PLANAR GRAPHS HAVE INDEPENDENCE RATIO AT LEAST 3/13

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Abstract. The 4 Color Theorem (4CT) implies that every n-vertex planar graph has an independent set of size at least $\frac{n}{4}$; this is best possible, as shown by the disjoint union of many copies of K_4. In 1968, Erdős asked whether this bound on independence number could be proved more easily than the full 4CT. In 1976 Albertson showed (independently of the 4CT) that every n-vertex planar graph has an independent set of size at least $\frac{2n}{9}$. Until now, this remained the best bound independent of the 4CT. Our main result improves this bound to $\frac{3n}{13}$.

1. Introduction

An *independent set* is a subset of vertices that induce no edges. The independence number $\alpha(G)$ of a graph G is the size of a largest independent set in G. Determining the independence nubmer of an arbitrary graph G is widely-studied and well-known to be NP-complete. Thus, much work in this area focuses on proving lower bounds for the indepence number of some special class of graphs, often in terms of $|V(G)|$. The *independence ratio* of a graph G is the quantity $\frac{\alpha(G)}{|V(G)|}$.

An immediate consequence of the 4 Color Theorem is that every planar graph has independence ratio at least $\frac{1}{4}$; simply take the largest color class. In fact, this bound is best possible, as shown by the disjoint union of many copies of K_4. In 1968, Erdős [2] suggested that perhaps this corollary could be proved more easily than the full 4 Color Theorem. And in 1976, Albertson [1] showed (independently of the 4 Color Theorem) that every planar graph has independence ratio at least $\frac{2}{9}$. Our main theorem improves this bound to $\frac{3}{13}$.

Theorem 1. *Every planar graph has independent ratio at least* $\frac{3}{13}$.

The proof of Theorem 1 is heavily influenced by Albertson's proof. One apparent difference is that our proof uses the discharging method, while his does not. However, this distinction is largely cosmetic. To demonstrate this point, we include an appendix with a short discharging version of the final step in Albertson's proof, which he verified using edge-counting (the reader unfamiliar with discharging arguments may prefer to start with this appendix). Although the arguments are essentially equivalent, the discharging method is somewhat more flexible. In part it was this added flexibility that allowed us to push his ideas further.

Our proof has the following outline. The bulk of the work consists in showing that certain configurations are *reducible*, i.e., they cannot appear in a minimal counterexample to the theorem. The remainder of the proof is a counting argument (called *discharging*), where we

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show that every planar graph contains one of the forbidden configurations; hence, it is not a minimal counterexample.

In the discharging section, we give each vertex v initial charge $d(v)-6$; by Euler's formula the sum of the initial charges is -12. Our goal is to redistribute charge, without changing the sum, (assuming that G contains no reducible configuration) so that every vertex finishes with nonnegative charge. This contradiction proves that, in fact, G must contain a reducible configuration. To this end, we want to show that G contains a reducible configuration whenever it has many 6^--vertices near each other, since 5-vertices will need to receive charge and 6-vertices will have no spare charge to give away. (We will see in Lemma 6 that any 4^--vertex is reducible.) Most of the work in the reducibility section goes into proving various formalizations of this intuition.

Typically, proofs like ours present the reducibility portion before the discharging portion. However, because many of our reducibility arguments are quite technical, we make the unusual choice to give the discharging first, with the goal of providing context for the reducible configurations. (Usually the process of finding a proof switches back and forth between discharging and reducibility. By necessity, though, the proof must present one of these first.)

Before the proof, we need a few definitions. A *k-vertex* is a vertex of degree k; similarly, a k^--vertex (resp. k^+-vertex) has degree at most (resp. at least) k. A k-neighbor of a vertex v is a k-vertex that is a neighbor of v; and k^--neighbors and k^+-neigbors are defined analogously. A k-cycle is a cycle of length k. A vertex set V_1 in a connected graph G is *separating* if $G \setminus V_1$ has at least two components. A cycle C is separating if $V(C)$ is separating. Finally, an *independent k-set* is an independent set of size k.

2. Discharging

Theorem 1. *Every planar graph G has independence ratio at least $\frac{3}{13}$.*

Proof. We will use discharging with initial charge $ch(v) = d(v) - 6$. We use the following discharging rules to guarantee that each vertex finishes with nonnegative charge.

- (R1) Each 6-vertex gives $\frac{1}{2}$ to each 5-neighbor unless either they share a common 6-neighbor and no common 5-neighbor or else the 5-neighbor receives charge from at least four vertices; in either of these cases, the 6-vertex gives the 5-neighbor $\frac{1}{4}$.
- (R2) Each 8^+-vertex v gives gives $\frac{1}{4} + \frac{h_w}{8}$ to each 6^--neighbor w where h_w is the number of 7^+-vertices in $N(v) \cap N(w)$.
- (R3) Each 7-vertex gives $\frac{1}{2}$ to each isolated 5-neighbor; gives 0 to each crowded 5-neighbor; gives $\frac{1}{4}$ to each other 5-neighbor; and gives $\frac{1}{4}$ to each 6-neighbor unless neither the 7-vertex nor the 6-vertex has a 5-neighbor.
- (R4) After applying (R1)–(R3), each 5-vertex with positive charge splits it equally among its 6-neighbors that gave it $\frac{1}{2}$.
- (R5) After applying (R1)–(R4), each 6-vertex with positive charge splits it equally among its 6-neighbors with negative charge.

$\mathbf{d(v) \geq 8}$: We will show that v gives away charge at most $\frac{d(v)}{4}$. To see that it does, let v first give charge $\frac{1}{4}$ to each neighbor. Now let each 6^--neighbor w take $\frac{1}{8}$ from each 7^+-vertex in $N(v) \cap N(w)$. Since $G[N(v)]$ is a cycle, each 7^+-neighbor gives away at most the $\frac{1}{4}$ it got from v. Each neighbor of v has received at least as much charge as by rule (R2) and v has given away charge $\frac{d(v)}{4}$. Now $d(v) \geq 8$ implies $\frac{d(v)}{4} \leq d(v) - 6$, so $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 7}$: Let $u_1, \ldots, u_7$ denote the neighbors of v in clockwise order. First suppose that v has an isolated 5-neighbor. Now the subgraph induced by the remaining 6^--neighbors must have independence number at most 1, by Lemma 20. Hence v gives away charge at most either $\frac{1}{2} + \frac{1}{2}$ or $\frac{1}{2} + 2(\frac{1}{4})$; in either case, $\text{ch}^*(v) \geq 0$. Assume instead that v has no isolated 5-neighbor. Suppose first that v has a (non-isolated) 5-neighbor. Now v has at most five total 6^--neighbors, again by Lemma 20. If v has at most four 6^- neighbors, then, since each 6^--neighbor receives charge at most $\frac{1}{4}$, v gives away at most $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$. By Lemma 20, if v has exactly five 6^--neighbors, then one is a crowded 5-neighbor, which receives no charge from v. So, again, v gives away charge at most $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$.

Finally, suppose that v has only 6^+-neighbors. By Lemma 23, v gives charge to at most four 6-neighbors, so $\text{ch}^*(v) \geq 0$.

$\mathbf{d(v) = 5}$: We must show that v receives total charge at least 1. Let $u_1, \ldots, u_5$ be the neighbors of v. First suppose that v has five 6^+-neighbors. It will receive charge at least $4(\frac{1}{4})$ unless exactly two of these are 7-vertices for which v is a crowded 5-neighbor. However, in this case the other three neighbors are all 6-neighbors, so v receives $2(\frac{1}{4}) + (\frac{1}{2})$. Now suppose that v has exactly four 6^+-neighbors, say $u_1, \ldots, u_4$. If v receives charge from each, then v receives at least $4(\frac{1}{4})$; so suppose that v receives charge from at most three neighbors. In total, v receives charge at least $\frac{1}{2}$ from u_1 and u_2: at least $2(\frac{1}{4})$ if u_1 is not a 6-vertex and at least $\frac{1}{2} + 0$ if u_1 is a 6-vertex. Similarly, v receives at least $\frac{1}{2}$ in total from u_3 and u_4; so, v receives total charge at least $2(\frac{1}{2})$. Now suppose that v has exactly three 6^+-neighbors, say u_1, u_2, u_3. If u_1 and u_3 are both 6-vertices, then v receives charge $\frac{1}{2}$ from each. If both are 7^+-vertices, then v receives charge $\frac{1}{4}$ from each and charge $\frac{1}{2}$ from u_2. So assume that exactly one of u_1 and u_3 is a 6-vertex, say u_1. Now v receives charge $\frac{1}{2}$ from u_1 and charge $\frac{1}{4}$ from each of u_2 and u_3, for a total of $\frac{1}{2} + 2(\frac{1}{4})$.

$\mathbf{d(v) = 6}$: Note that (R5) will never cause a 6-vertex to have negative charge. Thus, in showing that a 6-vertex has nonnegative charge, we need not consider it.

Clearly, a 6-vertex with no 5-neighbor finishes (R1)–(R3) with nonnegative charge. Suppose that v is a 6-vertex with exactly one 5-neighbor. We will show that v finishes (R1)–(R3) with charge at least $\frac{1}{4}$. Let $u_1, \ldots, u_6$ denote the neighbors of v and assume that u_1 is the only 5-vertex. By Lemma 17, at least one of u_1, u_3, u_5 is a 7^+-vertex, so it gives v charge $\frac{1}{4}$. If one of u_6 and u_2 is a 6-vertex, then v gives charge only $\frac{1}{4}$ to u_1, finishing with charge at least $2(\frac{1}{4}) - \frac{1}{4}$. Otherwise, v receives charge at least $\frac{1}{4}$ from each of u_6 and u_2, so finishes with charge at least $3(\frac{1}{4}) - \frac{1}{2}$. Similarly, if v has no 5-neighbor and at least one 8^+-neighbor, then v finishes (R1)–(R3) with charge at least $\frac{1}{4}$.

Now suppose that v has at least two 5-neighbors. By Lemma 9, At most one of u_1, u_3, u_5 can be a 5-vertex. Similarly, for u_2, u_4, u_6; hence, assume that v has exactly two 5-neighbors. These 5-neighbors can either be "across", say u_1 and u_4, or "adjacent", say u_1 and u_2.

Suppose that v has 5-neighbors u_1 and u_4. Note that all of its remaining neighbors must be 6^+-vertices. At least one of u_1, u_3, u_5 must be a 7^+-vertex; similarly for u_2, u_4, u_6. Now we show that the total net charge that v gives to u_3, u_4, u_5 is 0. Similarly, the total net charge that v gives to u_6, u_1, u_2 is 0. If both u_3 and u_5 are 7^+-vertices, then v gets $\frac{1}{4}$ from each and gives $\frac{1}{2}$ to u_4. Otherwise, one of u_3 and u_5 is a 6-vertex and the other is a 7^+-vertex; now v gets $\frac{1}{4}$ from the 7^+-vertex and gives only $\frac{1}{4}$ to u_4. The same is true for u_6, u_1, u_2. Thus, v finishes with charge 0.

Suppose instead that v has 5-neighbors u_1 and u_2. By Lemmas 17 and 18 either both of u_3 and u_5 are 7^+-vertices or one is a 6-vertex and the other an 8^+-vertex. The same holds for u_4 and u_6. Let $w_1, \ldots, w_5$ be the common neighbors of successive pairs of vertices in the list $u_6, u_1, u_2, u_3, u_4, u_5$. Consider the possible degrees for u_3, u_4, u_5, u_6. Up to symmetry, they are (i) $7^+, 7^+, 7^+, 7^+$, (ii) $7^+, 8^+, 7^+, 6$, (iii) $7^+, 6, 7^+, 8^+$, (iv) $8^+, 6, 6, 8^+$, (v) $8^+, 8^+, 6, 6$, and (vi) $6, 8^+, 8^+, 6$.

