Coloring graphs from almost maximum degree sized palettes

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INTRODUCTION

Graph coloring is instantiated by many partitioning problems that arise in practice. Any time we encounter a relation between things of some sort and wish to group those things so that no group contains a related pair, we have to solve a graph coloring problem. On some such encounters, the possible relations between our sorted things are restricted in a way that allows a small number of groups to be used. This is good. If the sort in question were a sort of tasks to be performed and the relation were can’t be performed at the same time, then such a grouping, taken in some order, gives an order in which to perform the tasks. Fewer groups means faster task completion. How many groups are needed for a certain sort of task collection? How can we find a grouping with the minimum number of groups? If we can’t find such a grouping, how can we at least determine the number of groups in a minimum grouping? This dissertation is primarily concerned with the first type of question—but an answer to the first type of question can have bearing on the other questions when coupled with a method of sorting. When we move from the instantiation to abstract graph coloring we call the things vertices and represent the relation of two vertices by an edge between them. The vertices together with the edges constitute a graph and one of our groupings corresponds to a labeling of the vertices with 1, 2, . . . , k so that there are no edges between vertices receiving the same label. Such a labeling is called a coloring or more precisely a k-coloring. For a graph G we write χ(G) for the minimum k for which G has a k-coloring—this corresponds to the number of groups in a minimum grouping. This study concerns the relation of χ to other graph parameters. Our terminology and notation are basically standard (for all notation, see Appendix A).

1.1 A short history

Here we collect statements of the results and conjectures that have bearing on this inquiry woven together with some historical remarks and our improvements. The first non-trivial result about coloring graphs with around $\Delta$ colors is Brooks’ theorem from 1941.

**Theorem 1.1.1** (Brooks [12]). Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings. Counterexamples exist showing that the $\Delta \geq 9$ condition is tight (see Figure 1.1).

**Conjecture 1.1.2** (Borodin and Kostochka [10]). Every graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$.

Note that another way of stating this is that for $\Delta \geq 9$, the only obstruction to $(\Delta - 1)$-coloring is a $K_\Delta$. In the same paper, Borodin and Kostochka prove the following weaker statement.
CHAPTER 1. INTRODUCTION

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(a) $\Delta = 6$

(b) $\Delta = 7$

(c) $\Delta = 7$

(d) $\Delta = 8$

Figure 1.1: Counterexamples to the Borodin-Kostochka Conjecture for small $\Delta$.

Theorem 1.1.3 (Borodin and Kostochka [10]). Every graph satisfying $\chi \geq \Delta \geq 7$ contains a $K_{\lfloor \Delta+1 \rfloor}$.

The proof is quite simple once you have a decomposition lemma of Lovász from the 1960’s [51].

Lemma 1.1.4 (Lovász [51]). Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$.

Proof. For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} (\|G[V_i]\| - r_i |V_i|).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$. Suppose there is $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) > r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$, there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$ and thus moving $x$ from $V_i$ to $V_j$ gives a new partition violating minimality of $f(P)$. Hence $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. □

Now to prove Borodin and Kostochka’s result, let $G$ be a graph with $\chi \geq \Delta \geq 7$ and use $r_1 := \lfloor \Delta + 1 \rfloor$ and $r_2 := \lfloor \Delta + 1 \rfloor$ in Lovász’s lemma to get a partition $(V_1, V_2)$ of $V(G)$ with $\Delta(G[V_i]) \leq r_i$ for each $i \in [2]$. Since $r_1 + r_2 = \Delta - 1$ and $\chi \geq \Delta$, it must be that $\chi(G[V_i]) \geq r_i + 1$ for some $i \in [2]$. But $\Delta \geq 7$, so $r_i \geq 3$ and hence by Brooks’ theorem $G[V_i]$ contains a $K_{\lfloor \Delta+1 \rfloor}$.

A decade later, Catlin [14] showed that bumping the $\Delta(G) + 1$ to $\Delta(G) + 2$ allowed for shuffling vertices from one partition set to another and thereby proving stronger decomposition results. A few years later Kostochka [47] modified Catlin’s algorithm to show that every triangle-free graph $G$ can be colored with at most $\frac{2}{3} \Delta(G) + 2$ colors. In [64], we generalized Kostochka’s modification to prove the following.

Lemma 3.2.1 (Rabern [64]). Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no incomplete $r_i$-regular components for each $i \in [k]$.
Setting \( k = \left\lceil \frac{\Delta(G) + 2}{3} \right\rceil \) and \( r_i = 2 \) for each \( i \) gives a slightly more general form of Kostochka’s triangle-free coloring result.

**Corollary 3.2.2** (Rabern [64]). The vertex set of any graph \( G \) can be partitioned into \( \left\lceil \frac{\Delta(G) + 2}{3} \right\rceil \) sets, each of which induces a disjoint union of triangles and paths.

For coloring, this actually gives the bound \( \chi(G) \leq 2 \left\lceil \frac{\Delta(G) + 2}{3} \right\rceil \) for triangle free graphs.

To get \( \frac{3}{2} \Delta(G) + 2 \), just use \( r_k = 0 \) when \( \Delta \equiv 2 \pmod{3} \). Similarly, for any \( r \geq 2 \), setting \( k = \left\lceil \frac{\Delta(G) + 2}{r + 1} \right\rceil \) and \( r_i = r \) for each \( i \) gives the following.

**Corollary 3.2.3** (Rabern [64]). Fix \( r \geq 2 \). The vertex set of any \( K_{r+1} \)-free graph \( G \) can be partitioned into \( \left\lceil \frac{\Delta(G) + 2}{r + 1} \right\rceil \) sets each inducing an \((r-1)\)-degenerate subgraph with maximum degree at most \( r \).

In fact, we proved a lemma stronger than Lemma 3.2.1 allowing us to forbid a larger class of components coming from any so-called \( r \)-permissible collection. In section 3.2 we will explore a result that both simplifies and generalizes this latter result.

Also in the 1980’s, Kostochka proved the following using a complicated recoloring argument together with a technique for reducing \( \Delta \) in a counterexample based on hitting every maximum clique with an independent set.

**Theorem 1.1.5** (Kostochka [46]). Every graph satisfying \( \chi \geq \Delta \) contains a \( K_{\Delta-28} \).

Kostochka [46] proved the following result which shows that graphs having clique number sufficiently close to their maximum degree contain an independent set hitting every maximum clique. In [62] we improved the antecedent to \( \omega \geq \frac{3}{4}(\Delta + 1) \). Finally, King [42] made the result tight.

**Lemma 4.1.1** (Kostochka [46]). If \( G \) is a graph satisfying \( \omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta} \), then \( G \) contains an independent set \( I \) such that \( \omega(G - I) < \omega(G) \).

**Lemma 4.1.3** (Rabern [62]). If \( G \) is a graph satisfying \( \omega \geq \frac{3}{4}(\Delta + 1) \), then \( G \) contains an independent set \( I \) such that \( \omega(G - I) < \omega(G) \).

**Lemma 4.1.5** (King [42]). If \( G \) is a graph satisfying \( \omega > \frac{3}{4}(\Delta + 1) \), then \( G \) contains an independent set \( I \) such that \( \omega(G - I) < \omega(G) \).

If \( G \) is a vertex critical graph satisfying \( \omega > \frac{3}{4}(\Delta + 1) \) and we expand the independent set \( I \) produced by Lemma 4.1.5 to a maximal independent set \( M \) and remove \( M \) from \( G \), we see that \( \Delta(G - M) \leq \Delta(G) - 1 \). Using this, the proof of many coloring results can be reduced to the case of the smallest \( \Delta \) for which they work. In Chapter 4, we give three such applications.

A little after Kostochka proved his bound, Mozhan [55] used a function minimization and vertex shuffling procedure different than, but related to Catlin’s, to prove the following.

**Theorem 1.1.6** (Mozhan [55]). Every graph satisfying \( \chi \geq \Delta \geq 10 \) contains a \( K_{\left\lceil \frac{28}{\Delta} + 1 \right\rceil} \).

Finally, in his dissertation Mozhan proved the following. We don’t know the method of proof as we were unable to obtain a copy of his dissertation. However, we suspect the method is a more complicated version of the above proof.

**Theorem 1.1.7** (Mozhan). Every graph satisfying \( \chi \geq \Delta \geq 31 \) contains a \( K_{\Delta-3} \).

In [63], we used part of Mozhan’s method to prove the following result. For a graph \( G \) let \( \mathcal{H}(G) \) be the subgraph of \( G \) induced on the vertices of degree at least \( \chi(G) \).
Theorem 1.1.8 (Rabern [63]). \( K_{\chi(G)} \) is the only vertex critical graph \( G \) with \( \chi(G) \geq \Delta(G) \geq 6 \) and \( \omega(H(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2 \).

Setting \( \omega(H(G)) = 1 \) proved a conjecture of Kierstead and Kostochka [38].

Corollary 1.1.9 (Rabern [63]). \( K_{\chi(G)} \) is the only vertex critical graph \( G \) with \( \chi(G) \geq \Delta(G) \geq 6 \) such that \( H(G) \) is edgeless.

In joint work with Kostochka and Stiebitz [48], we generalized and improved this result, again using Mozhan’s technique. In section 3.1, we will improve these results further and simplify the proofs by using Catlin’s vertex shuffling algorithm in place of Mozhan’s.

In 1999, Reed used probabilistic methods to prove that the Borodin-Kostochka conjecture holds for graphs with very large maximum degree.

Theorem 1.1.10 (Reed [67]). Every graph satisfying \( \chi \geq \Delta \geq 10^{14} \) contains a \( K_\Delta \).

A lemma from Reed’s proof of the above theorem is generally useful.

Lemma 1.1.11 (Reed [67]). Let \( G \) be a critical graph satisfying \( \chi = \Delta \geq 9 \) having the minimum number of vertices. If \( H \) is a \( K_{\Delta-1} \) in \( G \), then any vertex in \( G - H \) has at most 4 neighbors in \( H \). In particular, the \( K_{\Delta-1} \)'s in \( G \) are pairwise disjoint.

In Chapter 7, we improve this lemma by showing that under the same hypotheses, any vertex in \( G - H \) has at most 1 neighbor in \( H \). Moreover, we lift the result out of the context of a minimal counterexample to graphs satisfying a certain criticality condition—we refer to such graphs as mules. This allows meaningful results to be proved for values of \( \Delta \) less than 9. Also in Chapter 7, we prove that the following, prima facie weaker, conjecture is equivalent to the Borodin-Kostochka conjecture.

Conjecture 1.1.12 (Cranston and Rabern [20]). If \( G \) is a graph with \( \chi = \Delta = 9 \), then \( K_3 \ast K_6 \subseteq G \).

At the core of these results are the list coloring lemmas proved in section 5. There we classify graphs of the form \( A \ast B \) that are not \( f \)-choosable where \( f(v) := d(v) - 1 \) for each vertex \( v \). In Chapter 6 we use these list coloring results together with Chudnovsky and Seymour’s decomposition theorem for claw-free graphs [18] and our proof in [60] of the Borodin-Kostochka conjecture for line graphs of multigraphs to prove the conjecture for claw-free graphs.

Theorem 1.1.13 (Cranston and Rabern [19]). Every claw-free graph with \( \Delta \geq 9 \) satisfies \( \chi \leq \max\{\omega, \Delta - 1\} \).

In Chapter 8, we adapt a recoloring trick previously used for strong coloring and prove the following.

Corollary 8.4.2. Every graph with \( \chi \geq \Delta \geq 9 \) such that every vertex is in a clique on \( \frac{2}{3} \Delta + 2 \) vertices contains \( K_\Delta \).

Using this we show that to prove the Borodin-Kostochka conjecture it is enough to prove it for irregular graphs; more precisely, we prove the following.

Theorem 8.5.1. Every graph satisfying \( \chi \geq \Delta = k \geq 9 \) either contains \( K_k \) or contains an irregular critical subgraph satisfying \( \chi = \Delta = k - 1 \).

In particular, the Borodin-Kostochka conjecture would follow from the following. This is at least somewhat plausible since the only known critical (or connected even) counterexample to Borodin-Kostochka for \( \Delta = 8 \) is regular (see Figure 7.4).

Conjecture 8.5.2. Every critical graph satisfying \( \chi \geq \Delta = 8 \) is regular.
As our final application of the recoloring trick, we prove the following bounds on the chromatic number. The first generalizes the result of Beutelspacher and Hering [5] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. This result was generalized in another direction in [19] (also Chapter 6) where the conjecture was proved for claw-free graphs.

\textbf{Theorem 8.7.2.} Every graph satisfies 
\[ \chi \leq \max \{ \omega, \Delta - 1, 4\alpha \} \].

The second bound shows that the Borodin-Kostochka conjecture holds for graphs with maximum degree on the order of the square root of their order. This improves on prior bounds of \( \Delta > \frac{n+1}{2} \) from Beutelspacher and Hering [5] and \( \Delta > \frac{n-6}{3} \) of Naserasr [56].

\textbf{Theorem 8.7.3.} Every graph satisfies 
\[ \chi \leq \max \{ \omega, \Delta - 1, \left\lfloor \frac{15 + \sqrt{48n + 73}}{4} \right\rfloor \} \].

Borodin and Kostochka also conjectured [45] that their conjecture holds for list coloring. In Chapter 9, we prove that this conjecture holds for large \( \Delta \).

