can get the 11 down to 9. Using Lemma 5.10, we can inductively reduce to the $\Delta = 9$ case.

Theorem 7.7. *If G is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 9$, then G contains a $K_{\Delta(G)}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $\Delta(G)$. Then G is vertex critical. By Lemma 5.10, $\Delta(G) = 9$.

Let H be such that $G = L(H)$. Then by Lemma 7.1 and Lemma 7.4 we know that $\mu(H) = 3$. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. Then we have $z_1, z_2 \in N_H(x)$ as in Lemma 7.3. This gives

$$\begin{aligned} 9 &\leq d_H(z_1) + \mu(xz_1), & (1) \\ 18 &\leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2). & (2) \end{aligned}$$

In addition, we have for $i = 1, 2$,

$$9 \geq d_H(x) + d_H(z_i) - \mu(xz_i) - 1 = \Delta(H) + d_H(z_i) - \mu(xz_i) - 1.$$

Thus,

$$\begin{aligned} \Delta(H) &\leq 2\mu(xz_1) + 1 \leq 7, & (3) \\ \Delta(H) &\leq \mu(xz_1) + \mu(xz_2) + 1. & (4) \end{aligned}$$

Now, let $ab \in E(H)$ with $\mu(ab) = 3$. Then, since G is vertex critical, we have $8 = \Delta(G) - 1 \leq d_H(a) + d_H(b) - \mu(ab) - 1 \leq 2\Delta(H) - 4$. Thus $\Delta(H) \geq 6$. Hence we have $6 \leq \Delta(H) \leq 7$. Thus, by (3), we must have $\mu(xz_1) = 3$.

First, suppose $\Delta(H) = 7$. Then, by (4) we have $\mu(xz_2) = 3$. Let y be the other neighbor of x. Then $\mu(xy) = 1$ and thus $d_H(x) + d_H(y) - 2 \leq 9$. That gives $d_H(y) \leq 4$. Then we have vertices $w_1, w_2 \in N_H(y)$ guaranteed by Lemma 7.3. Note that $x \notin \{w_1, w_2\}$. Now $4 \geq d_H(y) \geq 1 + \mu(yw_1) + \mu(yw_2)$. Thus $\mu(yw_1) + \mu(yw_2) \leq 3$. This gives $d_H(w_1) + d_H(w_2) \geq 2\Delta(G) - 3 = 15$ contradicting $\Delta(H) \leq 7$.

Thus we must have $\Delta(H) = 6$. By (1) we have $d_H(z_1) = 6$. Then, applying (2) gives $\mu(xz_2) = 3$ and $d_H(z_2) = 6$. Since x was an arbitrary vertex of maximum degree and H is connected we conclude that $G = L(3 \cdot C_n)$ for some $n \geq 4$. But no such graph is 9-chromatic by Brooks' theorem. □

The graphs $G_t = L(t \cdot C_5)$ discussed above show that the following upper bound would be tight. Creating a counterexample would require some new construction technique that might lead to more counterexamples to Borodin-Kostochka for $\Delta = 8$.

Conjecture 7.8. *If G is the line graph of a multigraph, then*

$$\chi(G) \leq \max\left\{\omega(G), \frac{5\Delta(G) + 8}{6}\right\}.$$

7.2 Circular interval graphs

A *path* in a topological space T is a continuous map $p\colon [0,1] \to T$. A *representation* of a graph G in a topological space T is an injection $f\colon V(G) \hookrightarrow T$ together with a set of paths $\{p_{xy}\}_{xy \in E(G)}$ in T such that $p_{xy}(0) = f(x)$, $p_{xy}(1) = f(y)$ and $f^{-1}(im(p_{xy}))$ is a clique in G. A graph is a *circular interval graph* if it is representable in the unit circle. We note that this class coincides with the class of proper circular arc graphs. A graph is a *linear interval graph* if it is representable on the unit interval.