In **Case (i)**, v receives charge at least $4(\frac{1}{4})$, so $\text{ch}^*(v) \geq 0$. In **Case (ii)**, v receives charge at least $\frac{1}{4} + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8}) + \frac{1}{4}$, so $\text{ch}^*(v) \geq 0$. In **Case (iii)**, v receives charge at least $(\frac{1}{4} + \frac{1}{8}) + \frac{1}{4} + \frac{1}{4} = \frac{7}{8}$. If w_2 is a 6^+-vertex, then v gives only $\frac{1}{4}$ to u_2, so $\text{ch}^*(v) \geq 0$. So suppose that w_2 is a 5-vertex. Recall that w_3 is a 6^+-vertex by Lemma 9. Now in each case v get charge at least $\frac{1}{8}$ back from u_2. If w_3 is a 6-vertex, then u_3 receives charge $2(\frac{1}{2}) + \frac{1}{4}$ and sends back $\frac{1}{8}$ to each of v and w_3. Otherwise, w_3 is a 7^+-vertex, so u_3 sends v charge at least $\frac{3}{8}$, and v gets back at least $\frac{1}{8}$. Thus, in each instance of Case (iii), we have $\text{ch}^*(v) \geq 0$. So we are in Cases (iv), (v), or (vi).

Case (iv): $8^+, 6, 6, 8^+$. If w_2 is a 6^+-vertex, then both u_1 and u_2 are sent charge by four vertices and hence v gives away at most $\frac{1}{2}$. Since v gets at least $\frac{1}{2}$ from u_3 and u_6, we have $\text{ch}^*(v) \geq 0$. Hence, we assume that w_2 is a 5-vertex.

Now if w_1 is a 6-vertex, then u_1 receives charge $\frac{5}{4}$, so gives back $\frac{1}{8}$ to v. If instead w_1 is a 7^+-vertex, then u_1 receives charge at least $\frac{3}{4}$ from v and w_1 together and then charge at least $\frac{1}{4} + \frac{1}{8}$ from u_6 for a total of $\frac{9}{8}$. Since u_1 has only one 6-neighbor, it gives the extra $\frac{1}{8}$ back to v by (R4). The same holds for u_2, so v gets $\frac{1}{8}$ back from each of u_1 and u_2; so v gets charge at least $\frac{3}{4}$.

Suppose that u_4 has at least two 5-neighbors. Now one of them, call it x, is a common neighbor with either u_3 or u_5, so we can apply Lemma 19 to $\{v, w_2, x\}$ (again $x \not\leftrightarrow w_2$, since w_2 has two other 5-neighbors; x cannot be identified with one of these other 5-neighbors, since G has no separating 3-cycle). Similarly, u_5 has at most one 5-neighbor. Hence, by our argument above, both u_4 and u_5 finish (R1)–(R3) with charge at least $\frac{1}{4}$. Now we show that u_4 has at most three 6-neighbors; similarly for u_5.

Suppose w_4 is a 6-vertex. We can apply Lemma 19 to v, w_2, w_4 unless $w_2 \leftrightarrow w_4$. In that case, we apply Lemma 21 to u_1, u_4, w_4. Thus, w_4 is a 7^+-vertex. If w_5 is a 6-vertex, then we can apply Lemma 19 to v, w_2, w_5 unless $w_2 \leftrightarrow w_5$, so assume this. Now if the final neighbor of u_4, call it y, is a 6-vertex, then we apply Lemma 19 to u_5, y, u_1; we must have $y \not\leftrightarrow u_1$, since $w_5 \leftrightarrow w_2$. Thus, we conclude that u_4 has at most two 6-neighbors other than u_5, so at most two 6-neighbors that finish (R1)–(R3) with negative charge. An analogous argument holds for u_5. Hence v gets at least $\frac{1}{8}$ from each of u_4 and u_5 from (R5), for a total of $\frac{3}{4} + \frac{1}{4}$ as needed.

Case (v): $8^+, 8^+, 6, 6$. Note that v receives charge at least $2(\frac{3}{8}) = \frac{3}{4}$ from u_5 and u_6. If w_2 is a 6^+-vertex, then u_1 receives charge from four neighbors, so v gives away charge at most $\frac{1}{4} + \frac{1}{2}$. Thus $\text{ch}^*(v) \geq 0$. So assume w_2 is a 5-vertex. First, we show that v gets back at least $\frac{1}{8}$ from u_1. If $d(w_1) = 6$, then u_1 gets charge $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$, so returns charge $\frac{1}{8}$ to each of v and w_1. Otherwise $d(w_1) \geq 7$, so u_6 sends charge $\frac{3}{8}$ to u_1, and u_1 returns at least $\frac{1}{8}$ to v. Thus, v gets charge at least $\frac{7}{8}$.

If w_3 is a 6-vertex, then v gets back charge $\frac{1}{8}$ from u_2, so $\mathrm{ch}^*(v) \geq 0$. Instead, assume w_3 is a 7^+-vertex. Now we show that v gets charge at least $\frac{1}{8}$ from u_3 by (R5). Let y be the neighbor of u_3 other than v, u_2, w_3, w_4, u_4. By Lemma 18, y is an 8^+-vertex. If w_4 is a 5-vertex, then we apply Lemma 19 to v, w_2, w_4 to get a contradiction (w_4 cannot be adjacent to w_2, since w_2 already has two other 5-neighbors, and w_4 cannot be identified with u_1 or w_2, since G has no separating 3-cycles). Hence w_4 is a 6^+-vertex. So u_3 receives charge at least $\frac{1}{4}$ from w_3 and at least $\frac{1}{4} + \frac{1}{8}$ from y. After u_3 gives charge $\frac{1}{4}$ to u_2, it has charge at least $\frac{3}{8}$. So, by (R5), it gives each of its at most three 6-neighbors charge at least $\frac{1}{3}(\frac{3}{8}) = \frac{1}{8}$. Thus, $\mathrm{ch}^*(v) \geq 0$.

Case (vi): $6, 8^+, 8^+, 6$. First suppose that w_2 is a 6^+-vertex. Note that v gets charge at least $2(\frac{3}{8})$ from u_4 and u_5, so it suffices to show that v gives net charge at most $\frac{3}{8}$ to each of u_1 and u_2. We consider u_1; the case for u_2 is symmetric. If w_1 gives charge to u_1, then u_1 receives charge from four neighbors, so it gets charge only $\frac{1}{4}$ from v. Suppose instead that w_1 is a 7-vertex and w_2 is a 6-vertex. Now u_1 gets charge $\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$, so returns charge $\frac{1}{8}$ to each of v and w_2. Thus $\mathrm{ch}^*(v) \geq 0$. So instead, assume that w_2 is a 5-vertex.

If w_1 and w_3 are 6-vertices, then v gets back $\frac{1}{8}$ from each of u_1 and u_2, since each receives $\frac{1}{2} + \frac{1}{4} + \frac{1}{2}$ and returns $\frac{1}{8}$ to each vertex that gave it $\frac{1}{2}$. Since v gets $2(\frac{3}{8})$ from u_4 and u_5, we have $\mathrm{ch}^*(v) \geq 0$. So assume, by symmetry, that w_3 is a 7^+-vertex. If $w_4 \leftrightarrow w_2$, then we apply Lemma 8 to w_2 and u_2; so $w_4 \not\leftrightarrow w_2$. If w_4 is a 6^--vertex, then we apply Lemma 19 to v, w_2, w_4 to get a contradiction (as above, w_4 cannot be identified with u_1 or w_2, since G has no separating 3-cycle). Thus, w_4 is a 7^+-vertex. So u_3 has at least three 7^+-neighbors and at most two 6-neighbors. Thus, after u_3 gives charge $\frac{1}{4}$ to u_2, by (R5) it gives charge $\frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ to v. Thus, $\mathrm{ch}^*(v) \geq 0$. □

3. Reducibility

It is quite useful to know that a minimal counterexample has no separating 3-cycle; we prove this in Lemma 2. When proving coloring results, such a lemma is nearly trivial. However, for independece results, it requires much more work. Albertson proved an analogous lemma when showing that planar graphs have independence ratio at least $\frac{2}{9}$. Our proof generalizes his to the broader context of showing that all minor-closed graphs have independence ratio at least c for some rational c. We will apply this lemma to planar graphs and will let $c = \frac{3}{13}$.

Lemma 2. *Let $c > 0$ be rational. Let $\mathcal{G}$ be a minor-closed family of graphs. If G is a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn, then G has no separating 3-cycle.*

Proof. Suppose to the contrary that G has a separating 3-cycle X. Let A_1 and A_2 be induced subgraphs of G with $V(A_1) \cap V(A_2) = X$ and $A_1 \cup A_2 = G$.

Our plan is to find big independent sets in two smaller graphs in $\mathcal{G}$ (by minimality) and piece those independent sets together to get an independent set in G of size at least $c|G|$.

More precisely, we consider independent sets in each A_i, either with X deleted, or with some pair of vertices in X contracted. In Claims 1–3, we prove lower bounds on $\alpha(G)$ in terms of $|A_1|$ and $|A_2|$. In Claim 4, we examine $|A_1|$ and $|A_2|$ modulo b, where $c = \frac{a}{b}$ in lowest terms. In each case, we show that one of the independent sets constructed in Claims 1–3 has size at least $c|G|$. Our proof relies heavily on the fact that $\alpha(H)$ is an integer (for every graph H), which often allows us to gain slightly over $c|H|$.

Claim 1. $\alpha(G) \geq \lceil c(|A_1| - 3)\rceil + \lceil c(|A_2| - 3)\rceil$.

The union of the independent sets obtained by applying minimality of G to $A_1 \setminus X$ and $A_2 \setminus X$ is independent in G.

Claim 2. $\alpha(G) \geq \lceil c(|A_i| - 2)\rceil + \lceil c|A_j|\rceil - 1$ *whenever* $\{i, j\} = \{1, 2\}$.

For concreteness, let $i = 1$ and $j = 2$; the other case is analogous. Apply minimality to A_2 to get an independent set I_2 in A_2 with $|I_2| \geq \lceil c|A_2|\rceil$. Form A_1' from A_1 by contracting X to a single vertex u. Apply minimality to A_1' to get an independent set I_1 in A_1' with $|I_1| \geq \lceil c(|A_1| - 2)\rceil$. If $u \in I_1$, then $I_1 \cup I_2 \setminus \{u\}$ is independent in G and has the desired size. Otherwise, $I_1 \cup I_2 \setminus X$ is an independent set of the desired size in G.

Claim 3. $\alpha(G) \geq \lceil c(|A_1| - 1)\rceil + \lceil c(|A_2| - 1)\rceil - 1$.

Let $X = \{x_1, x_2, x_3\}$. For each $k \in \{1, 2\}$ and $t \in \{2, 3\}$, form $A_{k,t}$ from A_k by contracting x_1x_t to a vertex $x_{k,t}$. Applying minimality to $A_{k,t}$ gives an independent set $I_{k,t}$ in $A_{k,t}$ with $|I_{k,t}| \geq \lceil c(|A_k| - 1)\rceil$.

If at most one of $I_{1,t}$ and $I_{2,t}$ contains a vertex of X (or a contraction of two vertices in X), then to get a big independent set, we take their union, discarding this at most one vertex. Formally, if $\{x_{k,t}, x_{5-t}\} \cap I_{k,t} = \emptyset$, then $I_{k,t} \cup I_{3-k,t} \setminus X$ is an independent set in G of the desired size. So assume that each of $I_{1,t}$ and $I_{2,t}$ contains a vertex (or a contraction of an edge) of X.

Now we look for a vertex x_ℓ of X such that each of $I_{1,t}$ and $I_{2,t}$ contains x_ℓ or a contraction of x_ℓ. Formally, if $x_{5-t} \in I_{k,t} \cap I_{3-k,t}$, then $I_{k,t} \cup I_{3-k,t} \setminus X$ is an independent set in G of the desired size. Similarly, if $x_{k,t} \in I_{k,t}$ and $x_{3-k,t} \in I_{3-k,t}$, then $I_{k,t} \cup I_{3-k,t} \cup \{x_1\} \setminus \{x_{k,t}, x_{3-k,t}\}$ is an independent set in G of the desired size.