\textbf{Conjecture 1.1.14} (Borodin and Kostochka [45]). Every graph with \( \Delta \geq 9 \) satisfies 
\[ \chi_l \leq \max\{\omega, \Delta - 1\} \].
In Chapter 7 we will rely heavily on the technique of adding edges to a proper induced subgraph of a minimum counterexample. We first learned of this technique when reading Reed’s proof of the Borodin-Kostochka conjecture for large $\Delta$ (see [67]). To introduce the idea we give a short proof of Brooks’ theorem. The proof is completely different from Lovász’s short proof in [52]. We first reduce to the cubic case and then add edges to a proper induced subgraph to get a coloring we can complete. The reduction to the cubic case is an immediate consequence of more general lemmas on hitting all maximum cliques with an independent set that we prove in Chapter 4 (see also [46], [62] and [42]). Additionally, this reduction was demonstrated by Tverberg in [73]. One interesting feature of the proof is that it doesn’t use any connectivity concepts. We’ll give two versions of the proof, the first is shorter but uses the extra idea of excluding diamonds ($K_4$ less an edge).

**Theorem 2.0.15** (Brooks [12]). Every graph satisfies $\chi \leq \max\{3, \omega, \Delta\}$.

**Proof.** Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, $G$ is $\Delta$-regular.

First, suppose $G$ is 3-regular. If $G$ contains a diamond $D$, then we may 3-color $G - D$ and easily extend the coloring to $D$ by first coloring the nonadjacent vertices in $D$ the same. So, $G$ doesn’t contain diamonds. Since $G$ is not a forest it contains an induced cycle $C$. Since $K_4 \not\subseteq G$ we have $|N(C)| \geq 2$. So, we may take different $x, y \in N(C)$ and put $H := G - C$ if $x$ is adjacent to $y$ and $H := (G - C) + xy$ otherwise. Then, $H$ doesn’t contain $K_4$ as $G$ doesn’t contain diamonds. By minimality of $|G|$, $H$ is 3-colorable. That is, we have a 3-coloring of $G - C$ where $x$ and $y$ receive different colors. We can easily extend this partial coloring to all of $G$ since each vertex of $C$ has a set of two available colors and some pair of vertices in $C$ get different sets.

Hence we must have $\Delta \geq 4$. Consider a $\Delta$-coloring of $G - v$ for some $v \in V(G)$. Each color must be used on every $K_{\Delta}$ in $G - v$ and hence some color must be used on every $K_{\Delta}$ in $G$. Let $M$ be such a color class expanded to a maximal independent set. Then $\chi(G - M) = \chi(G) - 1 = \Delta > \max\{3, \omega(G - M), \Delta(G - M)\}$, a contradiction. \[\square\]

Here is the other version, not excluding diamonds and doing the reduction differently.

**Theorem 2.0.16** (Brooks [12]). Every graph $G$ with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K_{\Delta(G)+1}$.

**Proof.** Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, $G$ is $\Delta$-regular.
First, suppose $\Delta \geq 4$. Pick $v \in V(G)$ and let $w_1, \ldots, w_\Delta$ be $v$'s neighbors. Since $K_{\Delta+1} \not\subseteq G$, by symmetry we may assume that $w_2$ and $w_3$ are not adjacent. Choose a $(\Delta + 1)$-coloring $\{\{v\}, C_1, \ldots, C_\Delta\}$ of $G$ where $w_i \in C_i$ so as to maximize $|C_1|$. Then $C_1$ is a maximal independent set in $G$ and in particular, with $H := G - C_1$, we have $\chi(H) = \chi(G) - 1 = \Delta = \Delta(H) + 1 \geq 4$. By minimality of $|G|$, we get $K_\Delta \subseteq H$. But $\{\{v\}, C_2, \ldots, C_\Delta\}$ is a $\Delta$-coloring of $H$, so any $K_\Delta$ in $H$ must contain $v$ and hence $w_2$ and $w_3$, a contradiction.

Therefore $G$ is 3-regular. Since $G$ is not a forest it contains an induced cycle $C$. Put $T := N(C)$. Then $|T| \geq 2$ since $K_4 \not\subseteq G$. Take different $x, y \in T$ and put $H_{xy} := G - C$ if $x$ is adjacent to $y$ and $H_{xy} := (G - C) + xy$ otherwise. Then, by minimality of $|G|$, either $H_{xy}$ is 3-colorable or adding $xy$ created a $K_4$ in $H_{xy}$.

Suppose the former happens. Then we have a 3-coloring of $G - C$ where $x$ and $y$ receive different colors. We can easily extend this partial coloring to all of $G$ since each vertex of $C$ has a set of two available colors and some pair of vertices in $C$ get different sets.

Whence adding $xy$ created a $K_4$, call it $A$, in $H_{xy}$. We conclude that $T$ is independent and each vertex in $T$ has exactly one neighbor in $C$. Hence $|T| \geq |C| \geq 3$. Pick $z \in T - \{x, y\}$. Then $x$ is contained in a $K_4$, call it $B$, in $H_{xz}$. Since $d(x) = 3$, we must have $A - \{x, y\} = B - \{x, z\}$. But then any $w \in A - \{x, y\}$ has degree at least 4, a contradiction. \qed
The material in this chapter appeared in [65], [64] and [61].

Let \( G \) be the collection of all finite simple connected graphs. For a graph \( G \), \( x \in V(G) \) and \( D \subseteq V(G) \) we use the notation \( N_D(x) := N(x) \cap D \) and \( d_D(x) := |N_D(x)| \). Let \( c_G \) be the components of \( G \) and \( c(G) := |c_G| \). If \( h: G \to \mathbb{N} \), we define \( h \) for any graph as \( h(G) := \sum_{D \in c_G} h(D) \). An ordered partition of \( G \) is a sequence \((V_1, V_2, \ldots, V_k)\) where the \( V_i \) are pairwise disjoint and cover \( V(G) \). Note that we allow the \( V_i \) to be empty. When there is no possibility of ambiguity, we call such a sequence a partition.

3.1 Coloring when the high vertex subgraph has small cliques

In [38] Kierstead and Kostochka investigated the Brooks bound with the Ore-degree \( \theta \) in place of \( \Delta \).

**Definition 3.1.1.** The Ore-degree of an edge \( xy \) in a graph \( G \) is \( \theta(xy) := d(x) + d(y) \). The Ore-degree of a graph \( G \) is \( \theta(G) := \max_{xy \in E(G)} \theta(xy) \).

**Theorem 3.1.1** (Kierstead and Kostochka [38] 2010). If \( G \) is a graph with \( \chi(G) \geq \left\lceil \frac{\theta(G)}{2} \right\rceil + 1 \geq 7 \) then \( G \) contains \( K_{\chi(G)} \).

This statement about Ore-degree is equivalent to the following statement about vertex critical graphs.

**Theorem 3.1.2** (Kierstead and Kostochka [38] 2010). The only vertex critical graph \( G \) with \( \chi(G) \geq \Delta(G) \geq 7 \) such that \( H(G) \) is edgeless is \( K_{\chi(G)} \).

In [63], we improved the 7 to 6 by proving the following generalization.

**Theorem 3.1.3** (Rabern 2012 [63]). The only vertex critical graph \( G \) with \( \chi(G) \geq \Delta(G) \geq 6 \) and \( \omega(H(G)) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil - 2 \) is \( K_{\chi(G)} \).

This result and those in [59] were improved by Kostochka, Rabern and Stiebitz in [48]. In particular, the following was proved.

**Theorem 3.1.4** (Kostochka, Rabern and Stiebitz [48] 2012). The only vertex critical graphs \( G \) with \( \chi(G) \geq \Delta(G) \geq 5 \) such that \( H(G) \) is edgeless are \( K_{\chi(G)} \) and \( O_5 \).

Here \( O_n \) is the graph formed from the disjoint union of \( K_n - xy \) and \( K_{n-1} \) by joining \( \left\lceil \frac{n-1}{2} \right\rceil \) vertices of the \( K_{n-1} \) to \( x \) and the other \( \left\lfloor \frac{n-1}{2} \right\rfloor \) vertices of the \( K_{n-1} \) to \( y \) (see Figure 3.1). In [65] we proved a result that implies all of the results in [48]. The proof replaces
an algorithm of Mozhan \cite{55} with the original, more general, algorithm of Catlin \cite{14} on which it is based. This allows for a considerable simplification. Moreover, we prove two preliminary partitioning results that are of independent interest. All coloring results follow from the first of these, the second is a generalization of a lemma due to Borodin \cite{8} (and independently Bollobás and Manvel \cite{6}) about partitioning a graph into degenerate subgraphs. The following is the main coloring result in \cite{65}.

**Corollary 3.1.5.** Let $G$ be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(H(G)) \leq \frac{\chi(G)+p}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.

In the sections that follow we will prove this Corollary. First, we give a non-standard proof of Brooks’ theorem to illustrate the technique.

### 3.1.1 A weird proof of Brooks’ theorem

Let $G$ be a graph. A partition $P := (V_0, V_1)$ of $V(G)$ will be called normal if it achieves the minimum value of $(\Delta(G) - 1) \|V_0\| + \|V_1\|$. Note that if $P$ is a normal partition, then $\Delta(G[V_0]) \leq 1$ and $\Delta(G[V_1]) \leq \Delta(G) - 1$. The $P$-components of $G$ are the components of $G[V_i]$ for $i \in [2]$. A $P$-component is called an obstruction if it is a $K_2$ in $G[V_0]$ or a $K_{\Delta(G)}$ in $G[V_1]$ or an odd cycle in $G[V_1]$ when $\Delta(G) = 3$. A path $x_1x_2\cdots x_k$ is called $P$-acceptable if $x_1$ is contained in an obstruction and for different $i,j \in [k]$, $x_i$ and $x_j$ are in different $P$-components. For a subgraph $H$ of $G$ and $x \in V(G)$, we put $N_H(x) := N(x) \cap V(H)$.

**Lemma 3.1.6.** Let $G$ be a graph with $\Delta(G) \geq 3$. If $G$ doesn’t contain $K_{\Delta(G)+1}$, then $V(G)$ has an obstruction-free normal partition.

**Proof.** Suppose the lemma is false. Among the normal partitions having the minimum number of obstructions, choose $P := (V_0, V_1)$ and a maximal $P$-acceptable path $x_1x_2\cdots x_k$ so as to minimize $k$.

Let $A$ and $B$ be the $P$-components containing $x_1$ and $x_k$ respectively. Put $X := N_A(x_k)$. First, suppose $|X| = 0$. Then moving $x_1$ to the other part of $P$ creates another normal partition $P'$ having the minimum number of obstructions. But $x_2x_3\cdots x_k$ is a maximal $P'$-acceptable path, violating the minimality of $k$. Hence $|X| \geq 1$.

Pick $z \in X$. Moving $z$ to the other part of $P$ destroys the obstruction $A$, so it must create an obstruction containing $x_k$ and hence $B$. Since obstructions are complete graphs or odd cycles, the only possibility is that $\{z\} \cup V(B)$ induces an obstruction. Put $Y := N_B(z)$. Then, since obstructions are regular, $N_B(x) = Y$ for all $x \in X$ and $|Y| = \delta(B) + 1$. In particular, $X$ is joined to $Y$ in $G$.

Suppose $|X| \geq 2$. Then, similarly to above, switching $z$ and $x_k$ in $P$ shows that $\{x_k\} \cup V(A - z)$ induces an obstruction. Since obstructions are regular, we must have $|N_{A - z}(x_k)| = \Delta(A)$ and hence $|X| \geq \Delta(A) + 1$. Thus $|X \cup Y| = \Delta(A) + \delta(B) + 2 = \Delta(G) + 1$. Suppose $X$ is not a clique and pick nonadjacent $v_1, v_2 \in X$. It is easily seen that moving $v_1, v_2$ and then $x_k$ to their respective other parts violates normality of $P$. Hence $X$ is a clique. Suppose $Y$ is not a clique and pick nonadjacent $w_1, w_2 \in Y$. Pick $z' \in X - \{z\}$. Now moving $z$ and

![Figure 3.1: The graph $O_n$.](image)
then \( w_1, w_2 \) and then \( z' \) to their respective other parts again violates normality of \( P \). Hence \( Y \) is a clique. But \( X \) is joined to \( Y \), so \( X \cup Y \) induces a \( K_{\Delta(G)+1} \) in \( G \), a contradiction.

Hence we must have \(|X| = 1\). Suppose \( X \neq \{x_1\} \). First, suppose \( A \) is \( K_2 \). Then moving \( z \) to the other part of \( P \) creates another normal partition \( Q \) having the minimum number of obstructions. In \( Q \), \( x_kx_{k-1} \cdots x_1 \) is a maximal \( Q \)-acceptable path since the \( Q \)-components containing \( x_2 \) and \( x_k \) contain all of \( x_1 \)'s neighbors in that part. Running through the above argument using \( Q \) gets us to the same point with \( A \) not \( K_2 \). Hence we may assume \( A \) is not \( K_2 \).

Move each of \( x_1, x_2, \ldots, x_k \) in turn to their respective other parts of \( P \). Then the obstruction \( A \) was destroyed by moving \( x_i \) and for \( 1 \leq i < k \), the obstruction created by moving \( x_i \) was destroyed by moving \( x_{i+1} \). Thus, after the moves, \( x_k \) is contained in an obstruction. By minimality of \( k \), it must be that \( \{x_k\} \cup V(A-x_1) \) induces an obstruction and hence \(|X| \geq 2 \), a contradiction.

Therefore \( X = \{x_1\} \). But then moving \( x_1 \) to the other part of \( P \) creates an obstruction containing both \( x_2 \) and \( x_k \). Hence \( k = 2 \). Since \( x_1x_2 \) is maximal, \( x_2 \) can have no neighbor in the other part besides \( x_1 \). But now switching \( x_1 \) and \( x_2 \) in \( P \) creates a partition violating the normality of \( P \). \( \square \)

**Theorem 3.1.7** (Brooks 1941). If a connected graph \( G \) is not complete and not an odd cycle, then \( \chi(G) \leq \Delta(G) \).

