A NOTE ON VERTEX PARTITIONS

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Abstract. We prove a general lemma about partitioning the vertex set of a graph into subgraphs of bounded degree. This lemma extends a sequence of results of Lovász, Catlin, Kostochka and Rabern.

1. Introduction

In the 1960’s Lovász [4] proved the following decomposition lemma for graphs by considering a partition minimizing a certain function.

Lovász’s Decomposition Lemma. Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.

A decade later, Catlin [1] showed that bumping the $\Delta(G) + 1$ to $\Delta(G) + 2$ allowed for shuffling vertices from one partition set to another and thereby proving stronger decomposition results. A few years later Kostochka [3] modified Catlin’s algorithm to show that every triangle-free graph $G$ can be colored with at most $\frac{2}{3}\Delta(G) + 2$ colors. Around the same time, Mozhan [5] used a different, but related, function minimization and vertex shuffling procedure to prove coloring results. In [6], we generalized Kostochka’s modification to prove the following.

Lemma 1. Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete $r_i$-regular components for each $i \in [k]$.

In fact, we proved a stronger lemma allowing us to forbid a larger class of components coming from any so-called $r$-permissible collection. The purpose of this note is to simplify and generalize this latter result. The definition of an $r$-height function will be given in the following section.

Main Lemma. Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. If $h_i$ is an $r_i$-height function for each $i \in [k]$, then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that for each $i \in [k]$, $\Delta(G[V_i]) \leq r_i$ and $h_i(D) = 0$ for each component $D$ of $G[V_i]$.

2. The proof

Our notation follows Diestel [2] unless otherwise specified. The natural numbers include zero; that is, $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$. We also use the shorthand $[k] := \{1, 2, \ldots, k\}$. Let $G$ be the collection of all finite simple connected graphs.
Definition 1. For \( h : G \to \mathbb{N} \) and \( G \in \mathcal{G} \), a vertex \( x \in V(G) \) is called \( h \)-critical in \( G \) if \( G - x \in \mathcal{G} \) and \( h(G - x) < h(G) \).

Definition 2. For \( h : G \to \mathbb{N} \) and \( G \in \mathcal{G} \), a pair of vertices \( \{x, y\} \subseteq V(G) \) is called an \( h \)-critical pair in \( G \) if \( G - \{x, y\} \in \mathcal{G} \) and \( x \) is \( h \)-critical in \( G - y \) and \( y \) is \( h \)-critical in \( G - x \).

Definition 3. For \( r \in \mathbb{N} \) a function \( h : G \to \mathbb{N} \) is called an \( r \)-height function if it has each of the following properties:

1. if \( h(G) > 0 \), then \( G \) contains an \( h \)-critical vertex \( x \) with \( d(x) \geq r \);
2. if \( G \in \mathcal{G} \) and \( x \in V(G) \) is \( h \)-critical with \( d(x) \geq r \), then \( h(G - x) = h(G) - 1 \);
3. if \( G \in \mathcal{G} \) and \( x \in V(G) \) is \( h \)-critical with \( d(x) \geq r \), then \( G \) contains an \( h \)-critical vertex \( y \notin \{x\} \cup N(x) \) with \( d(y) \geq r \);
4. if \( G \in \mathcal{G} \) and \( \{x, y\} \subseteq V(G) \) is an \( h \)-critical pair in \( G \) with \( d_{G - y}(x) \geq r \) and \( d_{G - x}(y) \geq r \), then there exists \( z \in N(x) \cap N(y) \) with \( d(z) \geq r + 1 \).

For \( r \geq 2 \), the function \( h : G \to \mathbb{N} \) which gives 1 for all non-complete \( r \)-regular graphs and 0 for everything else is an \( r \)-height function. Applying the Main Lemma using this height function proves Lemma 1.

The proof of the Main Lemma uses ideas similar to those in [3] and [6]. For a graph \( G \), \( x \in V(G) \) and \( D \subseteq V(G) \) we use the notation \( N_D(x) := N(x) \cap D \) and \( d_D(x) := |N_D(x)| \). Let \( \mathcal{C}(G) \) be the components of \( G \) and \( c(G) := |\mathcal{C}(G)| \). If \( h : G \to \mathbb{N} \), we define \( h \) for any graph as \( h(G) := \sum_{D \in \mathcal{C}(G)} h(D) \).

Proof of Main Lemma. For a partition \( P := (V_1, \ldots, V_k) \) of \( V(G) \) let

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\begin{align*}
f(P) &:= \sum_{i=1}^k (|G[V_i]| - r_i |V_i|), \\
c(P) &:= \sum_{i=1}^k c(G[V_i]), \\
h(P) &:= \sum_{i=1}^k h_i(G[V_i]).
\end{align*}
\]

Let \( P := (V_1, \ldots, V_k) \) be a partition of \( V(G) \) minimizing \( f(P) \), and subject to that \( c(P) \), and subject to that \( h(P) \).