So, by symmetry, we may assume that $x_{1,2} \in I_{1,2}$ and $x_3 \in I_{2,2}$. Also, either $x_{1,3} \in I_{1,3}$ or $x_{2,3} \in I_{2,3}$. If $x_{1,3} \in I_{1,3}$, then $I_{2,2} \cup I_{1,3} \setminus \{x_{1,3}\}$ is an independent set in G of the desired size. Otherwise, $x_{2,3} \in I_{2,3}$ and $I_{1,2} \cup I_{2,3} \cup \{x_1\} \setminus \{x_{1,2}, x_{2,3}\}$ is an independent set in G of the desired size.

Claim 4. *The lemma holds.*

Let a and b be positive integers such that $c = \frac{a}{b}$ and $\gcd(a, b) = 1$. For each $i \in \{1, 2\}$, let $N_i = |A_i| - 3$ and for each $j \in \{0, 1, 2, 3\}$, choose k_i^j such that $1 \leq k_i^j \leq b$ and $k_i^j \equiv a(N_i + j) \pmod{b}$. In other words, $\lceil c(N_i + j)\rceil = \frac{a}{b}(N_i + j) + \frac{b - k_i^j}{b}$. Intuitively, if there exist i and j such that k_i^j is small compared to b, then we improve our lower bound on the independence number (in some smaller graph) by the fact that the independence number is always an integer. In the present claim, we show that if some k_i^j is small, then G has an independent set of the desired size. In contrast, if all k_i^j are big, then we get a contradiction.

By symmetry, we may assume that $k_1^0 \leq k_2^0$.

Subclaim 4a. $k_1^0 + k_2^0 \geq 2b + 1 - 3a$ *and* $k_1^1 + k_2^3 \geq b + a + 1$ *and* $k_1^3 + k_2^1 \geq b + a + 1$ *and* $k_1^2 + k_2^2 \geq b + a + 1$.

If any independent set constructed in Claims 1–3 has size at least $c|G|$, then we are done. So we assume not; more precisely, we assume that each of these independent sets has size at most $\frac{a|G|-1}{b}$. Each of the four desired bounds follow from simplifying the inequalities in Claims 1–3. Note that $|G| = N_1 + N_2 + 3$.

By Claim 1, we have $\alpha(G) \geq \lceil c(|A_1| - 3)\rceil + \lceil c(|A_2| - 3)\rceil = \frac{a}{b}(N_1 + N_2) + \frac{b-k_1^0}{b} + \frac{b-k_2^0}{b} = \frac{a}{b}|G| + \frac{2b-3a-k_1^0-k_2^0}{b}$. Hence $k_1^0 + k_2^0 \geq 2b + 1 - 3a$.

By Claim 2, we have $\alpha(G) \geq \lceil c(|A_1| - 2)\rceil + \lceil c|A_2|\rceil - 1 = \frac{a}{b}(N_1 + 1 + N_2 + 3) + \frac{b-k_1^1}{b} + \frac{b-k_2^3}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^1-k_2^3}{b} - 1$. Hence $k_1^1 + k_2^3 \geq b + a + 1$. Similarly, $k_1^3 + k_2^1 \geq b + a + 1$.

By Claim 3, we have $\alpha(G) \geq \lceil c(|A_1| - 1)\rceil + \lceil c(|A_2| - 1)\rceil - 1 \geq \frac{a}{b}(N_1 + 2 + N_2 + 2) + \frac{b-k_1^2}{b} + \frac{b-k_2^2}{b} - 1 = \frac{a}{b}|G| + \frac{2b+a-k_1^2-k_2^2}{b} - 1$. Hence $k_1^2 + k_2^2 \geq b + a + 1$.

Now to get a contradiction, it suffices to show that $k_i^j \leq a$ for some $i \in \{1, 2\}$ and some $j \in \{1, 2, 3\}$; since $k_i^j \leq b$ for all i and j, this will contradict one of the equalities above.

Subclaim 4b. *Either $k_2^1 \leq a$ or $k_2^2 \leq a$. In each case we get a contradiction, so the claim is true, and the lemma holds.*

By Subclaim 4a, we have $k_1^0 + k_2^0 \geq 2b + 1 - 3a$. By symmetry, we assumed $k_2^0 \geq k_1^0$, so we have $k_2^0 \geq \frac{2b+1-3a}{2}$. Since, $k_2^2 \equiv k_2^0 + 2a \pmod b$ and $\frac{2b+1-3a}{2} + 2a > b$, we have $k_2^2 \leq k_2^0 + 2a - b$. Now we consider two cases, depending on whether $k_2^0 \leq b - a$ or $k_2^0 \geq b - a + 1$. If $k_2^0 \leq b - a$, then $k_2^2 \leq k_2^0 + 2a - b \leq (b - a) + 2a - b = a$, a contradiction. Suppose instead that $k_2^0 \geq b - a + 1$. Now $k_2^1 \equiv k_2^0 + a \pmod b$. Since $k_2^0 \geq b - a + 1$, we see that $k_2^0 + a \geq b + 1$, so $k_2^1 \leq k_2^0 + a - b \leq a$, a contradiction. □

Now we turn to proving a series of lemmas showing that G can't have too many 6^--vertices near each other. Many of these lemmas will rely on applications of the following result, which we think may be of independent interest. The idea for the proof is to find big independent sets for two smaller graphs, and piece them together to get a big independent set in G.

For $S \subseteq V(G)$, let the *interior* of S be $\mathcal{I}(S) = \{x \in S | N(x) \subseteq S\}$. For vertex sets $V_1, V_2 \subset V(G)$ we write $V_1 \leftrightarrow V_2$ if there exists an edge $v_1v_2 \in E(G)$ with $v_1 \in V_1$ and $v_2 \in V_2$; otherwise, we write $V_i \not\leftrightarrow V_j$.

Lemma 3. *Let $\mathcal{G}$ be a minor-closed family of graphs. Let G be a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn (for some fixed $c > 0$). Let $S_1, \ldots, S_t$ be pairwise disjoint subsets of a nonempty set $S \subseteq V(G)$ such that $t < |S|$ and $G[S_i]$ is connected for all $i \in \{1, \ldots, t\}$. Now there exists $X \subseteq \{1, ..., t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$ and $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{i\in X} S_i\right]\right) < |X| + \lceil c(|S| - t)\rceil$.*

Proof. Suppose to the contrary that $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{i\in X} S_i\right]\right) \geq |X| + \lceil c(|S| - t)\rceil$ for all $X \subseteq \{1, ..., t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$. Create G' from G by contracting S_i to a single vertex w_i for each $i \in \{1, \ldots, t\}$ and removing the rest of S. (Note that we allow $t = 0$.) Since $t < |S|$, we have $|G'| < |G|$ and hence minimality of G gives an independent set I in G' with $|I| \geq c|G'| = c(|G| - |S| + t)$. Let $W = I \cap \{w_1, \ldots, w_t\}$. By assumption, we have $\alpha\left(G\left[\mathcal{I}(S) \cup \bigcup_{w_i\in W} S_i\right]\right) \geq |W| + \lceil c(|S| - t)\rceil$. If T is a maximum independent set in $G\left[\mathcal{I}(S) \cup \bigcup_{w_i\in W} S_i\right]$, then $(I \setminus W) \cup T$ is an independent set in G of size at least $|I| - |W| + |T| \geq c(|G| - |S| + t) - |W| + (|W| + \lceil c(|S| - t)\rceil) \geq c|G|$, a contradiction. □

We will often apply Lemma 3 with $S = J \cup N(J)$ for an independent set J. In this case, we always have $J \subseteq \mathcal{I}(S)$. We state this case explicitly in Lemma 4

Lemma 4. *Let $\mathcal{G}$ be a minor-closed family of graphs. Let G be a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn (for some fixed $c > 0$). No independent set J of G and nonnegative integer k satisfy the following conditions.*

1. $|J| \geq c(|N(J)| + k)$.
2. *For at least $|J| - k$ vertices $x \in J$, there is an independent set $\{u_x, v_x\}$ of size 2 in* $N(x) \setminus \bigcup_{y \in J \setminus \{x\}} N(y)$.

Proof. Suppose the lemma is false. Let $S = J \cup N(J)$ and $t = |J| - k$. Pick $x_1, \ldots, x_t \in J$ satisfying condition (2). For $i \in \{1, \ldots, t\}$, let $S_i = \{x_i, u_{x_i}, v_{x_i}\}$. Applying Lemma 3, we get $X \subseteq \{1, \ldots, t\}$ such that $S_i \not\leftrightarrow S_j$ for all distinct $i, j \in X$ and $\alpha\left(G\left[J \cup \bigcup_{i \in X} S_i\right]\right) < |X| + \lceil c(|S| - t)\rceil$. By (2), we have $\alpha\left(G\left[J \cup \bigcup_{i \in X} S_i\right]\right) \geq |(J \setminus X) \cup \bigcup_{x \in X} \{u_x, v_x\}| \geq (|J| - |X|) + 2|X| = |X| + |J|$. Hence $|X| + \lceil c(|S| - t)\rceil > |X| + |J|$, giving $\lceil c(|S| - t)\rceil > |J| \geq \lceil c(|N(J)| + k)\rceil$ by (1). But $|S| - t = (|J| + |N(J)|) - (|J| - k) = |N(J)| + k$; so $\lceil c(|S| - t)\rceil = \lceil c(|N(J)| + k)\rceil$, contradicting the previous inequality. This contradiction finishes the proof. $\square$

As a simple example of how to apply Lemma 4, we note that it immediately implies that every planar graph G has independence ratio at least $\frac{1}{5}$. By Euler's theorem, G has a 5^--vertex v. If $d(v) \leq 4$, then let $G' = G \setminus (v \cup N(v))$. Let I' be an independent set in G' of size at least $(n-5)/5$, and let $I = I' \cup \{v\}$. If instead $d(v) = 5$, then apply Lemma 4, with $c = \frac{1}{5}$, $J = \{v\}$, and $k = 0$; since K_6 is non-planar, v has some pair of non-adjacent neighbors. This completes the proof.

Lemma 5. *Let $\mathcal{G}$ be a minor-closed family of graphs. Let G be a minimal counterexample to the statement that every n-vertex graph in $\mathcal{G}$ has an independent set of size at least cn (for some fixed $c > 0$). For any non-dominating independent set J in G, we have*

$$|N(J)| \geq \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 2.$$

Proof. Assume the lemma is false and choose a counterexample J minimizing $|J|$. Suppose $G[J \cup N(J)]$ is not connected. Now we choose a partition $\{J_1, \ldots, J_k\}$ of J, minimizing k, such that $k \geq 2$ and $G[J_i \cup N(J_i)]$ is connected for each $i \in \{1, \ldots, k\}$. Applying the minimality of $|J|$ to each J_i we conclude that $|N(J_i)| \geq \left\lfloor \frac{1-c}{c}|J_i| \right\rfloor + 2$ for each $i \in \{1, \ldots, k\}$. The minimality of k gives $|N(J)| = \left|\bigcup_{i=1}^{k} N(J_i)\right| = \sum_{i=1}^{k} |N(J_i)|$, so $|N(J)| \geq 2k + \sum_{i=1}^{k} \left\lfloor \frac{1-c}{c}|J_i| \right\rfloor \geq k + \sum_{i=1}^{k} \frac{1-c}{c}|J_i| \geq 2 + \frac{1-c}{c}|J|$, a contradiction. Hence, $G[J \cup N(J)]$ is connected.

Let $S = J \cup N(J)$. Apply Lemma 3 with $t = 1$ and $S_1 = S$. This shows that either $|J| \leq \alpha(G[\mathcal{I}(S)]) < \lceil c(|S| - 1)\rceil$ or $\alpha(G[S]) < 1 + \lceil c(|S| - 1)\rceil$, since the only possibilities are $X = \emptyset$ and $X = \{1\}$. By assumption J is a counterexample, so $|N(J)| \leq \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 1$, which implies that $|S| = |J| + |N(J)| \le |J| + \left\lfloor \frac{1-c}{c}|J| \right\rfloor + 1 = \left\lfloor \frac{|J|}{c} \right\rfloor + 1$. Now $\lceil c(|S|-1) \rceil \le \left\lceil c((\left\lfloor \frac{|J|}{c} \right\rfloor + 1) - 1) \right\rceil = \left\lceil c \left\lfloor \frac{|J|}{c} \right\rfloor \right\rceil \le \lceil |J| \rceil = |J|$. Hence, we cannot have $X = \emptyset$ in Lemma 3.