**Proof.** Suppose not and choose a counterexample \( G \) minimizing \( \Delta(G) \). Plainly, \( \Delta(G) \geq 3 \). By Lemma 3.1.6, \( V(G) \) has an obstruction-free normal partition \( (V_0, V_1) \). Since \( G[V_0] \) has maximum degree at most one and contains no \( K_2 \)'s, we see that \( V_0 \) is independent. Since \( G[V_1] \) is obstruction-free, applying minimality of \( \Delta(G) \) gives \( \chi(G[V_1]) \leq \Delta(G[V_1]) < \Delta(G) \). Hence \( \chi(G) \leq \Delta(G) \), a contradiction. \( \square \)

### 3.1.2 The partitioning theorems

An *ordered partition* of a graph \( G \) is a sequence \( (V_1, V_2, \ldots, V_k) \) where the \( V_i \) are pairwise disjoint and cover \( V(G) \). Note that we allow the \( V_i \) to be empty. When there is no possibility of ambiguity, we call such a sequence a *partition*. For a vector \( r \in \mathbb{N}^k \) we take the coordinate labeling \( r = (r_1, r_2, \ldots, r_k) \) as convention. Define the *weight* of a vector \( r \in \mathbb{N}^k \) as \( w(r) := \sum_{i \in [k]} r_i \). Let \( G \) be a graph. An *\( r \)-partition* of \( G \) is an ordered partition \( P := (V_1, \ldots, V_k) \) of \( V(G) \) minimizing

\[
    f(P) := \sum_{i \in [k]} (|G[V_i]| - r_i |V_i|).
\]

It is a fundamental result of Lovász [51] that if \( P := (V_1, \ldots, V_k) \) is an \( r \)-partition of \( G \) with \( w(r) \geq \Delta(G)+1-k \), then \( \Delta(G[V_i]) \leq r_i \) for each \( i \in [k] \). The proof is simple: if there is a vertex in a part violating the condition, then there is some part it can be moved to that decreases \( f(P) \). As Catlin [14] showed, with the stronger condition \( w(r) \geq \Delta(G)+2-k \), a vertex of degree \( r_i \) in \( G[V_i] \) can always be moved to some other part while maintaining \( f(P) \). Since \( G \) is finite, a well-chosen sequence of such moves must always wrap back on itself. Many authors, including Catlin [14], Bollobás and Manvel [6] and Mozhan [55] have used such techniques to prove coloring results. We generalize these techniques by taking into account the degree in \( G \) of the vertex to be moved—a vertex of degree less than the maximum needs a weaker condition on \( w(r) \) to be moved.

For \( x \in V(G) \) and \( D \subseteq V(G) \) we use the notation \( N_D(x) := N(x) \cap D \) and \( d_D(x) := |N_D(x)| \). Let \( C(G) \) be the components of \( G \) and \( c(G) := |C(G)| \). For an induced subgraph \( H \) of \( G \), define \( d_G(H) := \min_{v \in V(H)} d_G(v) \). We also need the following notion of a movable subgraph.

**Definition 3.1.2.** Let \( G \) be a graph and \( H \) an induced subgraph of \( G \). For \( d \in \mathbb{N} \), the *\( d \)-movable subgraph* of \( H \) with respect to \( G \) is the subgraph \( H^d \) of \( G \) induced on

\[
    \{v \in V(H) \mid d_G(v) = d \text{ and } H-v \text{ is connected}\}.
\]
CHAPTER 3. DOING THE VERTEX SHUFFLE

We prove two partition theorems of similar form. All of our coloring results will follow from the first theorem, the second theorem is a degeneracy result from which Borodin’s result in [8] follows. For unification purposes, define a $t$-obstruction as an odd cycle when $t = 2$ and a $K_{t+1}$ when $t ≥ 3$.

**Theorem 3.1.8.** Let $G$ be a graph, $k, d ∈ \mathbb{N}$ with $k ≥ 2$ and $r ∈ \mathbb{N}^k_{≥2}$. If $ω(r) ≥ \max\{Δ(G) + 1 - k, d\}$, then at least one of the following holds:

1. $w(r) = d$ and $G$ contains an induced subgraph $Q$ with $|Q| = d+1$ that can be partitioned into $k$ cliques $F_1, ..., F_k$ where

   (a) $|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i ≥ 2$,

   (b) $|F_d^d| ≥ 2$, $|F_i| ≥ 1$ for $i ≥ 2$,

   (c) for $i ∈ [k]$, each $v ∈ V(F_i)$ is universal in $Q$;

2. there exists an $r$-partition $P := (V_1, ..., V_k)$ of $G$ such that if $C$ is an $r_i$-obstruction in $G[V_i]$, then $δ_G(C) ≥ d$ and $C_d$ is edgeless.

**Proof.** For $i ∈ [k]$, call a connected graph $G$ $i$-bad if $C$ is an $r_i$-obstruction such that $C_d$ has an edge. For a graph $H$ and $i ∈ [k]$, let $b_i(H)$ be the number of $i$-bad components of $H$. For an $r$-partition $P := (V_1, ..., V_k)$ of $G$ let

$$b(P) := \sum_{i ∈ [k]} b_i(G[V_i]).$$

Let $P := (V_1, ..., V_k)$ be an $r$-partition of $G$ minimizing $b(P)$.

Let $i ∈ [k]$ and $x ∈ V_i$ with $d_v(x) ≥ r_1$. Suppose $d_G(x) = d$. Then, since $w(r) ≥ d$, for every $j ≠ i$ we have $d_v(x) ≤ r_j$. Moving $x$ from $V_i$ to $V_j$ gives a new partition $P^*$ with $f(P^*) ≤ f(P)$. Note that if $d_G(x) < d$ we would have $f(P^*) < f(P)$ contradicting the minimality of $P$.

Suppose (2) fails to hold. Then $b(P) > 0$. By symmetry, we may assume that there is a 1-bad component $A_1$ of $G[V_1]$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i ∈ [k]$. Since $A_1$ is 1-bad we have $x_1 ∈ V(A_1^1)$ which has a neighbor in $V(A_1^d)$. By the above we can move $x_1$ from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, ..., V_{2,k})$ where $f(P_2) = f(P_1)$. Since removing $x_1$ from $A_1$ decreased $b_1(G[V_1])$, minimality of $b(P_1)$ implies that $x_1$ is in a 2-bad component $A_2$ in $V_{2,2}$. Now, we may choose $x_2 ∈ V(A_2^2) - \{x_1\}$ having a neighbor in $A_2^d$ and move $x_2$ from $V_{2,2}$ to $V_{2,1}$ to get a new partition $P_3 := (V_{3,1}, V_{3,2}, ..., V_{3,k})$ where $f(P_3) = f(P_1)$. We continue on this way to construct sequences $A_1, A_2, ..., P_1, P_2, P_3, ...$ and $x_1, x_2, ...$

This process can be defined recursively as follows. For $t ∈ \mathbb{N}$, put $j_t := 1$ for odd $t$ and $j_t := 2$ for even $t$. Put $P_t := P$ and $V_{t,i} := V_i$ for $i ∈ [k]$. Pick $x_t ∈ V(A_t^d)$ which has a neighbor in $V(A_t^1)$. Move $x_t$ from $V_{t,1}$ to $V_{t,2}$ to get a new partition $P_{t+1} := (V_{t+1,1}, V_{t+1,2}, ..., V_{t+1,k})$ where $f(P_{t+1}) = f(P_t)$ and $A_{t+1}$ be the 2-bad component in $V_{t+1,2}$ containing $x_t$. Then for $t ≥ 2$, pick $x_t ∈ V(A_t^d - x_{t-1})$ which has a neighbor in $V(A_t^1)$. Move $x_t$ from $V_{t,j_t}$ to $V_{t-1,j_t}$ to get a new partition $P_{t+1} := (V_{t+1,1}, V_{t+1,2}, ..., V_{t+1,k})$ where $f(P_{t+1}) = f(P_t)$ and let $A_{t+1}$ be the $(j_t-1)$-bad component in $V_{t+1,1} - x_{t-1}$ containing $x_t$.

Since $G$ is finite, at some point we will need to reuse a leftover component; that is, there is a smallest $t$ such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Let $j ∈ [2]$ be such that $V(A_s) ⊆ V_{t,j}$. Then $V(A_s) ⊆ V_{t-1,j}$. Note

**Claim 1.** $N(x_t) ∩ V(A_s - x_s) = N(x_s) ∩ V(A_s - x_s)$.

This is immediate since $A_s$ is $r_j$-regular.

**Claim 2.** $s = 1, t = 2$, both $A_s$ and $A_t$ are complete, $A_3^d$ is joined to $A_t - x_{t-1}$ and $A_t^d$ is joined to $A_s - x_s$.

Subclaim 2a. $N(x_s) ∩ V(A_t^d) ≠ \emptyset$.

In the construction of the sequence, $x_s$ was chosen such that it had a neighbor in $A_t^d$. 


Subclaim 2b. For any \( z \in N(x_s) \cap V(A^d_i) \) we have \( N(z) \cap V(A_t - x_t - 1) = N(z) \cap V(A_t - x_t - 1) \). Moreover, if \( x_s \) is adjacent to \( x_t \), then \( N(x_s) \cap V(A_t - x_t - 1) = N(x_t - 1) \cap V(A_t - x_t - 1) \) and \( x_s = x_t - 1 \).

In \( P_z \), move \( z \) to \( V_{y,3-j} \) to get a new partition \( P^\gamma := (V_{y,1}, V_{y,2}, \ldots, V_{y,k}) \). Then \( z \) must create an \( r_{3-j} \)-obstruction with \( A_t - x_t - 1 \) in \( V_{y,3-j} \) since \( z \) is adjacent to \( x_t \) by Claim 1. In particular, \( N(z) \cap V(A_t - x_t - 1) = N(z) \cap V(A_t - x_t - 1) \). If \( x_s \) is adjacent to \( x_t \), the same argument (with \( x_s \) in place of \( z \)) gives \( N(x_s) \cap V(A_t - x_t - 1) = N(x_t - 1) \cap V(A_t - x_t - 1) \) and \( x_s = x_t - 1 \).

Subclaim 2c. \( A_s \) is complete and \( x_s \) is adjacent to \( x_t \).

By Subclaim 2a, \( N(x_s) \cap V(A^d_i) \) and let \( P^\gamma \) be as in Subclaim 2b. In \( P^\gamma \), move \( x_t \) to \( V_{y,j} \) to get a new partition \( P^\gamma := (V_{y,1}, V_{y,2}, \ldots, V_{y,k}) \). Since \( s \) has at least two neighbors in \( A_s \), by Claim 1, \( x_t \) has a neighbor in \( A_s - z \). Hence \( x_t \) must create an \( r_{3-j} \)-obstruction with \( A_s - z \) in \( V_{y,j} \). In particular, \( N(z) \cap V(A_s - z) = N(x_t - 1) \cap V(A_s - z) \). Thus \( x_s \) is adjacent to \( x_t \) and we have \( N(x_s) \cap V(A_s) = N(x_s) \cap V(A_s) \).

Thus, if \( A_t \) is an odd cycle, it must be a triangle. Hence \( A_t \) is complete.

Subclaim 2d. \( A^d_i \) is joined to \( N(x_t - 1) \cap V(A_t - x_t - 1) \) and \( x_s = x_t - 1 \).

Since \( A_t \) is complete by Subclaim 2c, we have \( N(x_s) \cap V(A^d_i) = V(A^d_i) \). Since \( x_s \) is adjacent to \( x_t \) by Subclaim 2c, applying Subclaim 2b shows that \( A^d_i \) is joined to \( N(x_t - 1) \cap V(A_t - x_t - 1) \) and \( x_s = x_t - 1 \).

Subclaim 2e. \( s = 1 \) and \( t = 2 \).

Suppose \( s > 1 \). Then, since \( x_s \in V(A^d_i) \), Subclaim 2d shows that \( x_{s-1} \) is joined to \( N(x_t - 1) \cap V(A_t - x_t - 1) \) and hence \( A_t - x_t - 1 = A_s - x_s - 1 \) violating minimality of \( t \). Whence, \( s = 1 \) and \( t = 2 \).

Subclaim 2f. \( A_t \) is complete and \( A^d_i \) is joined to \( A_t - x_t - 1 \).

Pick \( z \in N(x_s) \cap V(A^d_i) \). Then \( z \) is joined to \( A_t - x_t - 1 \) by Subclaim 2d. In \( P_{t+1} \), move \( z \) to \( V_{t+1,3-j} \) to get a new partition \( P^d := (V_{t+1,1}, V_{t+1,2}, \ldots, V_{t+1,k}) \). Then \( z \) must create an \( r_{3-j} \)-obstruction with \( A_t - x_t - 1 \) in \( V_{t+1,3-j} \). In particular, \( V(A_t - x_t - 1) = N(z) \cap V(A_t - x_t - 1) \). Thus, if \( A_t \) is an odd cycle, it must be a triangle. Hence \( A_t \) is complete. Now Subclaim 2d gives that \( A^d_i \) is joined to \( A_t - x_t - 1 \).

Subclaim 2g. \( A^d_i \) is joined to \( A_s - x_s \).

Since \( x_s = x_t - 1 \), the statement is clear for \( x_t - 1 \). Pick \( y \in V(A^d_i) \) and \( z \in V(A^d_i) \). In \( P_t \), move \( y \) to \( V_{t,j} \). Since \( y \) is adjacent to \( z \) by Subclaim 2f, \( y \) must create an \( r_{3-j} \)-obstruction with \( A_s - x_s \) and since \( A_t \) is complete, \( y \) must be joined to \( A_s - x_s \). Hence \( A^d_i \) is joined to \( A_s - x_s \).

Claim 3. (1) holds.