A *b-fold* coloring of a graph is an assignment of sets of size b to the vertices of the graph such that adjacent vertices receive disjoint sets. An $(a\!:\!b)$*-coloring* is a b-fold coloring out of a set of a available colors. The *b-fold chromatic number* $\chi_b(G)$ of a graph G is the least a such that G has an $(a\!:\!b)$-coloring. Then we define the *fractional chromatic number* of G as $\chi_f(G) := \lim_{b\to\infty} \frac{\chi_b(G)}{b}$.

Proving Borodin-Kostochka for circular interval graphs will be easier if we have a large clique to work with. To get this we will use the fact that Reed's conjecture holds for circular interval graphs. This fact is immediate once we have the following two lemmas.

Lemma 7.9 (Molloy and Reed [40])**.** *Every graph satisfies* $\chi_f \le \frac{\omega+\Delta+1}{2}$.

Lemma 7.10 (Niessen and Kind [42])**.** *Every circular interval graph satisfies* $\chi = \lceil \chi_f \rceil$.

Now we take a maximum clique in our circular interval graph and use our list coloring lemmas from section 6 to show that no vertex close to the middle (in cyclic order) of our clique can have more than one neighbor on either side of the clique. But then all the vertices close to the middle are low or we have a K_Δ. In the former case, we can use list coloring lemmas to get a contradiction. The details are left out of this prospectus.

7.3 Quasi-line graphs

A graph is *quasi-line* if every vertex is bisimplicial (its neighborhood can be covered by two cliques). We apply a version of Chudnovsky and Seymour's structure theorem for quasi-line graphs from King and Reed [31]. The undefined terms will be defined after the statement.

Lemma 7.11. *For any quasi-line graph G, at least one of the following is true:*

- *G contains a nonlinear homogeneous pair of cliques,*
- *G is a circular interval graph,*
- *G is a line graph,*
- *G admits a canonical interval 2-join.*

A *homogeneous pair of cliques* (A_1, A_2) in a graph G is a pair of cliques such that for each $i \in [2]$, every vertex in $G-(A_1 \cup A_2)$ is either joined to A_i or misses all of A_i, some A_i contains at least two vertices and $|G-(A_1 \cup A_2)| \geq 2$. A homogeneous pair of cliques is *linear* if $G[A_1 \cup A_2]$ is a linear interval graph.

A graph G admits a *canonical interval* 2*-join* if G has an induced subgraph H such that:

1. H is a linear interval graph,
2. The ends of H are disjoint nonempty cliques A_1, A_2,
3. $G-H$ contains cliques B_1, B_2 (not necessarily disjoint) such that A_1 is joined to B_1 and A_2 is joined to B_2,
4. there are no other edges between H and $G-H$.

Chudnovsky and Fradkin [13] proved a lemma allowing us to handle non-linear homogeneous pairs of cliques.

Lemma 7.12 (Chudnovsky and Fradkin [13])**.** *If G is a critical quasi-line graph, then G contains no nonlinear homogeneous pair of cliques.*

Thus, if we have a minimum counterexample to Borodin-Kostochka that is quasi-line, the only possibility is that it admits a canonical interval 2-join. By applying our list coloring lemmas similarly to how it is done for circular interval graphs, we can show this is impossible.

7.4 Handling five-wheels

It will follow from some of our list coloring lemmas that if a neighborhood of some vertex contains an induced C_4 then that vertex is bisimplicial. Since the neighborhoods are E_3-free, as long as there is no C_5 the neighborhood will be chordal and again we can conclude that the vertex is bisimplicial. So if we can exclude C_5 from all neighborhoods, our graph will be quasi-line and the conjecture will follow from the previous section.

8 Conjectures equivalent to the Borodin-Kostochka conjecture that are a priori weaker

In this section we exclude more induced subgraphs in a minimal counterexample than we can exclude using the list coloring lemmas in section 6 alone. In fact, we lift these results out of the context of a minimal counterexample to graphs satisfying a certain criticality condition defined in terms of the following ordering.