Instead, we must have $X = \{1\}$, which implies that $\alpha(G[S]) < 1 + \lceil c(|S|-1) \rceil$. Since J is non-dominating, we have $S \ne V(G)$, so we may apply minimality of G to $G[S]$ to conclude that $\alpha(G[S]) \ge \lceil c|S| \rceil$. Combining this inequality with the previous one, we have $\lceil c|S| \rceil = \lceil c(|S|-1) \rceil$. Now the bound on $\lceil c(|S|-1) \rceil$ gives $\lceil c|S| \rceil = \lceil c(|S|-1) \rceil \le \left\lceil c \left\lfloor \frac{|J|}{c} \right\rfloor \right\rceil \le |J|$. Finally, applying Lemma 3 with $t = 0$ (simply deleting $J \cup N(J)$) shows that $|J| < \lceil c(|S|) \rceil$. These two final inequalities contradict each other, which finishes the proof. □

Lemmas 2–5 hold in a more general setting than just $c = \frac{3}{13}$, as we showed. In the rest of this section, we consider only a planar graph G that is minimal among those with independence ratio less than $\frac{3}{13}$. To remind the reader of this, we often call it a *minimal* G. Applying Lemma 5 with $c = \frac{3}{13}$ gives the following corollary.

Lemma 6. *For any non-dominating independent set J in a minimal G, we have*

$$|N(J)| \ge \left\lfloor \frac{10}{3}|J| \right\rfloor + 2.$$

In particular, if $|J| = 1$, then $|N(J)| \ge 5$; if $|J| = 2$, then $|N(J)| \ge 8$; and if $|J| = 3$, then $|N(J)| \ge 12$.

The case $|J| = 1$ shows that G has minimum degree 5, and this is the best we can hope for when $|J| = 1$. Since G is a planar triangulation, we can improve the bound when $|J| = 2$ to $|N(J)| \ge 9$. Similarly, in many cases we can improve the bound when $|J| = 3$ to $|N(J)| \ge 13$. These improvements are the focus of the next ten lemmas. In many instances, the proofs are easy applications of Lemma 3. First, we need a few basic facts about planar graphs.

Lemma 7. *If G is a plane triangulation with no separating 3-cycle and $\delta(G) = 5$, then*

(a) *If $v \in V(G)$, then $G[N(v)]$ is a cycle; and*
(b) *G is 4-connected with $|G| \ge 12$; and*
(c) *If $v, w \in V(G)$ are distinct, then $G[N(v) \cap N(w)]$ is the disjoint union of copies of K_1 and K_2.*

Proof. Plane triangulations are well-known to be 3-connected. Property (a) follows by noting that $G \setminus \{v\}$ is 2-connected and hence each face boundary is a cycle; so $G[N(v)]$ has a hamiltonian cycle. This cycle must be induced since G has no separating 3-cycle.

For (b), suppose that G has a separating set $\{x, y, z\}$. Since G has no separating 3-cycle, we assume that $xy \notin E(G)$. Since G is 3-connected, and $G \setminus \{x, y, z\}$ is disconnected, the vertices of $N(x)$ must be disconnected in $G \setminus \{x, y, z\}$. Since $xy \notin E(G)$, we get that $G[N(x)]$ has a separating set contained in $\{z\}$, contradicting (1). Since G is a plane triangulation and $\delta(G) = 5$, we have $5|G| \le 2|E(G)| = 6|G| - 12$, so $|G| \ge 12$.

By (a) and $\delta(G) = 5$, it follows that no neighborhood contains K_3 or C_4. If $G[N(v) \cap N(w)]$ had an induced P_3, then the neighborhood of the center of this P_3 would contain K_3 or C_4. This proves (c). □

Lemma 8. *Every independent set J in a minimal G with $|J| = 2$, satisfies $|N(J)| \ge 9$.*

Proof. By Lemma 7(b), $|G| \geq 12$; so J cannot be dominating when $|N(J)| \leq 7$. Hence, by Lemma 6, we may assume $|N(J)| = 8$. Let $J = \{x, y\}$. If we can apply Lemma 4 with $k = 0$, then we are done. If we cannot, then by symmetry we may assume that there is no independent 2-set in $N(x) \setminus N(y)$. So $N(x) \setminus N(y)$ is a clique. Since $d(x) \geq 5$ and $N(x)$ induces a cycle, $|N(x) \setminus N(y)| \leq 2$. Now, since x is a 5^+-vertex, $G[N(x) \cap N(y)]$ induces P_3; this contradicts Lemma 7(c). □

A direct consequence of Lemma 8 is the following useful fact.

Lemma 9. *A minimal G has no non-adjacent 5-vertices u and w with at least two common neighbors. In particular, each vertex v in G has $\frac{d(v)}{2}$ or more 6^+-neighbors.*

Proof. The first statement follows immediately from Lemma 8. Now we consider the second. Let v be a vertex with $d(v) = k$ and neighbors $u_1, \dots, u_k$ in clockwise order. If more than $k/2$ neighbors of v are 5-vertices, then (by Pigeonhole) there exists an integer i such that u_i and u_{i+2} are 5-vertices (subscripts are modulo k). Now we apply Lemma 8 to u_i and u_{i+2}. □

Now we consider the case when $|J| = 3$. Lemma 6 gives $|N(J)| \geq 12$. Our next few lemmas show certain conditions under which we can conclude that $|N(J)| \geq 13$.

Lemma 10. *Let J be an independent set in a minimal G with $|J| = 3$ and $|N(J)| \geq 12$. Choose $S_1, S_2 \subseteq J \cup N(J)$ such that $S_1 \cap S_2 = \emptyset$ and both $G[S_1]$ and $G[S_2]$ are connected. If $\alpha(G[S_i \cup J]) \geq 4$ for each $i \in \{1, 2\}$, then $|N(J)| \geq 13$.*

Proof. Suppose not and choose a counterexample minimizing $|J \cup N(J)| - |S_1 \cup S_2|$. Clearly $|N(J)| = 12$. First we show that $S_1 \cup S_2 = J \cup N(J)$. It suffices to show that $G[J \cup N(J)]$ is connected, since then we can add to either S_1 or S_2 any vertex in $N(S_1 \cup S_2) \setminus (S_1 \cup S_2)$. In particular, we show that every $x \in J$ satisfies $(x \cup N(x)) \cap (\cup_{y \in (J \setminus \{x\})}(y \cup N(y))) \neq \emptyset$. If not, then $| \cup_{y \in J \setminus \{x\}} (y \cup N(y))| \leq |J \cup N(J)| - (d(x) + 1) \leq 15 - 6 = 9$. However, now $J \setminus \{x\}$ violates Lemma 8. So, we must have $S_1 \cup S_2 = J \cup N(J)$. Similarly, $S_1 \leftrightarrow S_2$.

Now we apply Lemma 3 with $S = J \cup N(J)$, $t = 2$, and S_1 and S_2 as above. Since $S_1 \leftrightarrow S_2$, we have $|X| \leq 1$. By hypothesis, $\alpha(G[S_i \cup J]) \geq 4$ for each $i \in \{1, 2\}$, so suppose that $X = \emptyset$. Now we have $\alpha(G[J]) \geq |J| = 3 = \lceil \frac{3}{13}(|J \cup N(J)| - 2) \rceil = \lceil \frac{3}{13}(3 + 12 - 2) \rceil$. This contradiction completes the proof. □

Lemma 11. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G where*

(1) *$N(u_1) \setminus (N(u_2) \cup N(u_3))$ contains an independent 2-set; and*
(2) *$\alpha(G[J \cup N(u_2) \cup N(u_3)]) \geq 4$,*

then $|N(J)| \geq 13$.

Proof. Since G is a planar triangulation with minimum degree 5 and at least three 6^+-vertices by Lemma 9, we have $5|G| + 3 \leq 2|E(G)| = 6|G| - 12$ and hence $|G| \geq 15$. Thus J cannot be dominating when $|N(J)| \leq 11$. So, by Lemma 6, we know that $|N(J)| \geq 12$. Let I be an independent set of size 2 in $N(u_1) \setminus (N(u_2) \cup N(u_3))$.

First, suppose $N(u_2) \cap N(u_3) \neq \emptyset$. We apply Lemma 10 with $S_1 = \{u_1\} \cup I$ and $S_2 = \{u_2, u_3\} \cup N(u_2) \cup N(u_3)$. Clearly, $G[S_1]$ is connected. Also, $G[S_2]$ is connected since $N(u_2) \cap N(u_3) \neq \emptyset$, by assumption. The set $I \cup \{u_2, u_3\}$ shows that $\alpha(G[S_1 \cup J]) \geq 4$ and hypothesis

(2) shows that $\alpha(G[S_2 \cup J]) \geq 4$. So the hypotheses of Lemma 10 are satisfied, giving $|N(J)| \geq 13$.

Instead, suppose $N(u_2) \cap N(u_3) = \emptyset$. This implies $N(u_2) \setminus (N(u_1) \cup N(u_3)) = N(u_2) \setminus N(u_1)$. If $N(u_2) \setminus N(u_1)$ contains an independent 2-set as well, then applying Lemma 4 with $k = 1$ gives $|N(J)| \geq 13$, as desired. Otherwise, $|N(u_2) \setminus N(u_1)| \leq 2$, so $G[N(u_2) \cap N(u_1)]$ contains P_3, contradicting Lemma 7(c). □

One particular case of Lemma 11 is easy to verify in our applications, so we state it separately, as Lemma 13. First, we need the following lemma.

Lemma 12. *Let v be a 7^+-vertex in G. If $S \subseteq V(G)$ with $\{v\} \cup N(v) \subseteq S$ and $|S| \geq 10$, then $\alpha(G[S]) \geq 4$.*

Proof. If $d(v) \geq 8$, then we are done, since the neighbors of v induce an 8^+-cycle, which has independence number at least 4. So suppose $d(v) = 7$. Let $u_1, \ldots, u_7$ denote the neighbors of v in clockwise order; recall that $G[N(v)]$ is a 7-cycle. Pick $w_1, w_2 \in S \setminus (\{v\} \cup N(v))$. Let $H_i = G[N(v) \setminus N(w_i)]$ for each $i \in \{1, 2\}$. If H_i contains an independent 3-set for some $i \in \{1, 2\}$, then $J \cup \{w_i\}$ is the desired independent 4-set, so we are done. Therefore, we must have $|H_i| \leq 4$ for each $i \in \{1, 2\}$. So, $|N(v) \cap N(w_i)| \geq 3$ and hence Lemma 7(c) shows that $N(v) \cap N(w_i)$ has at least two components; therefore, so does H_i. It must have exactly two components or we get an independent 3-set in H_i. Similarly, if $|H_i| = 4$, then H_i has no isolated vertex. So, either H_i is $2K_2$ or $|H_i| \leq 3$. Now in each case we get a subdivision of $K_{3,3}$; the branch vertices of one part are v, w_1, w_2 and the branch vertices of the other are three of the u_i. This contradiction finishes the proof. □

Lemma 13. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G where*

(1) $N(u_1) \setminus (N(u_2) \cup N(u_3))$ *contains an independent 2-set; and*

(2) $G[J \cup N(u_2) \cup N(u_3)]$ *contains a 7^+-vertex and its neighborhood,*

then $|N(J)| \geq 13$.

Proof. We apply Lemma 11 using Lemma 12 to verify hypothesis (2). To do so, we need that $|J \cup N(u_2) \cup N(u_3)| \geq 10$; this is immediate from Lemma 8. □

Lemma 14. *Let J be an independent 3-set in G. Choose $S_1, S_2, S_3 \subseteq J \cup N(J)$ such that $G[S_i]$ is connected and $S_i \cap S_j = \emptyset$ for all distinct $i, j \in \{1, 2, 3\}$. If $|N(J)| \leq 13$, then either*

(1) $S_i \not\leftrightarrow S_j$ *for some* $\{i, j\} \subseteq \{1, 2, 3\}$; *or*

(2) $\alpha(G[S_i \cup J]) \leq 3$ *for some* $i \in \{1, 2, 3\}$.