We can play the same game with \( V_1 \) and \( V_2 \) for any \( 3 \leq i \leq k \) as we did with \( V_1 \) and \( V_2 \) above. Let \( B_1 := A_t \), \( B_2 := A_2 \) and for \( i \geq 3 \), let \( B_i \) be the \( r_{i+1} \)-obstruction made by moving \( x_{i-1} \) into \( V_1 \). Then \( B_i \) is complete for each \( i \in [k] \). Applying Claim 2 to all pairs \( B_i, B_j \) shows that for any distinct \( i, j \in [k] \), \( B_i^d \) is joined to \( B_j - x_t \) for \( i \geq 2 \). Let \( Q \) be the union of the \( F_i \). Then (a), (b) and (c) of (1) are satisfied. Note that \( |Q| = w(r) + 1 \) and since any \( v \in B_i^d \) is universal in \( Q \), \( |Q| \leq d + 1 \). By assumption \( w(r) \geq d \), whence \( w(r) = d \). Hence, (1) holds.

The following result generalizes a lemma due to Borodin [8]. This lemma of Borodin was generalized in another direction in [11]. The proof that follows is basically the same as that of Theorem 3.1.8. For a reader who is only interested in the coloring results, this theorem can be safely skipped.

Theorem 3.1.9. Let \( G \) be a graph, \( k, d \in \mathbb{N} \) with \( k \geq 2 \) and \( r \in \mathbb{N}_{\geq 1} \) where at most one of the \( r_i \) is one. If \( w(r) \geq \max \{ d(G) + 1 - k, d \} \), then at least one of the following holds:

1. \( w(r) = d \) and \( G \) contains a \( K_t \ast E_{d+1-t} \) where \( t \geq d + 1 - k \), for each \( v \in V(K_t) \) we have \( d_G(v) = d \) and for each \( v \in V(E_{d+1-t}) \) we have \( d_G(v) > d \); or,
2. there exists an r-partition \( P := (V_1, \ldots, V_k) \) of \( G \) such that if \( C \) is an \( r_i \)-regular component of \( G[V_i] \), then \( \delta_G(C) \geq d \) and there is at most one \( x \in V(C^d) \) with \( d_{C^d}(x) \geq r_i - 1 \). Moreover, \( P \) can be chosen so that either:

(a) for all \( i \in [k] \) and \( r_i \)-regular component \( C \) of \( G[V_i] \), we have \( |C^d| \leq 1 \); or,

(b) for some \( i \in [k] \) and some \( r_i \)-regular component \( C \) of \( G[V_i] \), there is \( x \in V(C^d) \) such that \( y \in \mathcal{N}_C(x) \mid d_C(y) = d \) is a clique.

Proof. For \( i \in [k] \), call a connected graph \( C \) i-bad if \( C \) is \( r_i \)-regular and there are at least two \( x \in V(C^d) \) with \( d_{C^d}(x) \geq r_i - 1 \). For a graph \( H \) and \( i \in [k] \), let \( b_i(H) \) be the number of i-bad components of \( H \). For an r-partition \( P := (V_1, \ldots, V_k) \) of \( G \) let

\[
c(P) := \sum_{i \in [k]} c(G[V_i]),
\]

\[
b(P) := \sum_{i \in [k]} b_i(G[V_i]).
\]

Let \( P := (V_1, \ldots, V_k) \) be an r-partition of \( V(G) \) minimizing \( c(P) \) and subject to that \( b(P) \).

Let \( i \in [k] \) and \( x \in V_i \) with \( d_V(x) \geq r_i \). Suppose \( d_G(x) = d \). Then, since \( w(r) \geq d \), for every \( j \neq i \) we have \( d_{V_j}(x) \leq r_j \). Moving \( x \) from \( V_i \) to \( V_j \) gives a new partition \( P' \) with \( f(P') \leq f(P) \). Note that if \( d_G(x) < d \) we would have \( f(P') < f(P) \) contradicting the minimality of \( P \).

Suppose \( b(P) > 0 \). By symmetry, we may assume that there is a 1-bad component \( A_1 \) of \( G[V_1] \). Put \( P_1 := P \) and \( V_1,i := V_i \) for \( i \in [k] \). Since \( A_1 \) is 1-bad we have \( x_1 \in V(A_1^d) \) with \( d_{A_1^d}(x_1) \geq r_1 - 1 \). By the above we can move \( x_1 \) from \( V_{1,1} \) to \( V_{1,2} \) to get a new partition \( P_2 := (V_2,1, V_2,2, \ldots, V_2,k) \) where \( f(P_2) = f(P_1) \). By the minimality of \( c(P_1) \), \( x_1 \) is adjacent to only one component \( C_2 \) in \( G[V_1,2] \). Let \( A_2 := G[V(C_2) \cup \{x_1\}] \). Since removing \( x_1 \) from \( A_1 \) decreased \( b_1(G[V_1]) \), minimality of \( b(P_1) \) implies that \( A_2 \) is 2-bad. Now, we may choose \( x_2 \in V(A_2^d) \) with \( d_{A_2^d}(x_2) \geq r_2 - 1 \) and move \( x_2 \) from \( V_{2,2} \) to \( V_{2,1} \) to get a new partition \( P_3 := (V_3,1, V_3,2, \ldots, V_3,k) \) where \( f(P_3) = f(P_1) \).

Continue on this way to construct sequences \( A_1, A_2, \ldots, P_1, P_2, P_3, \ldots \) and \( x_1, x_2, \ldots \).

Since \( G \) is finite, at some point we will need to reuse a leftover component; that is, there is a smallest \( t \) such that \( A_{t+1} = A_t = A_s \) for some \( s < t \). Let \( j \in [2] \) be such that in \( V(A_s) \subseteq V_{s,j} \). Then \( V(A_j) \subseteq V_{s,3-j} \). Note that, since \( A_s \) is \( r_j \)-regular, \( N(x_1) \cap V(A_s - x_s) = N(x_s) \cap V(A_s - x_s) \).

We claim that \( s = 1, t = 2 \), both \( A_s \) and \( A_t \) are complete, \( A^d_s \) is joined to \( A_t - x_{t-1} \) and \( A^d_t \) is joined to \( A_s - x_s \).

Put \( X := N(x_1) \cap V(A^d_s) \). Since \( x_1 \) witnesses the \( j \)-badness of \( A_s \), \(|X| \geq \max \{1, r_j - 1\} \).

Pick \( z \in X \). In \( P_s \), move \( z \) to \( V_{s,3-j} \) to get a new partition \( P^\gamma := (V_\gamma,1, V_\gamma,2, \ldots, V_\gamma,k) \).

Then \( z \) must create an \( r_{3-j} \)-regular component with \( A_t - x_{t-1} \in V_{\gamma,3-j} \) since \( z \) is adjacent to \( x_1 \). In particular, \( N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1}) \).

Suppose \( s \geq 1 \). Then \( x_{t-1} \) is joined to \( N(x_{t-1}) \cap V(A_t - x_{t-1}) \) and hence \( A_t = A_{t-1} = A_{t-1} - x_{t-1} \) violating minimality of \( t \). Whence, if \( r_j \geq 2 \) then \( s = 1 \).

Note that \( K = V(A^d_s) \) and hence if \( r_j \geq 2 \) then \( A_s \) is complete and \( A^d_s \) is joined to \( N(x_{t-1}) \cap V(A_t - x_{t-1}) \). If \( r_{3-j} = 1 \), then \( A_t \) is a \( K_2 \) and \( N(x_{t-1}) \cap V(A_t - x_{t-1}) = V(A_t - x_{t-1}) = \{x_i\} \). We already know that \( x_i \) is joined to \( A_s - x_s \). Thus the cases when
Suppose $r_j \geq 2$ and $r_{3-j} = 1$ are taken care of. By assumption, at least one of $r_j$ or $r_{3-j}$ is at least two. Hence it remains to handle the cases with $r_{3-j} \geq 2$.

Suppose $r_{3-j} \geq 2$. In $P_{t+1}$, move $z$ to $V_{t+1,3-j}$ to get a new partition $P_{t+1}^b := (V_{t+1,1}, V_{t+1,2}, \ldots, V_{t+1,k})$. Then $z$ must create an $r_{3-j}$-regular component with $A_t - x_t$ in $V_{t+1,3-j}$. In particular, $N(z) \cap V(A_t - x_t) = N(x_t) \cap V(A_t - x_t)$. Since $N(z) \cap V(A_t - x_t - 1) = N(x_t - 1) \cap V(A_t - x_t - 1)$, we have $N[x_{t-1}] \cap V(A_t) = N(z) \cap V(A_t) = N[x_t - 1] \cap V(A_t)$. Put $W := N[x_t - 1] \cap V(A_t^d)$. Each $w \in W$ is adjacent to $z$ and running through the argument above with $w$ in place of $x_t$ shows that $W$ is a clique joined to $z$. Moreover, since $x_t$ witnesses the $(3-j)$-badness of $A_t$, $|W| \geq r_{3-j}$. As with $A_t$ above, we conclude that $A_t$ is complete. Since $x_s \in V_{t+1,3-j}$ and $x_s$ is adjacent to $z$, it must be that $x_s \in V(A_t - x_t)$. Hence $x_s$ is joined to $W$ and $x_s = x_{t-1}$.

Suppose that $r_j \geq 2$ as well. We know that $s = 1$, $A_t$ is complete and $A_t^d$ is joined to $N(x_{t-1}) \cap V(A_t - x_t - 1) = A_t - x_t - 1$. Also, we just showed that $A_t$ is complete and $A_t^d$ is joined to $A_s - x_s$.

Thus, we must have $r_j = 1$ and $r_{3-j} \geq 2$. Then, since $A_s$ is a $K_2$, by the above, $A_s$ is joined to $W$. Since $W = A_t^d$, it only remains to show that $s = 1$. Suppose $s > 1$. Then $x_{s-1}$ is joined to $W$ and hence $x_{s-1} = A_s - x_{s-1}$ violating minimality of $t$.

Therefore $s = 1$, $t = 2$, both $A_s$ and $A_t$ are complete, $A_t^d$ is joined to $A_t - x_t - 1$ and $A_t^d$ is joined to $A_t - x_t$. But we can play the same game with $V_1$ and $V_2$ for any $3 \leq i \leq k$ as well. Let $B_1 := A_1$, $B_2 := A_2$ and for $i \geq 3$, let $B_i$ be the $r_i$-regular component made by moving $x_1$ into $V_i$. Then $B_i$ is complete for each $i \in [k]$. Applying what we just proved to all pairs $B_i, B_j$ shows that for any distinct $i, j \in [k]$, $B_i^d$ is joined to $B_j - x_1$. Since $|B_i^d| \geq r_i$ and $x_1 \in V(B_i^d)$ for each $i$, this gives a $K_t \ast E_{w(r) + 1 - t}$ in $G$ where $t \geq w(r) + 1 - k$. Take such a subgraph $Q$ maximizing $t$. Since all the $B_i$ are complete, any vertex of degree $d$ will be in $B_i^d$; therefore, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{w(r) + 1 - t})$ we have $d_G(v) > d$. Note that $|Q| = w(r) + 1$ and since $d_G(v) = d$ for any $v \in V(K_t)$, $|Q| \leq d + 1$. By assumption $w(r) \geq d$, whence $w(r) = d$. Thus if (1) fails, then the first part of (2) holds.

It remains to prove that we can choose $P$ to satisfy one of (a) or (b). Suppose that (1) fails and $P$ cannot be chosen to satisfy either (a) or (b). For $i \in [k]$, call a connected graph $C$ $i$-ugly if $C$ is $r_i$-regular and $|C^d| \geq 2$ let $u_i(H)$ be the number of $i$-ugly components of $H$. Note that if $C$ is $i$-bad, then it is $i$-ugly. For an $r$-partition $P := (V_1, \ldots, V_k)$ of $G$ let

$$u(P) := \sum_{i \in [k]} u_i(G[V_i]).$$

Choose an $r$-partition $Q := (V_1, \ldots, V_k)$ of $G$ first minimizing $c(Q)$, then subject to that requiring $b(Q) \leq 1$ and then subject to that minimizing $u(Q)$. Since $Q$ does not satisfy (a), at least one of $b(Q) = 1$ or $u(Q) \geq 1$ holds. By symmetry, we may assume that $G[V_1]$ contains a component $D_1$ which is either 1-bad or 1-ugly (or both). If $D_1$ is 1-bad, pick $w_1 \in V(D_1^d)$ witnessing the 1-badness of $D_1$; otherwise pick $w_1 \in V(D_1^d)$ arbitrarily. Move $w_1$ to $V_2$, to form a new $r$-partition. This new partition still satisfies all of our conditions on $Q$. As above we construct a sequence of vertex moves that will wrap around on itself.

This can be defined recursively as follows. For $t \geq 2$, if $D_t$ is bad pick $w_t \in V(D_t^d - w_t - 1)$ witnessing the badness of $D_t$; otherwise, if $D_t$ is ugly pick $w_t \in V(D_t^d - w_t - 1)$ arbitrarily. Now move $w_t$ to the part from which $w_t - 1$ came to form $D_{t+1}$. Let $Q_t := Q, Q_2, Q_3, \ldots$ be the partitions created by a run of this process. Note that the process can never create a component that is not ugly lest we violate the minimality of $u(Q)$.

Since $G$ is finite, at some point we will need to reuse a leftover component; that is, there is a smallest $t$ such that $D_{t+1} - x_t = D_s - x_s$ for some $s < t$. First, suppose $D_s$ is not bad, but merely ugly. Then $D_{t+1}$ is not bad and hence $b(Q_{t+1}) = 0$ and $u(Q_{t+1}) < u(Q)$, a contradiction. Hence $D_s$ is bad.

