A NOTE ON VERTEX PARTITIONS

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Abstract. We prove a general lemma about partitioning the vertex set of a graph into subgraphs of bounded degree. This lemma extends a sequence of results of Lovász, Catlin, Kostochka and Rabern.

1. Introduction

In the 1960's Lovász [4] proved the following decomposition lemma for graphs by considering a partition minimizing a certain function.

Lovász's Decomposition Lemma. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.*

A decade later, Catlin [1] showed that bumping the $\Delta(G) + 1$ to $\Delta(G) + 2$ allowed for shuffling vertices from one partition set to another and thereby proving stronger decomposition results. A few years later Kostochka [3] modified Catlin's algorithm to show that every triangle-free graph G can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. Around the same time, Mozhan [5] used a different, but related, function minimization and vertex shuffling procedure to prove coloring results. In [6], we generalized Kostochka's modification to prove the following.

Lemma 1. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete r_i-regular components for each $i \in [k]$.*

In fact, we proved a stronger lemma allowing us to forbid a larger class of components coming from any so-called *r-permissible collection.* The purpose of this note is to simplify and generalize this latter result. The definition of an r-height function will be given in the following section.

Main Lemma. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 2 - k$. If h_i is an r_i-height function for each $i \in [k]$, then $V(G)$ can be partitioned into sets $V_1, \dots, V_k$ such that for each $i \in [k]$, $\Delta(G[V_i]) \leq r_i$ and $h_i(D) = 0$ for each component D of $G[V_i]$.*

2. The proof

Our notation follows Diestel [2] unless otherwise specified. The natural numbers include zero; that is, $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. We also use the shorthand $[k] := \{1, 2, \dots, k\}$. Let $\mathcal{G}$ be the collection of all finite simple connected graphs.

Definition 1. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a vertex $x \in V(G)$ is called *h-critical* in G if $G - x \in \mathcal{G}$ and $h(G-x) < h(G)$.

Definition 2. For $h\colon \mathcal{G} \to \mathbb{N}$ and $G \in \mathcal{G}$, a pair of vertices $\{x, y\} \subseteq V(G)$ is called an *h-critical pair* in G if $G - \{x, y\} \in \mathcal{G}$ and x is h-critical in $G - y$ and y is h-critical in $G - x$.

Definition 3. For $r \in \mathbb{N}$ a function $h\colon \mathcal{G} \to \mathbb{N}$ is called an *r-height function* if it has each of the following properties:

1. if $h(G) > 0$, then G contains an h-critical vertex x with $d(x) \geq r$;
2. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then $h(G - x) = h(G) - 1$;
3. if $G \in \mathcal{G}$ and $x \in V(G)$ is h-critical with $d(x) \geq r$, then G contains an h-critical vertex $y \notin \{x\} \cup N(x)$ with $d(y) \geq r$;
4. if $G \in \mathcal{G}$ and $\{x, y\} \subseteq V(G)$ is an h-critical pair in G with $d_{G-y}(x) \geq r$ and $d_{G-x}(y) \geq r$, then there exists $z \in N(x) \cap N(y)$ with $d(z) \geq r + 1$.

For $r \geq 2$, the function $h\colon \mathcal{G} \to \mathbb{N}$ which gives 1 for all non-complete r-regular graphs and 0 for everything else is an r-height function. Applying the Main Lemma using this height function proves Lemma 1.

The proof of the Main Lemma uses ideas similar to those in [3] and [6]. For a graph G, $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$. Let $\mathcal{C}(G)$ be the components of G and $c(G) := |\mathcal{C}(G)|$. If $h\colon \mathcal{G} \to \mathbb{N}$, we define h for any graph as $h(G) := \sum_{D \in \mathcal{C}(G)} h(D)$.

Proof of Main Lemma. For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(\|G[V_i]\| - r_i\,|V_i|\right),$$

$$c(P) := \sum_{i=1}^{k} c(G[V_i]),$$

$$h(P) := \sum_{i=1}^{k} h_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$, and subject to that $c(P)$, and subject to that $h(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$ there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_{V_i}(x) > r_i$ we would have $f(P^*) < f(P)$ contradicting the minimality of P. This proves that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.

Now suppose that for some i_1 there is a component A_1 of $G[V_{i_1}]$ with $h_{i_1}(A_1) > 0$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. By property 1 of height functions, we have an h_{i_1}-critical vertex $x_1 \in V(A_1)$ with $d_{A_1}(x_1) \geq r_{i_1}$. By the above we have $i_2 \neq i_1$ such that moving x_1 from V_{1,i_1} to V_{1,i_2} gives a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{2,i_2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since x_1 is h_{i_1}-critical, by the minimality of $h(P_1)$, it must be that

$h_{i_2}(A_2) > h_{i_2}(C_2)$. By property 2 of height functions we must have $h_{i_2}(A_2) = h_{i_2}(C_2) + 1$. Hence $h(P_2)$ is still minimum. Now, by property 3 of height functions, we have an h_{i_2}-critical vertex $x_2 \in V(A_2) - (\{x_1\} \cup N_{A_2}(x_1))$ with $d_{A_2}(x_2) \geq r_{i_2}$.

Continue on this way to construct sequences $i_1, i_2, \ldots, A_1, A_2, \ldots, P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. In particular, $\{x_s, x_{t+1}\}$ is an h_{i_s}-critical pair in $Q := G[\{x_{t+1}\} \cup V(A_s)]$ where $d_{Q-x_{t+1}}(x_s) \geq r_{i_s}$ and $d_{Q-x_s}(x_{t+1}) \geq r_{i_s}$. Thus, by property 4 of height functions, we have $z \in N_Q(x_s) \cap N_Q(x_{t+1})$ with $d_Q(z) \geq r_{i_s} + 1$.

We now modify P_s to contradict the minimality of $f(P)$. At step $t+1$, x_t was adjacent to exactly r_{i_s} vertices in V_{t+1,i_s}. This is what allowed us to move x_t into V_{t+1,i_s}. Our goal is to modify P_s so that we can move x_t into the i_s part without moving x_s out. Since z is adjacent to both x_s and x_t, moving z out of the i_s part will then give us our desired contradiction.

So, consider the set X of vertices that could have been moved out of V_{s,i_s} between step s and step $t+1$; that is, $X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,i_s}$. For $x_j \in X$, since $d_{A_j}(x_j) \geq r_{i_s}$ and x_j is not adjacent to x_{j-1} we see that $d_{V_{s,i_s}}(x_j) \geq r_{i_s}$. Similarly, $d_{V_{s,i_t}}(x_t) \geq r_{i_t}$. Also, by the minimality of t, X is an independent set in G. Thus we may move all elements of X out of V_{s,i_s} to get a new partition $P^* := (V_{*,1}, \ldots, V_{*,k})$ with $f(P^*) = f(P)$.

Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and $t+1$ are the elements of X, we see that $d_{V_{*,i_s}}(x_t) = r_{i_s}$. Since $d_{V_{*,i_t}}(x_t) \geq r_{i_t}$ we can move x_t from V_{*,i_t} to V_{*,i_s} to get a new partition $P^{**} := (V_{**,1}, \ldots, V_{**,k})$ with $f(P^{**}) = f(P^*)$. Now, recall that $z \in V_{**,i_s}$. Since z is adjacent to x_t we have $d_{V_{**,i_s}}(z) \geq r_{i_s} + 1$. Thus we may move z out of V_{**,i_s} to get a new partition P^{***} with $f(P^{***}) < f(P^{**}) = f(P)$. This contradicts the minimality of $f(P)$. □

References

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