Proof. This is an immediate corollary of Lemma 3 with $S = J \cup N(J)$ and $t = 0$. If $S_i \leftrightarrow S_j$ for all $\{i, j\} \in \{1, 2, 3\}$, then in Lemma 3 either $|X| = 1$ or $|X| = 0$. We cannot have $|X| = 0$, since $\alpha(G[\mathcal{I}(S)]) \geq \alpha(G[J]) \geq |J| = 3 = \left\lceil \frac{3}{13}(13 + 3 - 3) \right\rceil$. Hence $|X| = 1$, which implies (2). □

The next lemma can be viewed as a variant on the result we get by applying Lemma 4 with $|J| = 3$ and $k = 0$ (and $c = \frac{3}{13}$). As in that case, we require that each of $N(u_1) \setminus (N(u_2) \cup N(u_3))$ and $N(u_2) \setminus (N(u_1) \cup N(u_3))$ contains an independent 2-set. However, here we do not require that $N(u_3) \setminus (N(u_1) \cup N(u_2))$ contains an independent 2-set. Instead, we have hypothesis (2) below. Not surprisingly, the proof is similar to that of Lemma 4.

Lemma 15. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G such that*

(1) *$N(u_i) \setminus (N(u_j) \cup N(u_3))$ contains an independent 2-set M_i for all $\{i, j\} = \{1, 2\}$; and*
(2) *$\alpha(G[J \cup V(H)]) \geq 4$, where H is u_3's component in $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$,*

then $|N(J)| \geq 14$.

Proof. First, we show that u_3 is distance two from each of u_1 and u_2. Suppose not; by symmetry, assume that u_3 is distance at least three from u_1. Now $N(u_3)\setminus(N(u_1)\cup N(u_2)) = N(u_3) \setminus N(u_2)$. By Lemma 7, $N(u_3) \cap N(u_2)$ consists of disjoint copies of K_1 and K_2. Thus, since $d(u_3) \geq 5$, we see that $N(u_3) \setminus (N(u_1) \cup N(u_2))$ contains an independent 2-set. Now, if $|N(J)| \leq 13$, then applying Lemma 4 with $k = 0$ gives a contradiction. Hence, u_3 is distance two from each of u_1 and u_2.

Choose disjoint subsets $S_1, S_2, S_3 \subset J \cup N(J)$ where $G[S_i]$ is connected for all $i \in \{1, 2, 3\}$ and $\{u_i\} \cup M_i \subseteq S_i$ for each $i \in \{1, 2\}$ and $u_3 \in S_3$, first maximizing $|S_3|$ and subject to that maximizing $|S_1| + |S_2| + |S_3|$. Since $J \subseteq S_1 \cup S_2 \cup S_3$, maximality of $|S_1| + |S_2| + |S_3|$ gives $S_1 \cup S_2 \cup S_3 = J \cup N(J)$.

Now we apply Lemma 3, with $S = S_1 \cup S_2 \cup S_3$. To get a contradiction, we need only verify, for each possible X, that $\alpha(G[\mathcal{I}(S) \cup \bigcup_{i \in X} S_i]) \geq |X| + \left\lceil \frac{3}{13}(|S| - |J|) \right\rceil = |X| + 3$. Since $S_3 \not\leftrightarrow S_1$ and $S_3 \not\leftrightarrow S_2$, either $|X| \leq 1$ or else $X = \{1, 2\}$. In the latter case, $M_1 \cup M_2 \cup \{u_3\}$ is the desired independent 5-set. If instead $X = \emptyset$, then J is the desired independent 3-set.

So we must have $X = \{i\}$ for some $i \in \{1, 2, 3\}$. If $i \in \{1, 2\}$, then $M_i \cup \{u_3, u_{3-i}\})$ is the desired independent set. So instead assume that $X = \{3\}$. But, by the maximality of $|S_3|$, $G[J \cup S_3]$ contains u_3's component in $G[\{u_3\} \cup N(J)] \setminus M_1 \setminus M_2$. So by (2), $G[J \cup S_3]$ has an independent 4-set, as desired. □

Again, one particular case of Lemma 15 is easy to verify, so we state it separately.

Lemma 16. *Let $J = \{u_1, u_2, u_3\}$. If J is an independent set in a minimal G such that*

(1) *$N(u_i) \setminus (N(u_j) \cup N(u_3))$ contains an independent 2-set M_i for all $\{i, j\} = \{1, 2\}$; and*
(2) *u_3's component H in $G[\{u_3\} \cup N(J)] \setminus (M_1 \cup M_2)$ satisfies $|J \cup V(H)| \geq 10$ and $G[J \cup V(H)]$ contains a 7^+ vertex and its neighborhood,*

then $|N(J)| \geq 14$.

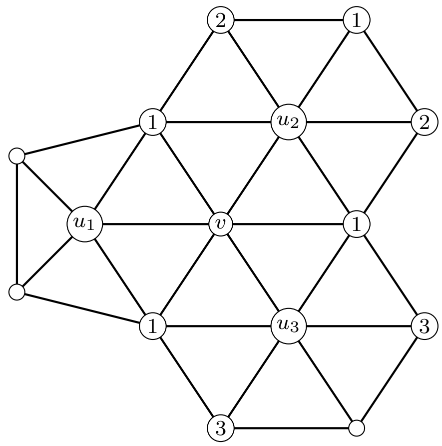
Proof. We apply Lemma 15, using Lemma 12 to verify hypothesis (2). □

Thus far, we our lemmas have not focused much on the actual planar embedding of G. At this point we transition and start analyzing the embedding, as well.

Lemma 17. *Every minimal G has no 6-vertex v with 6^--neighbors u_1, u_2 and u_3 that are pairwise non-adjacent.*

Proof. Lemma 6 yields $12 \leq |N(\{u_1, u_2, u_3\})| \leq d(u_1) + d(u_2) + d(u_3) - 5$. Hence, by symmetry, assume that the vertices are arranged as in Figure 1(A) with all vertices distinct as drawn or as in Figure 1(B) with at most one pair of vertices identified.

The first case is impossible by Lemma 4 with $k = 1$, using the vertices labeled 2 for u_2 and those labeled 3 for u_3. When the vertices in Figure 1(B) are distinct as drawn, we apply Lemma 4 with $k = 0$, using the vertices labeled 2 for u_2, the vertices labeled 3 for u_3, and those labeled 4 for u_1. Otherwise, by symmetry and the fact that G contains no



(A) A 6-vertex, v, with non-adjacent neighbors u_1, u_2, and u_3 such that $d(u_1) = 5$ and $d(u_2) = d(u_3) = 6$.

(B) A 6-vertex, v, with non-adjacent 6-neighbors u_1, u_2, and u_3.

FIGURE 1. The two cases of Lemma 17.

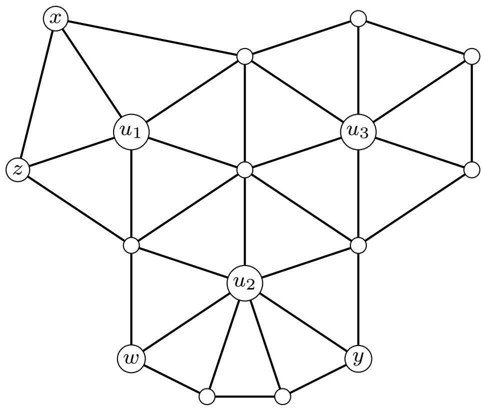
separating 3-cycle, assume that the vertices labeled 2 and 3 that are drawn at distance four are identified. Now the pairs of vertices labeled 1 each have a common neighbor, so the vertices labeled 1 must be an independent set, to avoid a separating 3-cycle. Now, to get a contradiction, apply Lemma 11, using the vertices labeled 4 for the independent 2-set. □

Lemma 18. *Every minimal G has no 6-vertex v with pairwise non-adjacent neighbors u_1, u_2, and u_3, where $d(u_1) = 5$, $d(u_2) \le 6$, and $d(u_3) = 7$.*

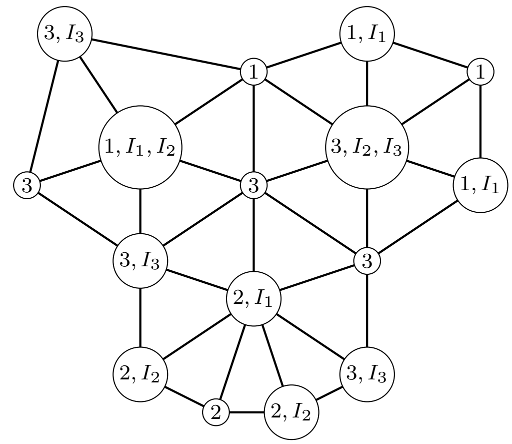
Proof. Let $J = \{u_1, u_2, u_3\}$. By Lemma 6, $12 \le |N(J)| \le 5 + 6 + 7 - 5 = 13$, so at most one pair of vertices in Figure 2(A) are identified.

First, suppose the vertices in the figure are distinct as drawn. Suppose $x \not\leftrightarrow y$, as in Figure 2(B). For each $i \in \{1, 2, 3\}$, let S_i consist of the vertices labeled i. Now for each $i \in \{1, 2, 3\}$, $G[S_i]$ is connected. Clearly, for each $i \in \{1, 2\}$ the vertices labeled I_i form an independent 4-set. Since $x \not\leftrightarrow y$, the vertices labeled I_3 also form an independent 4-set. Note that $S_1 \leftrightarrow S_3$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_2$. If $S_1 \leftrightarrow S_2$, then we can apply Lemma 14 to get a contradiction. So, we assume that $S_1 \not\leftrightarrow S_2$. But now we have an independent 5-set consisting of u_1, the two vertices labeled $\{1, I_1\}$ and the two vertices labeled $\{2, I_2\}$; hence $\alpha(G[S_1 \cup S_2 \cup J]) \ge 5$. So, we can apply Lemma 3 to get a contradiction. So, instead we assume $x \leftrightarrow y$.

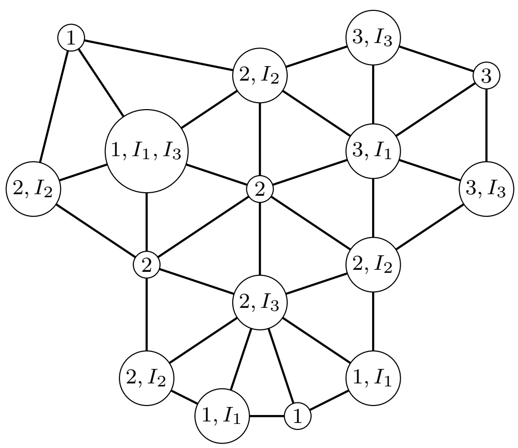
Suppose $w \not\leftrightarrow z$, as in Figure 2(C). For each $i \in \{1, 2, 3\}$, let S_i consist of the vertices labeled i . Clearly $G[S_i]$ is connected for each $i \in \{2, 3\}$. Also, $G[S_1]$ is connected because $x \leftrightarrow y$. Note that for each $i \in \{1, 3\}$, the vertices labeled I_i form an independent 4-set. Since $x \leftrightarrow y$ and $w \not\leftrightarrow z$, the vertices labeled I_2 also form an independent 4-set. Note that $S_1 \leftrightarrow S_2$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_3$. If $S_1 \leftrightarrow S_3$, then we apply Lemma 14 to get a contradiction. So instead we assume that $S_1 \not\leftrightarrow S_3$. But now we again have an independent 5-set, consisting of u_1, the two vertices labeled $\{1, I_1\}$, and the two vertices labeled $\{3, I_3\}$;



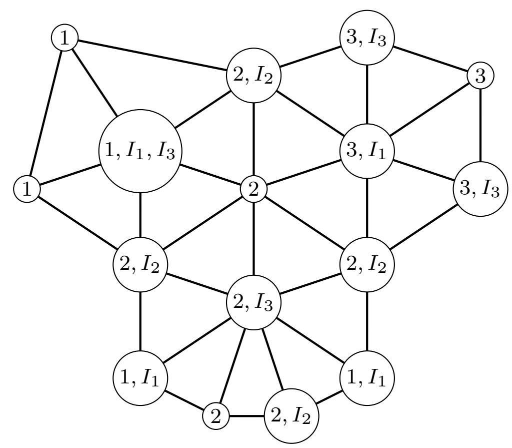
(A) The forbidden configuration.