Suppose $D_s$ is not bad. As in the proof of the first part of (2), we can conclude that $x_s = x_{t-1}$. Pick $z \in N(x_s) \cap V(D_s^d)$. Since $z$ is adjacent to $x_t$, by moving $z$ to the part containing $x_t$ in $P_s$ we conclude $N(z) \cap V(D_t - x_s) = N(x_s) \cap V(D_t - x_s)$. Put $T :=$
\( \{ y \in N_{D_i}(x) \mid d_{G}(y) = d \} \). Suppose \( T \) is not a clique and let \( w_1, w_2 \in T \) be nonadjacent. Now, in \( P_1 \), since \( z \) is adjacent to both \( w_1 \) and \( w_2 \), swapping \( w_1 \) and \( w_2 \) with \( z \) contradicts minimality of \( f(Q) \). Hence \( T \) is a clique and (b) holds, a contradiction.

Thus we may assume that \( D_3 \) is bad as well. Now we may apply the same argument as in the proof of the first part of (2) to show that (1) holds. This final contradiction completes the proof. \( \square \)

**Corollary 3.1.10** (Borodin [8]). Let \( G \) be a graph not containing a \( K_{\Delta(G)+1} \). If \( r_1, r_2 \in \mathbb{N}_{\geq 1} \) with \( r_1 + r_2 \geq \Delta(G) \geq 3 \), then \( V(G) \) can be partitioned into sets \( V_1, V_2 \) such that 
\[
\Delta(G[V_i]) \leq r_i \text{ and } \text{col}(G[V_i]) \leq r_i \text{ for } i \in [2].
\]

**Proof.** Apply Proposition 3.1.9 with \( r := (r_1, r_2) \) and \( d = \Delta(G) \). Since \( G \) doesn’t contain a \( K_{\Delta(G)+1} \) and no vertex in \( G \) has degree larger than \( d \), (1) cannot hold. Thus (2) must hold. Let \( P := (V_1, V_2) \) be the guaranteed partition and suppose that for some \( j \in [2] \), \( G[V_j] \) contains an \( r_j \)-regular component \( H \). Then every vertex of \( H \) has degree \( d \) in \( G \) and hence \( H^d \) contains all noncutvertices of \( H \). But \( H \) has maximum degree \( r_j \) and thus contains at least \( r_j \) noncutvertices. If \( r_j = 1 \), then \( H \) is \( K_2 \) and hence has 2 noncutvertices. In any case, we have \( |H^d| \geq 2 \). Hence (a) cannot hold for \( P \). Thus, by (b), we have \( i \in [2] \), an \( r_i \)-regular component \( C \) of \( G[V_i] \) and \( x \in V(C) \) such that \( N_{C}(x) \) is a clique. But then \( C \) is \( K_{r_i+1} \) violating (2), a contradiction.

Therefore, for \( i \in [2] \), each component of \( G[V_i] \) contains a vertex of degree at most \( r_i - 1 \). Whence \( \text{col}(G[V_i]) \leq r_i \) for \( i \in [2] \). \( \square \)

### 3.1.3 The coloring corollaries

Using Theorem 3.1.8, we can prove coloring results for graphs with only small cliques among the vertices of high degree. To make this precise, for \( d \in \mathbb{N} \) define \( \omega_d(G) \) to be the size of the largest clique in \( G \) containing only vertices of degree larger than \( d \); that is, \( \omega_d(G) := \omega(G[\{ v \in V(G) \mid d_{G}(v) > d \}]) \).

**Corollary 3.1.11.** Let \( G \) be a graph, \( k, d \in \mathbb{N} \) with \( k \geq 2 \) and \( r \in \mathbb{N}^k \). If \( w(r) \geq \max \{ \Delta(G) + 1 - k, d \} \) and \( r_i \geq \omega_d(G) + 1 \) for all \( i \in [k] \), then at least one of the following holds:

1. \( w(r) = d \) and \( G \) contains an induced subgraph \( Q \) with \( |Q| = d + 1 \) which can be partitioned into \( k \) cliques \( F_1, \ldots, F_k \) where
   
   (a) \( |F_1| = r_1 + 1, |F_i| = r_i \) for \( i \geq 2 \),
   
   (b) \( |F_i^d| \geq |F_i| - \omega_d(G) \) for \( i \in [k] \),
   
   (c) for \( i \in [k] \), each \( v \in V(F_i^d) \) is universal in \( Q \);

2. \( \chi(G) \leq w(r) \).

**Proof.** Apply Theorem 3.1.8 to conclude that either (1) holds or there exists an \( r \)-partition \( P := (V_1, \ldots, V_k) \) of \( G \) such that if \( C \) is an \( r_i \)-obstruction in \( G[V_i] \), then \( \delta_{C_i}(C) \geq d \) and \( C^d \) is edgeless. Since \( \Delta(G[V_i]) \leq r_i \) for all \( i \in [k] \), it will be enough to show that no \( G[V_i] \) contains an \( r_i \)-obstruction. Suppose otherwise that we have an \( r_i \)-obstruction \( C \) in some \( G[V_i] \). First, if \( r_i \geq 3 \), then \( C \) is \( K_{r_i+1} \) and hence \( C \) contains a \( K_{\omega_d(G) + 2} \). But \( C^d \) is edgeless, so \( \omega_d(G) > \omega_d(G) + 1 \), a contradiction. Thus \( r_i = 2 \) and \( C \) is an odd cycle. Since \( C^d \) is edgeless, the vertices of \( C \) are 2-colored by the properties ‘degree is \( d \)’ and ‘degree is greater than \( d \)’, impossible. \( \square \)

For a vertex critical graph \( G \), call \( v \in V(G) \) low if \( d(v) = \chi(G) - 1 \) and high otherwise. Let \( H(G) \) be the subgraph of \( G \) induced on the high vertices of \( G \).
Corollary 3.1.12. Let $G$ be a vertex critical graph with $\chi(G) = \Delta(G) + 2 - k$ for some $k \geq 2$. If $k \leq \frac{\chi(G) - 1}{\omega(H(G)) + 1}$, then $G$ contains an induced subgraph $Q$ with $|Q| = \chi(G)$ which can be partitioned into $k$ cliques $F_1, \ldots, F_k$ where

1. $|F_i| = \chi(G) - (k-1)(\omega(H(G)) + 1)$, $|F_i| = \omega(H(G)) + 1$ for $i \geq 2$;

2. for each $i \in [k]$, $F_i$ contains at least $|F_i| - \omega(H(G))$ low vertices that are all universal in $Q$.

Proof. Suppose $k \leq \frac{\chi(G) - 1}{\omega(H(G)) + 1}$. Put $r_i := \omega(H(G)) + 1$ for $i \in [k] - \{1\}$ and $r_1 := \chi(G) - 1 - (k-1)(\omega(H(G)) + 1)$. Set $r := (r_1, r_2, \ldots, r_k)$. Then $w(r) = \chi(G) - 1 = \Delta(G) + 1 - k$. Now applying Corollary 3.1.11 with $d := \chi(G) - 1$ proves the corollary. □

Corollary 3.1.13. Let $G$ be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(H(G)) \geq \frac{\chi(G) + 1}{p+1}$ then $G = K_{\chi(G)}$ or $G = O_5$.

Proof. Suppose not and choose a counterexample $G$ minimizing $|G|$. Put $G := \chi(G)$, $\Delta := \Delta(G)$ and $h := \omega(H(G))$. Then $p \geq 1$ and $h \geq 1$ by Brooks’ theorem. Hence $h \geq 5$. By assumption, we have $h \leq \frac{\chi(G) + 1}{p+1} - 2 = \frac{x-2p}{p+1} \leq \frac{x-2}{p+1}$ since $p \geq 1$. Thus $p + 1 \leq \frac{x+1}{h-3}$ and we may apply Corollary 3.1.13 with $k := p + 1$ to get an induced subgraph $Q$ of $G$ with $|Q| = \chi$ which can be partitioned into $p + 1$ cliques $F_1, \ldots, F_{p+1}$

1. $|F_i| = \chi - p(h+1)$, $|F_i| = h + 1$ for $i \geq 2$;

2. for each $i \in [p + 1]$, $F_i$ contains at least $|F_i| - h$ low vertices that are all universal in $Q$.

Let $T$ be the low vertices in $Q$, put $H := Q - T$ and $t := |T|$. Then $Q = K_t \ast H$ and $t \geq \chi - p(h + 1) + p(h + 1) - (p + 1)h = \chi - (p + 1)h$.

Take any $(\chi - 1)$-coloring of $G - Q$ and let $L$ be the resulting list assignment on $Q$. Then $|L(v)| = d_Q(v)$ for each $v \in T$ and $|L(v)| \geq d_Q(v) - p$ for each $v \in V(H)$. Since $t \geq \chi - (p + 1)h \geq 2p + 1 \geq p + 1$, if there are nonadjacent $x, y \in V(H)$ and $c \in L(x) \cap L(y)$, then we may color $x$ and $y$ both with $c$ and then greedily complete the coloring to the rest of $H$ and then to all of $Q$, a contradiction. Hence any nonadjacent pair in $H$ have disjoint lists.

Let $I$ be a maximal independent set in $H$. If there is an induced $P_3$ in $H$ with ends in $I$, set $o_I := 1$, otherwise set $o_I := 0$. Since each pair of vertices in $I$ have disjoint lists, we must have

$$\chi - 1 \geq \sum_{v \in I} |L(v)|$$

$$\geq \sum_{v \in I} t + d_H(v) - p$$

$$= (t - p) |I| + \sum_{v \in I} d_H(v)$$

$$\geq (t - p) |I| + |H| - |I| + o_I$$

$$= (t - (p + 1)) |I| + \chi - t + o_I.$$
**CHAPTER 3. DOING THE VERTEX SHUFFLE**

graph obtained by collapsing \( \{x, y\} \) to a single vertex \( v_{xy} \). If \( \chi(A') \leq \chi - 1 \), then we have a \((\chi - 1)\)-coloring of \( A \) in which \( x \) and \( y \) receive the same color. This is impossible as then we could complete the \((\chi - 1)\)-coloring to all of \( G \) greedily as above. Hence \( \chi(A') = \chi \) and thus we have a vertex critical subgraph \( Z \) of \( A' \) with \( \chi(Z) = \chi \). We must have \( v_{xy} \in V(Z) \) and since \( d_A(x) + d_A(y) \leq \chi - 1 \), \( v_{xy} \) is low. Hence, by minimality of \( |Z| \), \( Z = K_1 \) or \( Z = O_5 \).

First, suppose \( \chi \geq 6 \). Then \( h \geq 2 \) and thus we have \( z \in V(H) \setminus \{x, y\} \) nonadjacent to \( x \). Apply the previous paragraph to both pairs \( \{x, y\} \) and \( \{x, z\} \). The case \( Z = O_5 \) cannot happen, for then we would have \( \chi = \chi(Z) = 5 \), a contradiction. Put \( X_1 := N(x) \cap V(G - Q) \), \( X_2 := N(y) \cap V(G - Q) \), \( X_3 := N(z) \cap V(G - Q) \). Then \( |X_i| = \frac{\Delta - 1}{2} \) for \( i \in [3] \) and \( X_1 \) is joined to both \( X_2 \) and \( X_3 \). Since \( |X_1| - h > 0 \), each \( X_i \) contains a low vertex \( v_i \). But then \( N(v_1) = X_1 \cup X_2 \cup \{x\} \) and we must have \( X_3 = X_2 \). Whence \( N(v_2) = X_1 \cup X_2 \cup \{y, z\} \) giving \( \chi(v_2) \geq \chi \), a contradiction.

Therefore \( \chi = 5 \), \( h = 1 \) and \( V(H) = \{x, y\} \). If \( Z = K_2 \), then \( N[x] \cup N[y] \) induces an \( O_5 \) in \( G \) and hence \( G = O_5 \), a contradiction. Thus \( Z = O_5 \). But \( h = 1 \), so all of the neighbors of both \( x \) and \( y \) are low and hence all of the neighbors of \( v_{xy} \) in \( Z \) are low. But \( O_5 \) has no such low vertex \( v_{xy} \) with all low neighbors, so this is impossible.

**Question.** The condition on \( k \) needed in Corollary 3.1.12 is weaker than that in Corollary 3.1.13. **What do the intermediate cases look like? What are the extremal examples?**

### 3.2 Destroying incomplete components in vertex partitions

In [47] Kostochka modified an algorithm of Catlin [14] to show that every triangle-free graph \( G \) can be colored with at most \( \frac{2}{3} \Delta(G) + 2 \) colors. In fact, his modification proves that the vertex set of any triangle-free graph \( G \) can be partitioned into \( \left\lfloor \frac{\Delta(G) + 2}{3} \right\rfloor \) sets, each of which induces a disjoint union of paths. In [64] we generalized this as follows.

**Lemma 3.2.1** (Rabern [64]). Let \( G \) be a graph and \( r_1, \ldots, r_k \in \mathbb{N} \) such that \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \). Then \( V(G) \) can be partitioned into sets \( V_1, \ldots, V_k \) such that \( \Delta(G[V_i]) \leq r_i \) and \( G[V_i] \) contains no incomplete \( r_i \)-regular components for each \( i \in [k] \).

Setting \( k = \left\lfloor \frac{\Delta(G) + 2}{3} \right\rfloor \) and \( r_i = 2 \) for each \( i \) gives a slightly more general form of Kostochka’s theorem.

**Corollary 3.2.2** (Rabern [64]). The vertex set of any graph \( G \) can be partitioned into \( \left\lfloor \frac{\Delta(G) + 2}{3} \right\rfloor \) sets, each of which induces a disjoint union of triangles and paths.

For coloring, this actually gives the bound \( \chi(G) \leq 2 \left\lfloor \frac{\Delta(G) + 2}{3} \right\rfloor \) for triangle free graphs. To get \( \frac{2}{3} \Delta(G) + 2 \), just use \( r_k = 0 \) when \( \Delta \equiv 2(\mod 3) \). Similarly, for any \( r \geq 2 \), setting \( k = \left\lfloor \frac{\Delta(G) + 2}{r+1} \right\rfloor \) and \( r_i = r \) for each \( i \) gives the following.