Definition 8.1. If G and H are graphs, an *epimorphism* is a graph homomorphism $f\colon G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 8.2. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f\colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of finite simple graphs. We call this the *child order* and denote it by '$\prec$'. By definition, if $H \triangleleft G$ then $H \prec G$.

Lemma 8.1. *The ordering $\prec$ is well-founded; that is, every nonempty set of finite simple graphs has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty set of finite simple graphs. Pick $G \in \mathcal{T}$ minimizing $|G|$ and then maximizing $\|G\|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. □

Definition 8.3. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a *$\mathcal{T}$-mule*.

With the definition of mule we have captured the important properties (for coloring) of a counterexample first minimizing the number of vertices and then maximizing the number of edges. Viewing $\mathcal{T}$ as a set of counterexamples, we can add edges to or contract independent sets in induced subgraphs of a $\mathcal{T}$-mule and get a non-counterexample. We could do the same with a minimal counterexample, but with mules we have more minimal objects to work with. One striking consequence of this is that many of our proofs naturally construct multiple counterexamples to Borodin-Kostochka for small Δ.

For $k \in \mathbb{N}$, by a *k-mule* we mean a $\mathcal{C}_k$-mule. We will give the main results in simplified form.

Lemma 8.2. *Fix $k \geq 7$ and let G be a k-mule (excepting two specimens). If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most one neighbor in H.*

Lemma 8.3. *Fix $k \geq 8$ and let G be a k-mule (excepting one specimen). Let A and B be graphs with $3 \leq |A| \leq k-3$ and $|B| = k - |A|$ such that $A * B \trianglelefteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

This shows that the following weaker-looking conjecture is equivalent to Borodin-Kostochka.

Conjecture 8.4. *Any graph with $\chi \geq \Delta \geq 9$ contains $K_3 * E_{\Delta-3}$ as a subgraph.*

It also suggests that it might be worthwhile to look at the following weaker conjecture.

Conjecture 8.5. *Let G be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $3 \leq t \leq k-3$, then G can be $(k-1)$-colored.*

Notation

Symbology	Meaning
$\lvert G\rvert$	the number of vertices G has
$\lVert G\rVert$	the number of edges G has
$G[S]$	the subgraph of G induced on S
$E_G(X,Y)$	the edges in G with one end in X and the other in Y
$E_G(X)$	$E_G(X, V(G)-X)$
$\chi(G)$	the chromatic number of G
$\omega(G)$	the clique number of G
$\alpha(G)$	the independence number of G
$\Delta(G)$	the maximum degree of G
$\delta(G)$	the minimum degree of G
$\kappa(G)$	the vertex connectivity of G
$\overline{G}$	the complement of G
$A+B$	the disjoint union of graphs A and B
$A*B$	the join of graphs A and B (that is, $\overline{\overline{A}+\overline{B}}$)
kG	$\underbrace{G+G+\cdots+G}_{k \text{ times}}$
G^k	$\underbrace{G*G*\cdots*G}_{k \text{ times}}$
$H \subseteq G$	H is a subgraph of G
$H \subset G$	H is a proper subgraph of G
$H \trianglelefteq G$	H is an induced subgraph of G
$H \triangleleft G$	H is a proper induced subgraph of G
$H \prec G$	H is a child of G
$f\colon S \hookrightarrow T$	an injective function from S to T
$f\colon S \twoheadrightarrow T$	a surjective function from S to T
$X := Y$	X is defined as Y
K_k	the complete graph on k vertices
E_k	the edgeless graph on k vertices (that is, $\overline{K_k}$)
P_k	the path on k vertices
C_k	the cycle on k vertices
$K_{a,b}$	the complete bipartite graph with parts of size a and b (that is, $E_a * E_b$)
$[n]$	$\{1,2,\ldots,n\}$
$\mathbb{N}$	the natural numbers $(0,1,2,\ldots)$
$\mathbb{R}$	the real numbers

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