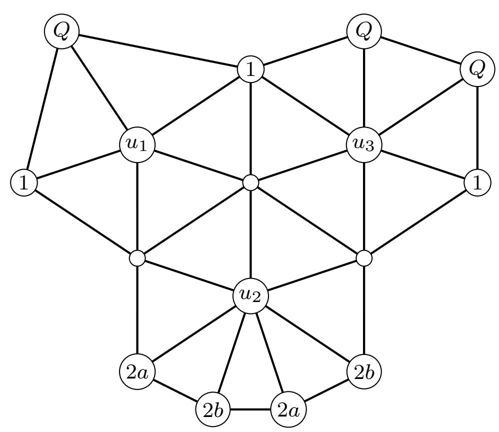
(B) The $x \not\leftrightarrow y$ case.



(C) The $x \leftrightarrow y$ and $w \not\leftrightarrow z$ case.



(D) The $x \leftrightarrow y$ and $w \leftrightarrow z$ case.



(E) The case of one identified pair.

FIGURE 2. The case of Lemma 18.

hence $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$. So, again we apply Lemma 3 to get a contradiction. Thus, we instead assume $w \leftrightarrow z$.

Now consider Figure 2(D). For each $i \in \{1,2,3\}$, let S_i consist of the vertices labeled i. Note that $G[S_i]$ is connected for each $i \in \{2,3\}$. Also, $G[S_1]$ is connected because $x \leftrightarrow y$ and $w \leftrightarrow z$. Clearly, the vertices labeled I_i form an independent 4-set for each $i \in \{1,3\}$. Since $x \leftrightarrow y$, the vertices labeled I_2 also form an independent 4-set. Note that $S_1 \leftrightarrow S_2$ and $S_2 \leftrightarrow S_3$; however, possibly $S_1 \not\leftrightarrow S_3$. If $S_1 \leftrightarrow S_3$, then we apply Lemma 14 to get a contradiction. So, instead we assume that $S_1 \not\leftrightarrow S_3$. But now we have an independent 5-set, consisting of u_1, the two vertices labeled $\{1, I_1\}$, and the two vertices labeled $\{3, I_3\}$; hence $\alpha(G[S_1 \cup S_3 \cup J]) \geq 5$. So, we apply Lemma 3 to get a contradiction.

Hence, we may assume that exactly one pair of vertices in Figure 2(A) is identified. No neighbor of u_1 can be identified with a neighbor of u_3, since then u_1 and u_3 would have three common neighbors, violating Lemma 8. Hence, to avoid separating 3-cycles, we assume that a vertex labeled $2a$ is identified with a vertex labeled Q (the case where a vertex labeled $2b$ is identified with a vertex lableled Q is nearly identical, so we omit the details). But now the rightmost vertex labeled 1 and the leftmost vertex labeled 1 are on opposite sides of a separating cycle and hence non-adjacent. Therefore, u_2 together with the vertices labeled 1 is an independent 4-set. So, now we apply Lemma 11 to get a contradiction, using the vertices labeled $2b$ for the independent 2-set. □

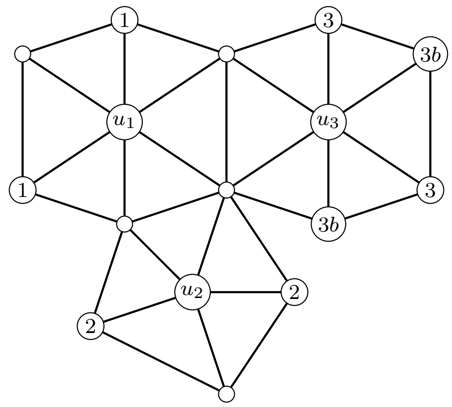
Lemma 19. *Let u_1 be a 6-vertex with non-adjacent vertices u_2 and u_3 each at distance two from u_1, where u_2 is a 5-vertex and u_3 is a 6^--vertex. A minimal G cannot have u_1 and u_2 with two common neighbors, and also u_1 and u_3 with two common neighbors.*

Proof. Figure 3 shows the possible arrangements when u_3 is a 6-vertex. The case when u_3 is a 5-vertex is similar, but easier. In particular, when u_3 is a 5-vertex, we already know $|N(\{u_1, u_2, u_3\})| \leq 12$, so all vertices in the corresponding figures must be distinct as drawn. Furthermore, it now suffices to apply Lemma 4 with $k = 1$. We omit further details. So suppose instead that $d(u_3) = 6$.

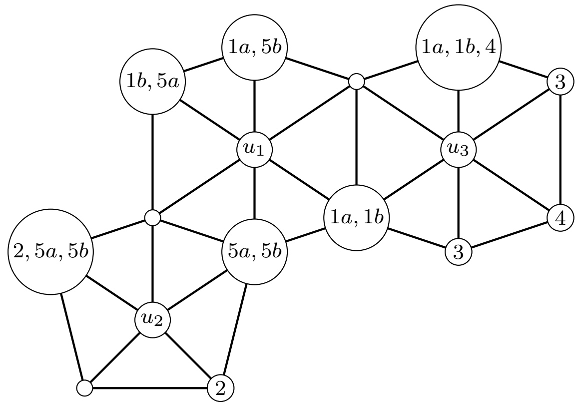
First, suppose all vertices in the figures are distinct as drawn. Now Figures 3(A,C) are impossible by Lemma 4 with $k = 0$; for each $i \in \{1,2,3\}$, we use the vertices labeled i as the independent 2-set for u_i. For Figure 3(B), let I_1 be the vertices labeled u_2 or $1a$ and let I_2 be the vertices labeled u_2 or $1b$. To avoid a separating 3-cycle, at least one of I_1 or I_2 is independent. Hence Figure 3(B) is impossible by Lemma 15; for the independent 4-set, use I_1 or I_2 and for each $i \in \{2,3\}$, use the vertices labeled i as the independent 2-set for u_i.

By Lemma 6, $|N(J)| \geq 12$, so exactly one pair of vertices is identified in one of Figures 3(A,B,C). First, consider Figures 3(A,C) simultaneously. Since G has no separating 3-cycle, the identified pair must contain a vertex labeled 3. Now we apply Lemma 4 with $k = 1$, using the vertices labeled $3b$ in place of those labeled 3.

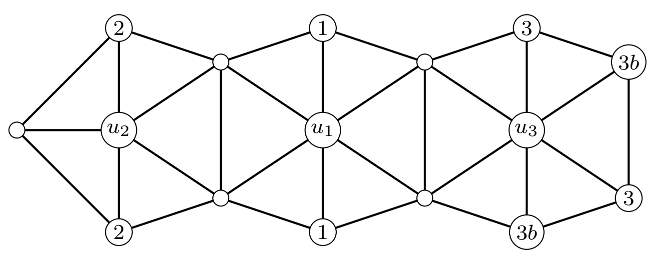
Finally, for Figure 3(B), we apply Lemma 11. For the independent 2-set we use either the vertices labeled 3 or the vertices labeled 4; at least one of these pairs contains no identified vertex. For the independent 4-set, we use either u_3 and the vertices labeled $5a$ or else u_3 and the vertices labeled $5b$. Since G has no separating 3-cycle, at least one of these 4-sets will be independent. □



(A) Here u_2 and u_3 have a common neighbor in $N(u_1)$.



(B) Here u_2 and u_3 have adjacent neighbors in $N(u_1)$.



(C) Here u_2 and u_3 have neighbors at distance 2 in $N(u_1)$.

FIGURE 3. The cases of Lemma 19. The three possibilities for an independent 3-set $\{u_1, u_2, u_3\}$ where $d(u_1) = 6$, $d(u_2) \leq 6$, $d(u_3) = 5$, and each of u_2 and u_3 has two neighbors in common with u_1.

Lemma 20. *Every minimal G has no 7-vertex v with a 5-neighbor and two other 6^--neighbors, u_1, u_2, and u_3, that are pairwise non-adjacent. In other words, Figures 4(A–E) are forbidden.*

Proof. Lemma 6 yields $12 \le |N(\{u_1, u_2, u_3\})| \le d(u_1)+d(u_2)+d(u_3)-4 \le 5+6+6-4 \le 13$. So, by symmetry, we assume that the vertices are arranged as in Figures 4(B,C) with all vertices distinct as drawn or as in Figures 4(D,E) with at most one pair of vertices identified.

First suppose the vertices are disinct as drawn. For Figures 4(B,C,D), we apply Lemma 4; for (B) and (C) we use $k = 1$, and for (D) we use $k = 0$. For Figure 4(E), we apply Lemma 16, using the vertices labeled 1 for M_1 and the those labeled 4 for M_2.

So, instead suppose that a single pair of vertices is identified in one of Figures 4(D,E). First consider (D). If a vertex labeled 1 is identified with another vertex, then we apply Lemma 13 using the vertices labeled 2 for the independent 2-set (vertices labeled 1 and 2 cannot be identified, since they are drawn at distance at most 3). Otherwise, the identified vertices must be those labeled 2 and 4 that are drawn at distance four. Now the vertices labeled w_3 or 3 are pairwise at distance two, so must be an independent 4-set. Now we get a contradiction, by applying Lemma 11 using the vertices labeled 1 for the independent 2-set.

Finally, consider (E). Again we apply Lemma 4, with $k = 1$. Since u_1 has three possibilities for its pair of non-adjacent neighbors, and no neighbor of u_1 appears in all three of these pairs, u_1 satisfies condition (2). Similarly, u_3 also satisfies condition (2). □

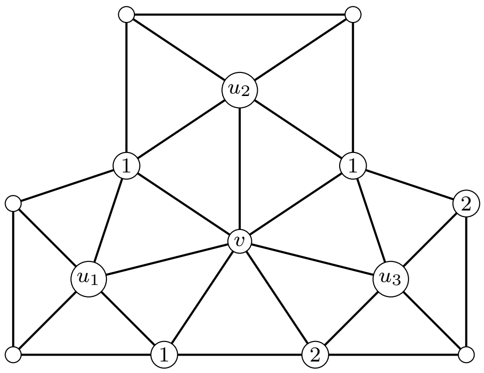
Lemma 21. *Let v_1, v_2, v_3 be the corners of a 3-face, each a 6^+-vertex. Let u_1, u_2, u_3 be the other pairwise common neighbors of v_1, v_2, v_3, i.e., u_1 is adjacent to v_1 and v_2, u_2 is adjacent to v_2 and v_3, and u_3 is adjacent to v_3 and v_1. We cannot have $|N(\{u_1, u_2, u_3\})| \le 13$. In particular, we cannot have $d(u_1) = d(u_2) = 5$ and $d(u_3) \le 6$.*

Proof. If the only pairwise common neighbors of the u_i are the v_i, then two u_i are 5-vertices and the third is a 6^--vertex. The case where the u_i have more pairwise common neighbors is nearly identical, and we remark on it briefly at the end of the proof. So suppose that $d(u_1) = d(u_2) = 5$ and $d(u_3) = 6$, as shown in Figure 5; the case where $d(u_3) = 5$ is nearly identical. We will apply Lemma 4 with $J = \{u_1, u_2, u_3\}$ and $k = 0$. Clearly, J is an independent set. Now we verify that each vertex of J satisfies condition (2). Since G has no separating 3-cycle, the two vertices in each pair with a common label (among $\{1, 2, 3\}$) are distinct and non-adjacent. Similarly, the vertices with labels in $\{1, 2, 3\}$ are distinct, since they are drawn at pairwise distance at most three, and G has no separating 3-cycle. Thus, we can apply Lemma 4, as desired.