**Corollary 3.2.3** (Rabern [64]). Fix \( r \geq 2 \). The vertex set of any \( K_{r+1} \)-free graph \( G \) can be partitioned into \( \left\lfloor \frac{\Delta(G) + 2}{r+1} \right\rfloor \) sets each inducing an \((r-1)\)-degenerate subgraph with maximum degree at most \( r \).

For the purposes of coloring it is more economical to split off \( \Delta + 2 - (r + 1) \left\lfloor \frac{\Delta + 2}{r+1} \right\rfloor \) parts with \( r_j = 0 \).

**Corollary 3.2.4** (Rabern [64]). Fix \( r \geq 2 \). The vertex set of any \( K_{r+1} \)-free graph \( G \) can be partitioned into \( \left\lfloor \frac{\Delta(G) + 2}{r+1} \right\rfloor \) sets each inducing an \((r-1)\)-degenerate subgraph with maximum degree at most \( r \) and \( \Delta(G) + 2 - (r + 1) \left\lfloor \frac{\Delta(G) + 2}{r+1} \right\rfloor \) independent sets. In particular, \( \chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G) + 2}{r+1} \right\rfloor \).
For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks’ Theorem. Lemma 3.2.1 is of the same form as Lovász’s Lemma 1.1.4, but it gives a more restrictive partition at the cost of replacing $\Delta(G) + 1$ with $\Delta(G) + 2$. For $r \geq 3$, combining Lovász’s Lemma 1.1.4 with Brooks’ theorem gives the following better bound for a $K_{r+1}$-free graph $G$ (first proved in [10], [15] and [50]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G) + 1}{r + 1} \right\rfloor.$$

### 3.2.1 A generalization

Here we prove a generalization of Lemma 3.2.1.

**Definition 3.2.1.** For $h: G \to \mathbb{N}$ and $G \in \mathcal{G}$, a vertex $x \in V(G)$ is called $h$-critical in $G$ if $G - x \in \mathcal{G}$ and $h(G - x) < h(G)$.

**Definition 3.2.2.** For $h: G \to \mathbb{N}$ and $G \in \mathcal{G}$, a pair of vertices $\{x, y\} \subseteq V(G)$ is called an $h$-critical pair in $G$ if $G - \{x, y\} \in \mathcal{G}$ and $x$ is $h$-critical in $G - y$ and $y$ is $h$-critical in $G - x$.

**Definition 3.2.3.** For $r \in \mathbb{N}$ a function $h: \mathcal{G} \to \mathbb{N}$ is called an $r$-height function if it has each of the following properties:

1. if $h(G) > 0$, then $G$ contains an $h$-critical vertex $x$ with $d(x) \geq r$;
2. if $G \in \mathcal{G}$ and $x \in V(G)$ is $h$-critical with $d(x) \geq r$, then $h(G - x) = h(G) - 1$;
3. if $G \in \mathcal{G}$ and $x \in V(G)$ is $h$-critical with $d(x) \geq r$, then $G$ contains an $h$-critical vertex $y \notin \{x\} \cup N(x)$ with $d(y) \geq r$;
4. if $G \in \mathcal{G}$ and $\{x, y\} \subseteq V(G)$ is an $h$-critical pair in $G$ with $d_{G-y}(x) \geq r$ and $d_{G-x}(y) \geq r$, then there exists $z \in N(x) \cap N(y)$ with $d(z) \geq r + 1$.

**Lemma 3.2.5.** Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. If $h_i$ is an $r_i$-height function for each $i \in [k]$, then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that for each $i \in [k]$, $\Delta(G[V_i]) \leq r_i$ and $h_i(D) = 0$ for each component $D$ of $G[V_i]$.

For each $r \in \mathbb{N}$, it is easy to see that the function $h_r: \mathcal{G} \to \mathbb{N}$ defined as follows is an $r$-height function:

$$h_r(G) := \begin{cases} 1 & G \text{ is incomplete and } r\text{-regular;} \\ 0 & \text{otherwise.} \end{cases}$$

Applying Lemma 3.2.5 with these height functions proves Lemma 3.2.1. Other height functions exist, but we don’t yet have a sense of their ubiquity or lack thereof.

**Proof of Lemma 3.2.5.** For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} (\|G[V_i]\| - r_i |V_i|),$$

$$c(P) := \sum_{i=1}^{k} c(G[V_i]),$$

$$h(P) := \sum_{i=1}^{k} h_i(G[V_i]).$$
Let \( P := (V_1, \ldots, V_k) \) be a partition of \( V(G) \) minimizing \( f(P) \), and subject to that \( c(P) \), subject to that \( h(P) \).

Let \( i \in [k] \) and \( x \in V_i \) with \( d_{V_i}(x) \geq r_i \). Since \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \) there is some \( j \neq i \) such that \( d_{V_j}(x) \leq r_j \). Moving \( x \) from \( V_i \) to \( V_j \) gives a new partition \( P^* \) with \( f(P^*) \leq f(P) \). Note that if \( d_{V_i}(x) > r_i \) we would have \( f(P^*) < f(P) \) contradicting the minimality of \( P \).

This proves that \( \Delta(G[V_i]) \leq r_i \) for each \( i \in [k] \).

Now suppose that for some \( i_1 \) there is a component \( A_1 \) of \( G[V_{i_1}] \) with \( h_{i_1}(A_1) > 0 \). Put \( P_1 := P \) and \( V_{1,i} := V_i \) for \( i \in [k] \). By property 1 of height functions, we have an \( h_{i_1} \)-critical vertex \( x_1 \in V(A_1) \) with \( d_{A_1}(x_1) \geq r_{i_1} \). By the above we have \( i_2 \neq i_1 \) such that moving \( x_1 \) from \( V_{1,i_1} \) to \( V_{1,i_2} \) gives a new partition \( P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k}) \) where \( f(P_2) = f(P_1) \).

By the minimality of \( c(P_1) \), \( x_2 \) is adjacent to only one component \( C_2 \) in \( G[V_{i_2}] \). Let \( A_2 := G[V(C_2) \cup \{x_1\}] \). Since \( x_1 \) is \( h_{i_2} \)-critical, by the minimality of \( h(P_1) \), it must be that \( h_{i_2}(A_2) > h_{i_2}(C_2) \). By property 2 of height functions we must have \( h_{i_2}(A_2) = h_{i_2}(C_2) + 1 \).

Hence \( h(P_2) \) is still minimum. Now, by property 3 of height functions, we have an \( h_{i_2} \)-critical vertex \( x_2 \in V(A_2) - (\{x_1\} \cup N_{A_2}(x_1)) \) with \( d_{A_2}(x_2) \geq r_{i_2} \).

Continue on this way to construct sequences \( i_1, i_2, \ldots, A_1, A_2, \ldots, P_1, P_2, P_3, \ldots \) and \( x_1, x_2, \ldots \). Since \( G \) is finite, at some point we will need to re-use a leftover component; that is, there is a smallest \( t \) such that \( A_{t+1} = A_t - x_t \) for some \( s < t \). In particular, \( \{x_s, x_{t+1}\} \) is an \( h_s \)-critical pair in \( Q := G[\{x_{t+1}\} \cup V(A_t)] \) where \( d_{Q-x_t+1}(x_s) \geq r_{i_s} \) and \( d_{Q-x_t}(x_{t+1}) \geq r_{i_t} \). Thus, by property 4 of height functions, we have \( z \in N_Q(x_s) \cap N_Q(x_{t+1}) \) with \( d_Q(z) \geq r_{i_s} + 1 \).

We now modify \( P_s \) to contradict the minimality of \( f(P) \). At step \( t + 1 \), \( x_t \) was adjacent to exactly \( r_{i_t} \) vertices in \( V_{t+1,i_t} \). This is what allowed us to move \( x_t \) into \( V_{t+1,i_t} \). Our goal is to modify \( P_s \) so that we can move \( x_t \) into the \( i_s \) part without moving \( x_s \) out. Since \( z \) is adjacent to both \( x_s \) and \( x_t \), moving \( z \) out of the \( i_s \) part will then give us our desired contradiction.

So, consider the set \( X \) of vertices that could have been moved out of \( V_{s,i_s} \) between step \( s \) and step \( t + 1 \); that is, \( X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,i_s} \). For \( x_j \in X \), since \( d_{A_j}(x_j) \geq r_{i_s} \) and \( x_j \) is not adjacent to \( x_{j-1} \) we see that \( d_{V_{s,i_s}}(x_j) \geq r_{i_s} \). Similarly, \( d_{V_{s,i_s}}(x_i) \geq r_{i_s} \). Also, by the minimality of \( t \), \( X \) is an independent set in \( G \). Thus we may move all elements of \( X \) out of \( V_{s,i_s} \) to get a new partition \( P^* := (V_{s,1}, \ldots, V_{s,k}) \) with \( f(P^*) = f(P) \).

Since \( x_t \) is adjacent to exactly \( r_{i_s} \) vertices in \( V_{t+1,i_s} \), and the only possible neighbors of \( x_t \) that were moved out of \( V_{s,i_s} \) between steps \( s \) and \( t + 1 \) are the elements of \( X \), we see that \( d_{V_{s,i_s}}(x_t) = r_{i_t} \). Since \( d_{V_{s,i_s}}(x_t) \geq r_{i_s} \), we can move \( x_t \) from \( V_{s,i_s} \) to \( V_{s,i_s} \) to get a new partition \( P^{**} := (V_{s,1}, \ldots, V_{s,k}) \) with \( f(P^{**}) = f(P^*) \). Now, recall that \( z \in V_{s,i_s} \). Since \( z \) is adjacent to \( x_t \), we have \( d_{V_{s,i_s}}(z) \geq r_{i_t} + 1 \). Thus we may move \( z \) out of \( V_{s,i_s} \) to get a new partition \( P^{***} \) with \( f(P^{***}) < f(P^{**}) = f(P) \). This contradicts the minimality of \( f(P) \).

\[ \square \]
4

REducing maximum degree

Some of the material in this chapter appeared in [62].

4.1 Hitting all maximum cliques

As part of his proof that every graph with $\chi \geq \Delta$ contains a $K_{\Delta-28}$, Kostochka proved the following lemma.

**Lemma 4.1.1** (Kostochka [46]). If $G$ is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then $G$ contains an independent set $I$ such that $\omega(G - I) < \omega(G)$.

To talk about the proof we first need a definition.

**Clique Graph.** Let $G$ be a graph. For a collection of cliques $Q$ in $G$, let $X_Q$ be the intersection graph of $Q$. That is, the vertex set of $X_Q$ is $Q$ and there is an edge between $Q_1 \neq Q_2 \in Q$ iff $Q_1$ and $Q_2$ intersect.

Kostochka’s proof proceeded in two stages. First show that the vertices in each component of the clique graph have a large intersection. Then find an independent transversal of these intersections. Such a transversal is an independent set hitting every maximum clique. Kostochka used a custom method to find a transversal. In [62], we applied the following lemma of Haxell [34] (proved long after Kostochka’s paper) to find the independent transversal.

**Lemma 4.1.2.** Let $H$ be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. Then $H$ has an independent set $\{v_1, \ldots, v_n\}$ where $v_i \in V_i$ for each $i \in [r]$.

Finding the independent transversal using this lemma gives the following.

**Lemma 4.1.3** (Rabern [62]). If $G$ is a graph satisfying $\omega \geq \frac{3}{4}(\Delta + 1)$, then $G$ contains an independent set $I$ such that $\omega(G - I) < \omega(G)$.

Aharoni, Berger and Ziv [1] showed that Haxell’s proof actually gets more than Lemma 4.1.2. From their extension, King [42] proved the following lopsided version of Haxell’s lemma.

**Lemma 4.1.4** (King [42]). Let $G$ be a graph partitioned into $r$ cliques $V_1, \ldots, V_r$. If there exists $k \geq 1$ such that for each $i$ every $v \in V_i$ has at most $\min\{k, |V_i| - k\}$ neighbors outside $V_i$, then $G$ contains an independent set with $r$ vertices.

Using this gives the best possible form of the lemma.

**Lemma 4.1.5** (King [42]). If $G$ is a graph satisfying $\omega > \frac{3}{2}(\Delta + 1)$, then $G$ contains an independent set $I$ such that $\omega(G - I) < \omega(G)$.
4.1.1 A simple proof of Kostochka’s first stage

The proofs for Kostochka’s first stage can be made much simpler than the originals and we do so here.

**Lemma 4.1.6** (Hajnal [30]). Let $G$ be a graph and $Q$ a collection of maximum cliques in $G$. Then

$$\left| \bigcup Q \right| + \left| \bigcap Q \right| \geq 2\omega(G).$$

**Proof.** Suppose the lemma is false and let $Q$ be a counterexample with $|Q|$ minimal. Put $r := |Q|$ and say $Q = \{Q_1, \ldots, Q_r\}$.

Consider the set $W := (Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i$. Plainly, $W$ is a clique. Thus we may derive a contradiction as follows.

$$\omega(G) \geq |W|$$

$$= (Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i$$

$$= Q_1 \cap \bigcup_{i=2}^r Q_i \cup \bigcap_{i=2}^r Q_i$$

$$= |Q_1| + \bigcup_{i=2}^r Q_i - \bigcap_{i=2}^r Q_i$$

$$\geq \omega(G) + 2\omega(G) - \left( \bigcup_{i=1}^r Q_i + \bigcap_{i=1}^r Q_i \right)$$

$$> \omega(G).$$

**Lemma 4.1.7** (Kostochka [46]). If $Q$ is a collection of maximum cliques in a graph $G$ with $\omega(G) > \frac{3}{5}(\Delta(G) + 1)$ such that $X_Q$ is connected, then $\cap Q \neq \emptyset$.