Let \( i \in [k] \) and \( x \in V_i \) with \( d_{V_i}(x) \geq r_i \). Since \( \sum_{i=1}^k r_i \geq \Delta(G) + 2 - k \) there is some \( j \neq i \) such that \( d_{V_j}(x) \leq r_j \). Moving \( x \) from \( V_i \) to \( V_j \) gives a new partition \( P^* \) with \( f(P^*) \leq f(P) \). Note that if \( d_{V_i}(x) > r_i \) we would have \( f(P^*) < f(P) \) contradicting the minimality of \( P \). This proves that \( \Delta(G[V_i]) \leq r_i \) for each \( i \in [k] \).

Now suppose that for some \( i_1 \) there is a component \( A_i \) of \( G[V_{i_1}] \) with \( h_{i_1}(A_i) > 0 \). Put \( P_1 := P \) and \( V_{i_1} := V_i \) for \( i \in [k] \). By property 1 of height functions, we have an \( h_{i_1} \)-critical vertex \( x_1 \in V(A_1) \) with \( d_{A_1}(x_1) \geq r_{i_1} \). By the above we have \( i_2 \neq i_1 \) such that moving \( x_1 \) from \( V_{i_1} \) to \( V_{i_2} \) gives a new partition \( P_2 := (V_{i_2}, V_{i_2}, \ldots, V_{i_2}) \) where \( f(P_2) = f(P_1) \). By the minimality of \( c(P_1) \), \( x_1 \) is adjacent to only one component \( C_2 \) in \( G[V_{i_2}] \). Let \( A_2 := G[V(C_2) \cup \{x_1\}] \). Since \( x_1 \) is \( h_{i_1} \)-critical, by the minimality of \( h(P_1) \), it must be that
h_{i_2}(A_2) > h_{i_2}(C_2). By property 2 of height functions we must have h_{i_2}(A_2) = h_{i_2}(C_2) + 1. Hence h(P_2) is still minimum. Now, by property 3 of height functions, we have an h_{i_2}-critical vertex x_2 ∈ V(A_2) - (\{x_1\} \cup N_{A_2}(x_1)) with d_{A_2}(x_2) ≥ r_{i_2}.

Continue on this way to construct sequences r_{i_2}, . . . , i_1, A_1, A_2, . . . , P_1, P_2, P_3, . . . and x_1, x_2, . . . . Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that A_{t+1} - x_t = A_s - x_s for some s < t. In particular, \{x_s, x_{t+1}\} is an h_{i_s}-critical pair in Q := G[\{x_{t+1}\} \cup V(A_s)] where d_{Q-x_{t+1}}(x_s) ≥ r_{i_s} and d_{Q-x_s}(x_{t+1}) ≥ r_{i_s}.

Thus, by property 4 of height functions, we have z ∈ N_Q(x_s) \cap N_Q(x_{t+1}) with d_Q(z) ≥ r_{i_s} + 1.

We now modify P_s to contradict the minimality of f(P). At step t+1, x_t was adjacent to exactly r_{i_s} vertices in V_{t+1,i_s}. This is what allowed us to move x_t into V_{t+1,i_s}. Our goal is to modify P_s so that we can move x_t into the i_s part without moving x_s out. Since z is adjacent to both x_s and x_t, moving z out of the i_s part will then give us our desired contradiction.

So, consider the set X of vertices that could have been moved out of V_{s,i_s} between step s and step t+1; that is, X := \{x_{s+1}, x_{s+2}, . . . , x_{t-1}\} \cap V_{s,i_s}. For x_j ∈ X, since d_{A_j}(x_j) ≥ r_{i_s} and x_j is not adjacent to x_{j-1} we see that d_{V_{s,i_s}}(x_j) ≥ r_{i_s}. Similarly, d_{V_{s,t}}(x_t) ≥ r_{i_t}. Also, by the minimality of t, X is an independent set in G. Thus we may move all elements of X out of V_{s,i_s} to get a new partition P^* := (V_{s,1}, . . . , V_{s,k}) with f(P^*) = f(P).

Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and t+1 are the elements of X, we see that d_{V_{t+1,i_s}}(x_t) = r_{i_s}. Since d_{V_{s,t}}(x_t) ≥ r_{i_t} we can move x_t from V_{s,i_t} to V_{t+1,i_s} to get a new partition P^{**} := (V_{s,1}, . . . , V_{s,k}, V_{t+1,i_s}) with f(P^{**}) = f(P^*). Now, recall that z ∈ V_{s,i_s}. Since z is adjacent to x_t we have d_{V_{s,i_s}}(z) ≥ r_{i_s} + 1. Thus we may move z out of V_{s,i_s} to get a new partition P^{***} with f(P^{***}) < f(P^{**}) = f(P). This contradicts the minimality of f(P).

References