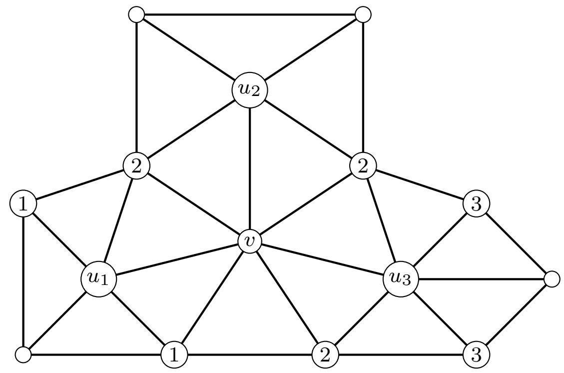
In the more general case where the u_i have pairwise common neighbors in addition to the v_i, the argument above still shows that the vertices with labels in $\{1, 2, 3\}$ are distinct. So again, we can apply Lemma 4 with $k = 0$. □

Lemma 22. *Let u_1 be a 7-vertex with non-adjacent 5-vertices u_2 and u_3 each at distance two from u_1. A minimal G cannot have u_1 and u_2 with two common neighbors and also u_1 and u_3 with two common neighbors.*

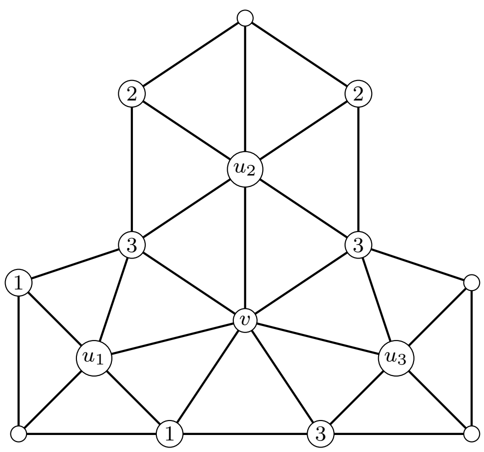
Proof. This situation is shown in Figures 6(A,B,C), where at most one pair of vertices drawn as distinct are identified. If all vertices labeled 2 or 3 are distinct as drawn, then we apply Lemma 16 and get a contradiction. By Lemma 6, the only other possibility is that exactly one pair of vertices is identified. Such a pair must consist of vertices labeled 2 and 3 that are drawn at distance four (otherwise we apply Lemma 4, with $k = 1$). In Figure 6(A),



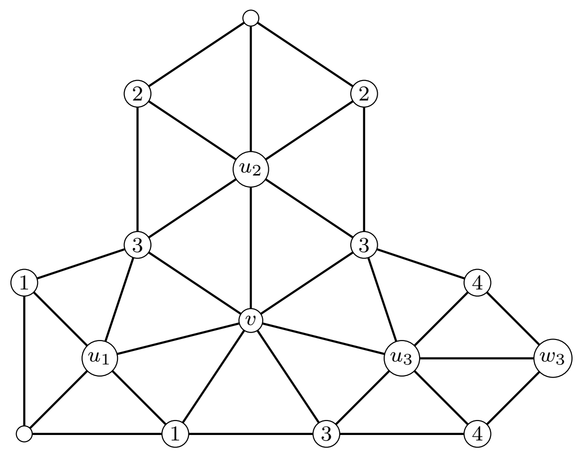
(A) A 7-vertex, v, with non-adjacent 5-neighbors, u_1, u_2, and u_3.



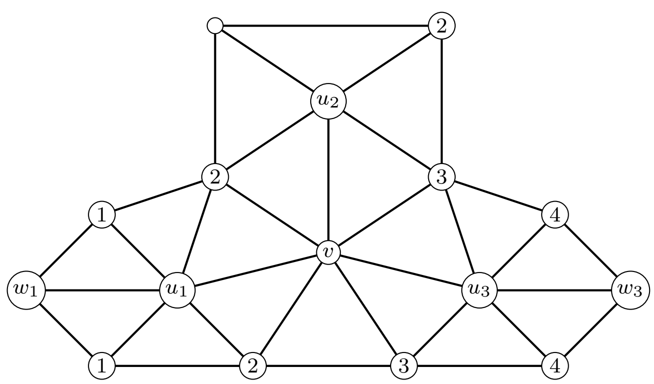
(B) A 7-vertex, v, with a 6-neighbor, u_3, and two 5-neighbors, u_1 and u_2, with all pairs of u_is non-adjacent.



(C) A 7-vertex, v, with a 6-neighbor, u_2, and two 5-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent.

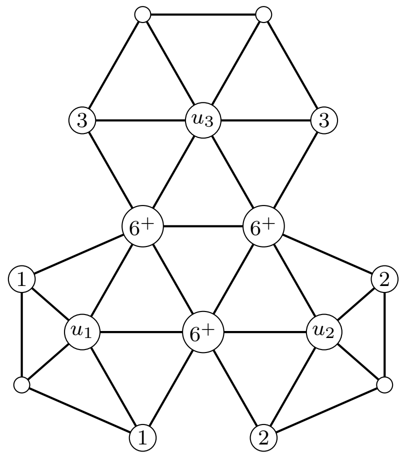


(D) A 7-vertex, v, with a 5-neighbor, u_1, and two 6-neighbors, u_2 and u_3, with all pairs of u_is non-adjacent.



(E) A 7-vertex, v, with a 5-neighbor, u_2, and two 6-neighbors, u_1 and u_3, with all pairs of u_is non-adjacent.

Figure 4. The five cases of Lemma 20.



(A) A 3-face $v_1v_2v_3$, such that the pairwise common neighbors of v_1, v_2, v_3 have degrees 5, 5, and at most 6.

FIGURE 5. The key case of Lemma 21.

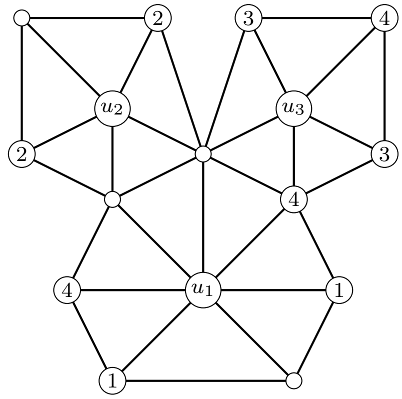
this is impossible, since the two 5-vertices u_2 and u_3 would have two neighbors in common, violating Lemma 9.

Now we consider the cases shown in Figures 6(B,C) simultaneously. We apply Lemma 11 using the vertices labeled 1 for the independent 2-set. Let I_1 be the set of vertices labeled 4. If I_1 is independent, then we are done; so assume not. Recall that a vertex labeled 2 is identified with a vertex labeled 3.

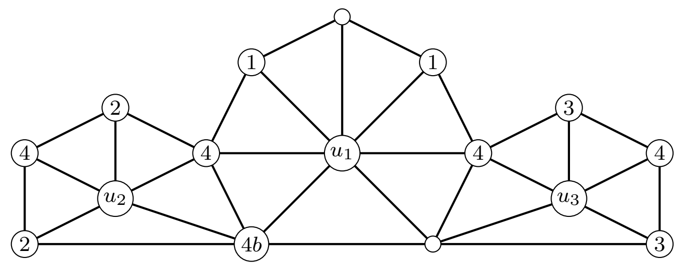
Suppose the vertices labeled 4 in $N(u_2)\setminus N(u_1)$ and $N(u_3)\setminus N(u_1)$ are not adjacent. Now by symmetry, we may assume that the vertex labeled 4 in $N(u_1)\cap N(u_2)$ is adjacent to the vertex labeled 4 in $N(u_3)\setminus N(u_1)$. Let I_2 be the set made from I_1 by replacing the vertex labeled 4 in $N(u_1)\cap N(u_2)$ with the vertex labeled $4b$. If I_2 is independent, then we are done; so assume not. Now the vertex labeled $4b$ must be adjacent to the vertex labeled 4 in $N(u_3)\setminus N(u_1)$, but this makes a separating 3-cycle (consisting of two vertices labeled 4 and one labeled $4b$), a contradiction.

So, we may assume that the vertices labeled 4 in $N(u_2)\setminus N(u_1)$ and $N(u_3)\setminus N(u_1)$ are adjacent. Suppose the topmost vertex labeled 2 is identified with the topmost vertex labeled 3. Now again we are done; our independent 4-set consists of the two neighbors of u_1 labeled 4, together with an independent 2-set from among the two leftmost and two rightmost vertices (by planarity, they cannot all four be pairwise adjacent).

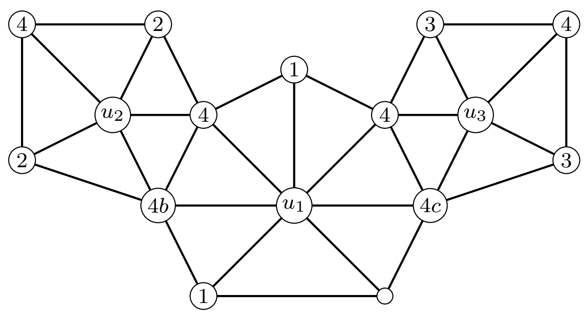
The only remaining possibility is that the bottommost vertex labeled 2 is identified with the bottommost vertex labeled 3 (since the two topmost vertices labeled 4 are adjacent). If we are in Figure 6(B), then the vertex labeled $4b$ is a 5-vertex; since it shares two neighbors with u_3, another 5-vertex, we contradict Lemma 9. Hence, we must be in Figure 6(C). Now our independent 4-set consists of the two neighbors of u_1 labeled 4b and 4c, together with an indpendent 2-set from among the four topmost vertices (again, by planarity, they cannot all be pairwise adjacent). $\square$



(A) Here u_2 and u_3 have a common neighbor in $N(u_1)$.



(B) Here u_2 and u_3 have adjacent neighbors in $N(u_1)$.



(C) Here u_2 and u_3 have neighbors at distance 2 in $N(u_1)$.

FIGURE 6. The cases of Lemma 22. The three possibilities for an independent 3-set $\{u_1, u_2, u_3\}$ where $d(u_1) = 7$, $d(u_2) = d(u_3) = 5$, and each of u_2 and u_3 has two neighbors in common with u_1.

Lemma 23. *Suppose that a minimal G contains a 7-vertex v with no 5-neighbor. Now v cannot have at least five 6-neighbors, each of which has a 5-neighbor.*

Proof. Suppose to the contrary. Denote the neighbors of v in clockwise order by $u_1, \ldots, u_7$.

Case 1: Vertices u_1, u_2, u_3, u_4 are 6-vertices, each with a 5-neighbor.

First, suppose that u_2 and u_3 have a common 5-neighbor, w_2. Consider the 5-neighbor w_1 of u_1. By Lemma 9, it cannot be common with u_2; similarly, the 5-neighbor w_4 of u_4 cannot be common with u_3. (We must have w_1 and w_4 distinct, since otherwise we apply Lemma 21 to $\{u_1, u_4, w_2\}$. Also, we must have w_1 and w_4 each distinct from w_2, since G has no separating 3-cycles.)

First, suppose that w_1 has two common neighbors with u_2. If $w_1 \not\leftrightarrow u_4$, then we apply Lemma 19 to $\{w_1, u_2, u_4\}$; so assume $w_1 \leftrightarrow u_4$. Now let $J = \{u_1, u_4, w_2\}$. Clearly, J is an independent 3-set. Also $|N(J)| \leq 6 + 6 + 5 - 4 = 13$, so we are done by Lemma 21. So w_1 cannot have two common neighbors with u_2. Similarly, w_4 cannot have two common neighbors with u_3. Hence, $w_1 \leftrightarrow u_7$ and also $w_4 \leftrightarrow u_5$. Now we must have $w_1 \leftrightarrow w_4$; otherwise we apply Lemma 22 to $\{v, w_1, w_4\}$. Similarly, we must have $w_1 \leftrightarrow w_2$ and $w_2 \leftrightarrow w_4$; these edges cut off w_4 from u_1, so $u_1 \not\leftrightarrow w_4$. Since u_1 and w_4 are non-adjacent, but have a 5-neighbor in common, they must have two neighbors in common. So we apply Lemma 19 to $\{u_1, u_3, w_4\}$. Hence, we conclude that the common neighbor of u_2 and u_3 is not a 5-neighbor.