**Proof.** Suppose not and choose a counterexample $Q := \{Q_1, \ldots, Q_r\}$ minimizing $r$. Plainly, $r \geq 3$. Let $A$ be a noncutvertex in $X_Q$ and $B$ a neighbor of $A$. Put $Z := Q - \{A\}$. Then $X_Z$ is connected and hence by minimality of $r$, $\cap Z \neq \emptyset$. In particular, $|\cup Z| \leq \Delta(G) + 1$. Hence $|\cup Q| \leq |\cup Z| + |A - B| \leq 2(\Delta(G) + 1) - \omega(G) < 2\omega(G)$. This contradicts Hajnal’s lemma.

With a little more work we can prove the following generalization of Kostochka’s lemma which has a Helly feel. We won’t use this result here, but it has some independent interest. The following example from King [40] shows that the condition $\omega > \frac{k+1}{2k+1}(\Delta + 1)$ is tight. Take $K_{k+1} \ast E_{k+1}$ and remove a perfect matching between $K_{k+1}$ and $E_{k+1}$ to get a graph $H$. Then $\omega(H) = k + 1$ and $\Delta(H) = 2k$ and thus $\omega(H) = \frac{k+1}{2k+1}(\Delta(H) + 1)$. Taking $Q$ to be all $(k + 1)$-cliques containing a vertex in the $E_{k+1}$, we see that $\cap Q = \emptyset$ but any $k$ elements of $Q$ have common intersection.

**Lemma 4.1.8.** Fix $k \geq 2$. Let $G$ be a graph satisfying $\omega > \frac{k+1}{2k+1}(\Delta + 1)$. If $Q$ is a collection of maximum cliques in $G$ such that any $k$ elements of $Q$ have common intersection, then $\cap Q \neq \emptyset$.

**Proof.** Suppose not and choose a counterexample $Q := \{Q_1, \ldots, Q_r\}$ minimizing $r$. Plainly, $r \geq k + 1$. Put $Z_i := Q - \{Q_i\}$. Then any $k$ elements of $Z_i$ have common intersection and hence by minimality $\cap Z_i \neq \emptyset$. In particular $\cup Z_i$ contains a universal vertex and thus
\[|\cup Z_i| \leq \Delta(G) + 1.\] Now, by Hajnal’s Lemma, \(|\cap Z_i| \geq 2\omega(G) - (\Delta(G) + 1) > 2\omega(G) - \frac{2k+1}{k+1}\omega(G) = \frac{1}{k+1}\omega(G).\]

Put \(m := \min_i |Q_i - \cup Z_i|\). Note that the \(\cap Z_i\) are pairwise disjoint since \(\cap Q = \emptyset\). Thus \(\cup Q\) contains the disjoint union of the \(\cap Z_i\) as well as at least \(m\) vertices in each clique outside the rest. In particular,

\[|\cup Q| \geq \frac{1}{k+1}\omega(G)r + mr \geq \omega(G) + (k+1)m.\]

In addition,

\[|\cup Q| \leq m + \Delta(G) + 1.\]

Hence,

\[m \leq \frac{\Delta(G) + 1 - \omega(G)}{k} < \frac{1}{k+1}\omega(G).\]

Finally,

\[|\cup Q| \leq m + \Delta(G) + 1 < \frac{1}{k+1}\omega(G) + \frac{2k+1}{k+1}\omega(G) = 2\omega(G).\]

Applying Hajnal’s Lemma gives a contradiction. \(\square\)

### 4.1.2 Independent transversals

In [33], Haxell and Szabó develop a technique for dealing with independent transversals. In [32], Haxell used this technique to give a simpler proof of her Lemma. The proof gives a bit more as the following lemma shows. This is just slightly more general than the extension given in [1] by Aharoni, Berger and Ziv and either gives enough to prove King’s lopsided symmetry we may assume

\[\sum_{i \in [k]} \mathbb{I}(x_i, y_i) = k \] as to minimize \(k\). Let \(xy \in E(G)\). By assumption \(\pi\) has a \((G - xy)\)-independent transversal \(T\). Note that we must have \(x, y \in T\) lest \(T\) be a \(G\)-independent transversal of \(\pi\).

By symmetry we may assume that \(\pi(x) = k - 1\) and \(\pi(y) = k\). Put \(X := \pi^{-1}(k - 1), Y := \pi^{-1}(k)\) and \(H := G - N(x, y) - E(X, Y)\). Define \(\zeta : V(H) \rightarrow [k - 1]\) by \(\zeta(v) := \min \{\pi(v), k - 1\}\). Note that since \(x, y \in T\), we have \(|\zeta^{-1}(i)| \geq 1\) for each \(i \in [k - 2]\). Put \(Z := \zeta^{-1}(k - 1)\). Then \(Z \neq \emptyset\) for otherwise \(M := \{xy\}\) totally dominates \(G[X \cup Y]\) giving a contradiction.

Suppose \(\zeta\) has an \(H\)-independent transversal \(S\). Then we have \(z \in S \cap Z\) and by symmetry we may assume \(z \in X\). But then \(S \cup \{y\}\) is a \(G\)-independent transversal of \(\pi\), a contradiction.

Let \(H' \subseteq H\) be a minimal spanning subgraph such that \(\zeta\) has no \(H'\)-independent transversal. Now \(d(z) \geq 1\) for each \(z \in Z\) for otherwise \(T - \{x, y\} \cup \{z\}\) would be an \(H'\)-independent transversal of \(\zeta\). Pick \(zw \in E(H')\). By minimality of \(k\), we have \(J \subseteq [k - 1]\) with \(\zeta(z), \zeta(w) \in J\) and an induced matching \(M\) of \(H' \left[\zeta^{-1}(J)\right]\) with \(zw \in M\) such that
1. $\bigcup M$ totally dominates $H'[\zeta^{-1}(J)],$

2. the multigraph with vertex set $J$ and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since $H'$ is a spanning subgraph of $H$, $\bigcup M$ totally dominates $H[\zeta^{-1}(J)]$ and hence $\bigcup M'$ totally dominates $G[\pi^{-1}(J')]$. Moreover, the multigraph in (2) for $M'$ and $J'$ is formed by splitting the vertex $k-1 \in J$ in two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. \[\square\]

**Lemma 4.1.4** (King [42]). Let $H$ be a graph partitioned into $k$ cliques $V_1, \ldots, V_k$. If there exists $r \geq 1$ such that for each $i$ every $v \in V_i$ has at most $\min \{r, |V_i| - r\}$ neighbors outside $V_i$, then $G$ contains an independent set with $k$ vertices.

**Proof.** Suppose not and choose a counterexample $H$ minimizing $||H||$. Remove all the edges from each $V_i$ to form a graph $G$. Pick $xy \in E(G)$ and apply Lemma 4.1.9 on $xy$ with $\pi: V(G) \to [r]$ given by $\pi(V_i) = i$ to get the guaranteed $J \subseteq [r]$ and induced matching $M$. Note that by our assumption, the ends of an edge from $V_i$ to $V_j$ together dominate at most $\min \{|V_i|, |V_j|\}$ vertices. Let $T$ be the tree with vertex set $J$ and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$. Choose a root $c$ of $T$. Traversing $T$ in leaf-first order and for each leaf $a$ with parent $b$ picking $|V_a|$ from $\min \{|V_a|, |V_b|\}$ we get that the vertices in $M$ together dominate at most $\sum_{i \in J - c} |V_i|$ vertices, a contradiction. \[\square\]

### 4.1.3 Putting it all together

Now we are in a position to prove Lemma 4.1.5.

**Lemma 4.1.5** (King [42]). If $G$ is a graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$, then $G$ contains an independent set $I$ such that $\omega(G - I) < \omega(G)$.

**Proof.** Put $\Delta := \Delta(G)$ and $\omega := \omega(G)$. Let $Q$ be all the maximum cliques in $G$ and $Q_1, \ldots, Q_k$ the vertex sets of the components of $X_Q$. Since the components of $X_Q$ satisfy the hypotheses of Lemma 4.1.7, we have $F_i := \cap Q_i \neq \emptyset$ for all $i \in [k]$. Put $D_i := \bigcup Q_i$.

Put $r := \frac{1}{3}(\Delta + 1)$. Note that the vertices in $F_i$ are universal in $D_i$. Since $|D_i| \geq \omega > \frac{2}{3}(\Delta + 1)$, each $v \in F_i$ has at most $r$ neighbors in the rest of the $F_i$. Applying Lemma 4.1.6 gives $|F_i'| + |D_i| \geq 2 \omega > \frac{2}{3}(\Delta + 1)$. Thus each $v \in F_i$ has at most $\Delta + 1 - |D_i| > \Delta + 1 - \frac{1}{3}(\Delta + 1) - |F_i| = |F_i| - r$ neighbors in the rest of the $F_i$. Applying Lemma 4.1.4 gives an independent set intersecting each $F_i$ and hence every maximum clique in $G$. \[\square\]

### 4.2 Example reductions

#### 4.2.1 The quintessential reduction example

Reed [66] has conjectured that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$ 

If we could always find an independent set whose removal decreased both $\omega$ and $\Delta$, then the conjecture would follow by simple induction since we can give the independent set a single color and use at most $\left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil - 1$ colors on what remains. Expanding the independent set given by Lemma 4.1.5 to a maximal one shows that this sort of argument goes through when $\omega > \frac{2}{3}(\Delta + 1)$. Thus a minimum counterexample to Reed's conjecture satisfies $\omega \leq \frac{2}{3}(\Delta + 1)$. 


4.2.2 Reducing for Brooks

The proof in Chapter 2 reduces Brooks' theorem to the $\Delta = 3$ case by an ad hoc argument. The reduction follows from the general lemmas on hitting maximum cliques as follows. Let $G$ be a counterexample to Brooks' theorem minimizing $\Delta(G)$. Suppose $\Delta(G) \geq 4$. We may assume $G$ is critical. If $\omega(G) < \Delta(G)$, then removing any maximal independent set from $G$ decreases $\chi(G)$ and $\Delta(G)$ both by one giving a counterexample with smaller $\Delta$. Hence $\omega(G) \geq \Delta(G)$. But then $\omega(G) > \frac{2}{3}(\Delta(G) + 1)$ and Lemma 4.1.5 gives us an independent set $I$ such that $\omega(G - I) < \omega(G)$. Let $M$ be a maximal independent set containing $I$. Then $G - M$ is a counterexample with smaller $\Delta$.

4.2.3 Reducing for Borodin-Kostochka

More generally, we can use the facts on hitting maximum cliques to prove the following reduction lemma.

**Definition 4.2.1.** For $k, j \in \mathbb{N}$, let $C_{k,j}$ be the collection of all vertex critical graphs satisfying $\chi = \Delta = k$ and $\omega < k - j$. Put $C_k := C_{k,0}$. Note that $C_{k,j} \subseteq C_{k,i}$ for $j \geq i$.

**Lemma 4.2.1.** Fix $k, j \in \mathbb{N}$ with $k \geq 3j + 6$. If $G \in C_{k,j}$, then there exists $H \in C_{k-1,j}$ such that $H \prec G$.

**Proof.** Let $G \in C_{k,j}$. We first show that there exists a maximal independent set $M$ such that $\omega(G - M) < k - (j + 1)$. If $\omega(G) < k - (j + 1)$, then any maximal independent set will do for $M$. Otherwise, $\omega(G) = k - (j + 1)$. Since $k \geq 3j + 6$, we have $\omega(G) = k - (j + 1) > \frac{2}{3}(k + 1) = \frac{2}{3}(\Delta(G) + 1)$. Thus by Lemma 4.1.5, we have an independent set $I$ such that $\omega(G - I) < \omega(G)$. Expand $I$ to a maximal independent set to get $M$.

Now $\chi(G - M) = k - 1 = \Delta(G - M)$, where the last equality follows from Brooks' theorem and $\omega(G - M) < k - (j + 1) \leq k - 1$. Since $\omega(G - M) < k - (j + 1)$, for any $(k-1)$-critical induced subgraph $H \leq G - M$ we have $H \in C_{k-1,j}$.

As a consequence we get the result of Kostochka that the Borodin-Kostochka conjecture can be reduced to the $k = 9$ case.

**Lemma 4.2.2.** Let $\mathcal{H}$ be a hereditary graph property. For $k \geq 5$, if $\mathcal{H} \cap C_k = \emptyset$, then $\mathcal{H} \cap C_{k+1} = \emptyset$. In particular, to prove the Borodin-Kostochka conjecture it is enough to show that $C_9 = \emptyset$. 


COLORING FROM ALMOST DEGREE Sized PALETTES

The material in this chapter appeared in [20] and is joint work with Dan Cranston.

In this section we use list-coloring lemmas to forbid a large class of graphs from appearing as induced subgraphs of vertex critical graphs satisfying \( \chi = \Delta \). In each case, we assume that such a graph \( H \triangleleft G \) appears as an induced subgraph of such a graph \( G \). By criticality of \( G \), we can color \( G \setminus H \) with \( \Delta - 1 \) colors. If \( H \) can be colored regardless of which colors are forbidden by its colored neighbors in \( G \setminus H \), then we can clearly extend this coloring to all of \( G \). Such \( H \) are precisely the \( d_1 \)-choosable graphs.

We characterize all graphs \( A \ast B \) with \(|A| \geq 2, |B| \geq 2 \) that are not \( d_1 \)-choosable. The characterization is somewhat lengthy, so we split it into a number of lemmas. For the case \(|A| \geq 4, |B| \geq 4 \), see Lemma 5.3.44. When \(|A| = 3 \), we consider the four cases \( A = E_3 \) (Lemma 5.3.39), \( A = \overline{P}_3 \) (Lemma 5.3.43), \( A = P_3 \) (Lemma 5.3.26), and \( A = K_3 \) (Lemma 5.3.29). When \(|A| = 2 \), we consider the case \( A = E_2 \) in Lemma 5.3.22 and the case \( A = K_2 \) in Lemma 5.3.47.