Since u_1 and u_3 are 6^--vertices, by Lemma 17, vertex u_2 cannot have another 6^--vertex that is nonadjacent to u_1 and u_3. Thus, the common neighbor w_1 of u_1 and u_2 is a 5-vertex; similarly, the common neighbor w_4 of u_3 and u_4 is a 5-vertex. We must have $w_1 \leftrightarrow w_4$, for otherwise we apply Lemma 22. We may assume that u_6 is a 6-vertex. If not, then v's five 6-neighbors, each with a 5-neighbor, are *successive*; so, by symmetry, we are in the case above, where u_2 and u_3 have a common 5-neighbor.

By planarity and symmetry, either $u_1 \not\leftrightarrow w_4$ or else $u_4 \not\leftrightarrow w_1$; assume the former. Since u_1 and w_4 share a 5-neighbor (and are non-adjacent), they have two common neighbors. Now if $u_6 \not\leftrightarrow w_4$, then we apply Lemma 19 to $\{u_1, u_6, w_4\}$. Hence, assume $u_6 \leftrightarrow w_4$. This implies that $u_4 \not\leftrightarrow w_1$. Now, the same argument implies that $u_6 \leftrightarrow w_1$. Now let $J = \{u_1, u_4, u_6\}$. Lemma 6 gives $12 \leq |N(J)| \leq 6 + 6 + 6 - 6 = 12$. Thus the vertices of J have no additional pairwise common neighbors. Hence, we have an independent 2-set M_1 in $N(u_1) \setminus (N(u_4) \cup N(u_6))$. Similarly, we have an independent 2-set M_4 in $N(u_4) \setminus (N(u_1) \cup N(u_6))$. Now we apply Lemma 10 with $J = \{u_1, u_4, u_6\}$ and $S_1 = M_1 \cup \{u_1\}$ and $S_2 = M_4 \cup \{u_4\}$. In each case, we have $\alpha(G[S_i \cup J]) \geq |M_i \cup \{u_{5-i}, u_6\}| = 4$. This implies that $|N(J)| \geq 13$, a contradiction. Hence, v cannot have four successive 6-neighbors, each with a 5-neighbor.

Case 2: Vertices u_1, u_2, u_3, u_5, u_6 are 6-vertices, each with a 5-neighbor.

Suppose that the common neighbor w_5 of u_5 and u_6 is a 5-vertex. By symmetry (between u_1 and u_3) and Lemma 17, assume that the common neighbor w_2 of u_2 and u_3 is a 5-vertex. If $w_2 \not\leftrightarrow w_5$, then we apply Lemma 22; so assume that $w_2 \leftrightarrow w_5$. If $u_6 \not\leftrightarrow w_2$, then apply Lemma 19 to $\{u_6, u_1, w_2\}$; note that u_6 and w_2 have two common neighbors, since they have a common 5-neighbor. So assume that $u_6 \leftrightarrow w_2$. Similarly, we assume that $u_3 \leftrightarrow w_5$, since otherwise we apply Lemma 19 to $\{u_3, u_1, w_5\}$. Now consider the 5-neighbor w_1 of u_1. By Lemma 9, it cannot be a common neighbor of u_2 (because of w_2). If it is a common neighbor of u_7, then we apply Lemma 22 to $\{w_1, w_5, v\}$; note that $w_1 \not\leftrightarrow w_5$, since they are cut off by edge w_2u_6. Hence, w_1 is neither a common neighbor of u_7 nor of u_2. Now we apply Lemma 19 to $\{u_2, w_1, w_5\}$. Thus, we conclude that the common neighbor of u_5 and u_6 is not a 5-vertex.

Let x denote the common neighbor of u_5 and u_6; as shown in the previous paragraph, x must be a 6^+-vertex. Suppose that the 5-neighbor w_5 of u_5 is also a neighbor of x. If

$w_5 \not\leftrightarrow u_1$, then we apply Lemma 19 to $\{u_6, u_1, w_5\}$; so assume that $w_5 \leftrightarrow u_1$. Now if the 5-neighbor w_6 of u_6 is also adjacent to x, then we apply Lemma 19 to $\{u_5, w_6, u_3\}$; we must have $w_6 \not\leftrightarrow u_3$ due to edge w_5u_1. So, by symmetry (between u_5 and u_6), we may assume that $w_5 \leftrightarrow u_4$. Now, by Lemma 17, the 5-neighbor w_2 of u_2 has a common neighbor with either u_1 or u_3. In either case, we must have $w_2 \leftrightarrow w_5$; otherwise, we apply Lemma 22 to $\{v, w_2, w_5\}$. If $w_6 \leftrightarrow u_7$, then $w_6 \leftrightarrow w_2$ and $w_6 \leftrightarrow w_5$; otherwise, we apply Lemma 22 to $\{v, w_6, w_2\}$ or $\{v, w_6, w_5\}$. Now we apply Lemma 19 to $\{u_5, u_3, w_6\}$. So instead $w_6 \not\leftrightarrow u_7$. Finally, we apply Lemma 19 to $\{u_5, w_6, u_3\}$. This completes the proof. $\square$

Acknowledgments

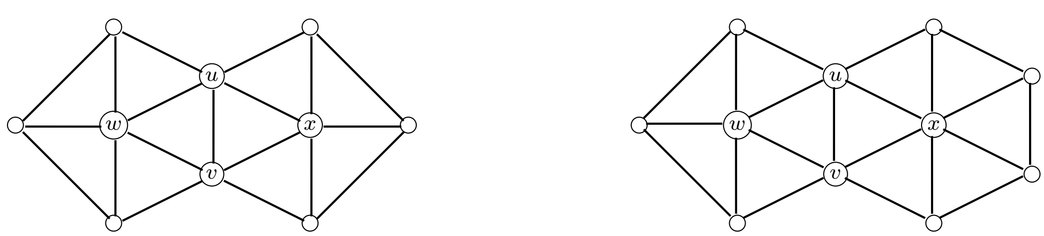
As we mentioned in the introduction, the ideas in this paper come largely from Albertson's proof [1] that planar graphs have independence ratio at least $\frac{2}{9}$. In fact, many of the reducible configurations that we use here are special cases of the reducible configurations in that proof. We very much like that paper, and so it was a pleasure to be able to extend Albertson's work. It seems that the part of his own proof that Albertson was least pleased with was verifying "unavoidability", i.e., showing that every planar graph contains a reducible configuration. In the introduction to [1], he wrote: "Finally Section 4 is devoted to a massive, ugly edge counting which demonstrates that every triangulation of the plane must contain some forbidden subgraph." In the appendix that follows, we give a short proof of this same unavoidability statement, via discharging. We think Mike might have liked it.

The first author thanks his Lord and Savior, Jesus Christ.

Appendix

Here we give a short discharging proof that every planar triangulation with minimum degree 5 and no separating 3-cycle must contain a certain configuration, which Albertson showed could not appear in a minimal planar graph with independence ratio less than $\frac{2}{9}$. (In fact, finding this proof helped encourage us to begin work on the present paper.)

Lemma A. *Let u and v be adjacent vertices, such that uvw and uvx are 3-faces and $d(w) = 5$ and $d(x) \leq 6$; call this configuration H. If G is a plane triangulation with minimum degree 5 and no separating 3-cycle, then G contains a copy of H.*



(A) Adjacent vertices u and v, with nonadjacent common 5-neighbors w and x.

(B) Adjacent vertices u and v, with nonadjacent common neighbors w and x, of degree 5 and 6.

FIGURE 7. The two instances of configuration H.

Proof. Assume that G has minimum degree 5 and no separating 3-cycle, but also has no copy of H. This assumption leads to a contradiction, which implies the result. An immediate consequence of this assumption (by Pigeonhole) is that the number of 5-neighbors of each vertex v is at most $\frac{d(v)}{2}$. Below, when we verify that each vertex finishes with nonnegative charge, we consider both the degree of v and its number of 5-neighbors. We write (a, b)*-vertex* to denote a vertex of degree a that has b 5-neighbors.

We assign to each vertex v a charge $\mathrm{ch}(v)$, where $\mathrm{ch}(v) = d(v)-6$. Note that $\sum_{v\in V} \mathrm{ch}(v) = 2|E(G)| - 6|V(G)|$. Since G is a plane triangulation, Euler's formula implies that $2|E(G)| - 6|V(G)| = -12$. Now we redistribute the charge, without changing the sum, so that each vertex finishes with nonnegative charge. This redistribution is called *discharging*, and we write $\mathrm{ch}^*(v)$ to denote the charge at each vertex v after discharging. Since each vertex finishes with nonnegative charge, we get the obvious contradiction $-12 = \sum_{v\in V} \mathrm{ch}(v) = \sum_{v\in V} \mathrm{ch}^*(v) \geq 0$. We redistribute the charge via the following three discharging rules, which we apply simultaneously everywhere they are applicable.

(R1) Each 7^+-vertex gives charge $\frac{1}{3}$ to each 5-neighbor.
(R2) Each 7^+-vertex gives charge $\frac{1}{7}$ to each 6-neighbor that has at least one 5-neighbor.
(R3) Each 6-vertex gives charge $\frac{2}{7}$ to each 5-neighbor.

We now verify that after discharging, each vertex v has nonnegative charge.

$\mathbf{d(v) = 5}$: Note that each $(5, 0)$-vertex has five 6^+-neighbors; each $(5, 1)$-vertex has four 6^+-neighbors, at least two of which are 7^+-neighbors; and each $(5, 2)$-vertex has three 7^+-neighbors. Thus, if v is a $(5, 0)$-vertex: $\mathrm{ch}^*(v) \geq -1 + 5\left(\frac{2}{7}\right) > 0$; if v is a $(5, 1)$-vertex: $\mathrm{ch}^*(v) \geq -1 + 2\left(\frac{1}{3}\right) + 2\left(\frac{2}{7}\right) > 0$; and if v is a $(5, 2)$-vertex: $\mathrm{ch}^*(v) = -1 + 3\left(\frac{1}{3}\right) = 0$;

$\mathbf{d(v) = 6}$: Note that each $(6, 1)$-vertex has at least two 7^+-neighbors; and each $(6, 2)$-vertex has four 7^+-neighbors. Thus, if v is a $(6, 0)$-vertex: $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 0$; if v is a $(6, 1)$-vertex: $\mathrm{ch}^*(v) \geq 0 + 2\left(\frac{1}{7}\right) - \left(\frac{2}{7}\right) = 0$; and if v is a $(6, 2)$-vertex: $\mathrm{ch}^*(v) = 0 + 4\left(\frac{1}{7}\right) - 2\left(\frac{2}{7}\right) = 0$.

$\mathbf{d(v) = 7}$: Note that each $(7, 1)$-vertex has six 6^+-neighbors, at least two of which are 7^+-vertices; each $(7, 2)$-vertex has five 6^+-neighbors, at least three of which are 7^+-vertices; and each $(7, 3)$-vertex has four 7^+-neighbors. Thus, if v is a $(7, 0)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 7\left(\frac{1}{7}\right) = 0$; if v is a $(7, 1)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 1\left(\frac{1}{3}\right) - 4\left(\frac{1}{7}\right) > 0$; if v is a $(7, 2)$-vertex, then $\mathrm{ch}^*(v) \geq 1 - 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{7}\right) > 0$; and if v is a $(7, 3)$-vertex, then $\mathrm{ch}^*(v) = 1 - 3\left(\frac{1}{3}\right) = 0$.

$\mathbf{d(v) = 8}$: v has at most four 5-neighbors, and gives each of these charge $\frac{1}{3}$; v gives each other neighbor charge at most $\frac{1}{7}$. Thus $\mathrm{ch}^*(v) \geq 8 - 6 - 4(\frac{1}{3}) - 4(\frac{1}{7}) > 0$.

$\mathbf{d(v) \geq 9}$: v gives each neighbor charge at most $\frac{1}{3}$, so $\mathrm{ch}^*(v) \geq d(v) - 6 - d(v)(\frac{1}{3}) = \frac{2}{3}(d(v) - 9) \geq 0$.

Thus $-12 = \sum_{v\in V} \mathrm{ch}(v) = \sum_{v\in V} \mathrm{ch}^*(v) \geq 0$. This contradiction implies the result. □

References

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[2] C. Berge. *Graphes et hypergraphes.* Dunod, Paris, 1970. Monographies Universitaires de Mathématiques, No. 37.