Let \( G \) be a graph. A list assignment to the vertices of \( G \) is a function from \( V(G) \) to the finite subsets of \( \mathbb{N} \). A list assignment \( L \) to \( G \) is good if \( G \) has a coloring \( c \) where \( c(v) \in L(v) \) for each \( v \in V(G) \). It is bad otherwise. We call the collection of all colors that appear in \( L \), the pot of \( L \). That is \( \text{Pot}(L) := \bigcup_{v \in V(G)} L(v) \). For a subgraph \( H \) of \( G \) we write \( \text{Pot}_H(L) := \bigcup_{v \in V(H)} L(v) \). For \( S \subseteq \text{Pot}(L) \), let \( G_S \) be the graph \( G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}] \). We also write \( G_c \) for \( G_{\{c\}} \). We let \( B(L) \) be the bipartite graph that has parts \( V(G) \) and \( \text{Pot}(L) \) and an edge from \( v \in V(G) \) to \( c \in \text{Pot}(L) \) iff \( c \in L(v) \). For \( f : V(G) \to \mathbb{N} \), an \( f \)-assignment on \( G \) is an assignment \( L \) of lists to the vertices of \( G \) such that \( |L(v)| = f(v) \) for each \( v \in V(G) \). We say that \( G \) is \( f \)-choosable if every \( f \)-assignment on \( G \) is good.

### 5.1 Shrinking the pot

In this section we prove a lemma about bad list assignments with minimum pot size. Some form of this lemma has appeared independently in at least two places we know of—Kierstead [37] and Reed and Sudakov [68]. We will use this lemma repeatedly in the arguments that follow.

Given a graph \( G \) and \( f : V(G) \to \mathbb{N} \), we have a partial order on the \( f \)-assignments to \( G \) given by \( L < L' \) iff \( |\text{Pot}(L)| < |\text{Pot}(L')| \). When we talk of minimal \( f \)-assignments, we mean minimal with respect to this partial order.
Lemma 5.1.1. Let $G$ be a graph and $f: V(G) \to \mathbb{N}$. Assume $G$ is not $f$-choosable and let $L$ be a minimal bad $f$-assignment. Assume $L(v) \neq \text{Pot}(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq \text{Pot}(L)$, any coloring of $G_S$ from $L$ uses some color not in $S$.

Proof. Suppose not and let $\emptyset \neq S \subseteq \text{Pot}(L)$ be such that $G_S$ has a coloring $\phi$ from $L$ using only colors in $S$. For $v \in V(G)$, let $h(v)$ be the smallest element of $\text{Pot}(L) - L(v)$ (this is well defined by assumption). Pick some $c \in S$ and construct a new list assignment $L'$ as follows.

$$L'(v) = \begin{cases} L(v) & \text{if } v \in V(G) - V(G_S) \\ L(v) & \text{if } v \in V(G_S) \text{ and } c \notin L(v) \\ (L(v) - \{c\}) \cup \{h(v)\} & \text{if } v \in V(G_S) \text{ and } c \in L(v) \end{cases}$$

Note that $L'$ is an $f$-assignment and $\text{Pot}(L') = \text{Pot}(L) - \{c\}$. Thus, by minimality of $L$, we can properly color $G$ from $L'$. In particular, we have a coloring of $V(G) - V(G_S)$ from $L$ using no color from $S$. We can complete this to a coloring of $G$ from $L$ using $\phi$. This contradicts the fact that $L$ is bad. \qed

When $|S| = 1$, we can say more. We will use the following lemma in the proof that the graph $D_8$ in Figure 6.2 is $d_1$-choosable. It should be useful elsewhere as well.

Lemma 5.1.2. Let $G$ be a graph and $f: V(G) \to \mathbb{N}$. Suppose $G$ is not $f$-choosable and let $L$ be a minimal bad $f$-assignment. Then for any $c \in \text{Pot}(L)$, there is a component $H$ of $G_c$ such that $\text{Pot}_H(L) = \text{Pot}(L)$. In particular, $\text{Pot}_{G_c}(L) = \text{Pot}(L)$.

Proof. Suppose otherwise that we have $c \in \text{Pot}(L)$ such that $\text{Pot}_H(L) \subsetneq \text{Pot}(L)$ for all components $H$ of $G_c$. Say the components of $G_c$ are $H_1, \ldots, H_t$. For $i \in [t]$, choose $\alpha_i \in \text{Pot}(L) - \text{Pot}_{H_i}(L)$. Now define a list assignment $L'$ on $G$ by setting $L'(v) := L(v)$ for all $v \in V(G) - V(G_c)$ and for each $i \in [t]$ setting $L'(v) := (L(v) - c) \cup \{\alpha_i\}$ for each $v \in V(H_i)$. Then $|\text{Pot}(L')| < |\text{Pot}(L)|$ and hence by minimality $L$ we have an $L'$-coloring $\pi$ of $G$. Plainly $Q := \{v \in V(G_c) \mid \pi(v) = \alpha_i \text{ for some } i \in [t]\}$ is an independent set. Since $c$ doesn’t appear outside $G_c$, we can recolor all vertices in $Q$ with $c$ to get an $L$-coloring of $G$. This contradicts the fact that $L$ is bad. \qed

Definition 5.1.1. A bipartite graph with parts $A$ and $B$ has positive surplus (with respect to $A$) if $|N(X)| > |X|$ for all $\emptyset \neq X \subseteq A$.

Lemma 5.1.3. Let $G$ be a graph and $f: V(G) \to \mathbb{N}$. Assume $G$ is not $f$-choosable and let $L$ be a minimal bad $f$-assignment. Assume $L(v) \neq \text{Pot}(L)$ for each $v \in V(G)$. Then $B(L)$ has positive surplus (with respect to $\text{Pot}(L)$).

Proof. Suppose not and choose $\emptyset \neq X \subseteq \text{Pot}(L)$ such that $|N(X)| \leq |X|$ minimizing $|X|$. If $|X| = 1$, then $G_X$ can be colored from $X$ contradicting Lemma 5.1.1. Hence $|X| \geq 2$. By minimality of $|X|$, for any $Y \subset X$, $|N(Y)| \geq |Y| + 1$. Hence, for any $x \in X$, we have $|N(X)| \geq |N(X - \{x\})| \geq |X - \{x\}| + 1 = |X|$. Thus, by Hall’s Theorem, we have a matching of $X$ into $N(X)$, but $|N(X)| \leq |X|$ so this gives a coloring of $G_X$ from $X$ contradicting Lemma 5.1.1. \qed

Our approach to coloring a graph (particularly a join) will often be to consider nonadjacent vertices $u$ and $v$ and show that their lists contain a common color. By the pigeonhole principle, this follows immediately when $|L(u)| + |L(v)| > |\text{Pot}(L)|$. We will use the following lemma frequently throughout the remainder of this paper.

Small Pot Lemma. Let $G$ be a graph and $f: V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If $G$ is not $f$-choosable, then $G$ has a minimal bad $f$-assignment $L$ such that $|\text{Pot}(L)| < |G|$.

Proof. Suppose not and let $L$ be a minimal bad $f$-assignment. For each $v \in V(G)$ we have $|L(v)| = f(v) < |G| \leq |\text{Pot}(L)|$ and hence $L(v) \neq \text{Pot}(L)$. Thus by Lemma 5.1.3 we have the contradiction $|G| \geq |N(\text{Pot}(L))| > |\text{Pot}(L)|$. \qed
5.2 Degree choosability

Definition 5.2.1. Let $G$ be a graph and $r \in \mathbb{Z}$. Then $G$ is $d_r$-choosable if $G$ is $f$-choosable where $f(v) = d(v) - r$.

Note that a vertex critical graph with $\chi = \Delta + 1 - r$ contains no induced $d_r$-choosable subgraph. Since we are working to prove the Borodin-Kostochka conjecture, we will focus on the case $r = 1$ and primarily study $d_1$-choosable graphs. For $r = 0$, we have the following well known generalization of Brooks’ Theorem due independently to Borodin [9] and Erdős, Rubin and Taylor [24].

5.2.1 Degree-choosable graphs

Definition 5.2.2. A Gallai tree is a graph all of whose blocks are complete graphs or odd cycles.

Classification of $d_0$-choosable graphs. For any connected graph $G$, the following are equivalent.

- $G$ is $d_0$-choosable.
- $G$ is not a Gallai tree.
- $G$ contains an induced even cycle with at most one chord.

In [49], Kostochka, Stiebitz and Wirth give a short proof of the equivalence of (1) and (2) as well as extending the result to hypergraphs. In [35], Hladký, Král and Schauz gave an algebraic proof (Rubin’s block theorem below plus the Alon-Tarsi theorem [3]), it also works for paintability (online list-choosability), see [69].

We give some lemmas about $d_0$-assignments that will be useful in the later study of general $d_k$-assignments. Combined with the following structural result, these lemmas give a quick proof of the classification of $d_0$-choosable graphs. The following lemma from [24] is due to Rubin. For other proofs, see [23] and [35].

Lemma 5.2.1. Any 2-connected graph is complete, an odd cycle or contains an induced even cycle with at most one chord.

Proof. Suppose not and choose a counterexample $G$ minimizing $|G|$. Since $G$ is 2-connected and not complete, it contains an induced cycle $C$ of length at least four. Then $C$ is an induced odd cycle and thus $G - C$ is not empty. Since $G$ is 2-connected, we may choose a shortest $C$-path in $G$ with distinct ends in $C$—call it $R$. Since $G[V(C) \cup V(R)]$ is 2-connected, by minimality of $|G|$, $V(G) = V(C) \cup V(R)$.

First suppose $R$ has length at least 3. Then since $R$ is shortest, $G = C \cup R$ and thus one of the small cycles in $C \cup R$ is an even induced cycle or the large cycle is an even induced cycle with at most one chord, giving a contradiction.

Thus $R$ has length 2. Let $z$ be the vertex on $R$ in $G - C$. If $z$ has only two neighbors in $C$, then we get a contradiction as in the previous paragraph. Thus $z$ has at least three neighbors $a, b, c \in V(C)$. Now $|C| \geq 4$ since $G$ is not complete. Thus, without loss of generality, the vertices between $a$ and $b$ on $C$ in cyclic order are $w_1, \ldots, w_k$ with $k \geq 1$. But $G - \{w_1, \ldots, w_k\}$ is 2-connected, not complete, and not an odd cycle. Hence, by minimality of $|G|$, $G - \{w_1, \ldots, w_k\}$ contains an induced even cycle with at most one chord. This final contradiction completes the proof.

The following lemma was used in [49].

Lemma 5.2.2. A connected graph is $d_0$-choosable iff it contains a $d_0$-choosable induced subgraph.
Proof. The forward direction is plain. For the reverse, let \( H \leq G \) be \( d_0 \)-choosable. Since \( G \) is connected, we can order \( V(G) \) such that each vertex in \( V(G - H) \) has a neighbor after it and \( V(H) \) comes last. Coloring \( V(G - H) \) greedily from the lists in this order leaves a \( d_0 \)-assignment on \( H \) which we can complete by assumption.

Lemma 5.2.3. Let \( L \) be a bad \( d_0 \)-assignment on a connected graph \( G \) and \( x \in V(G) \) a noncutvertex. Then \( L(x) \subseteq L(y) \) for each \( y \in N(x) \).

Proof. Suppose otherwise that we have \( c \in L(x) - L(y) \) for some \( y \in N(x) \). Coloring \( x \) with \( c \) leaves at worst a \( d_0 \)-assignment \( L' \) on the connected \( H := G - x \) where \( |L'(y)| > d_H(y) \). But then we can complete the coloring, a contradiction.

Lemma 5.2.4. Any even subdivision of a bridgeless \( d_0 \)-choosable graph is \( d_0 \)-choosable.

Proof. Since subdividing an edge cannot create a bridge, it suffices to show that subdividing an edge with two vertices preserves \( d_0 \)-choosability. Let \( G \) be a bridgeless \( d_0 \)-choosable graph. Suppose there exists \( xy \in E(G) \) such that subdividing \( xy \) with vertices \( w \) and \( z \) creates a graph \( H \) which is not \( d_0 \)-choosable. Let \( L \) be a bad \( d_0 \)-assignment on \( H \). Since \( G \) is bridgeless, \( w \) and \( z \) are not cutvertices of \( H \). By Lemma 5.2.3, \( L(w) = L(z) \). But \( L \) restricted to \( G \) is a \( d_0 \)-assignment, so we have a coloring \( \pi \) of \( H - \{w, z\} \) from \( L \) such that \( \pi(x) \neq \pi(y) \). Now \( L(w) - \{\pi(x)\} \neq L(z) - \{\pi(y)\} \) so we can complete the coloring to all of \( H \), a contradiction.

Using the Small Pot Lemma it is easy to prove that \( C_4 \) and \( K_4^- \) are \( d_0 \)-choosable which combined with Lemma 5.2.4 shows that every even cycle with at most one chord is \( d_0 \)-choosable. It turns out that the conclusion of the Small Pot Lemma holds for general bad \( d_0 \)-assignments, not just minimal ones. We will use the following lemma often in proofs when we end up with a bad \( d_0 \)-assignment that may not be minimal.

Lemma 5.2.5. If \( L \) a bad \( d_0 \)-assignment on a connected graph \( G \), \( |\text{Pot}(L)| < |G| \).

Proof. Suppose that the lemma is false and choose a connected graph \( G \) together with a bad \( d_0 \)-assignment \( L \) where \( |\text{Pot}(L)| \geq |G| \) minimizing \( |G| \). Plainly, \( |G| \geq 2 \). Let \( x \in G \) be a noncutvertex (any end block has at least one). By Lemma 5.2.3, \( L(x) \subseteq L(y) \) for each \( y \in N(x) \). Thus coloring \( x \) decreases the pot by at most one, giving a smaller counterexample. This contradiction completes the proof.

Proof of the classification of \( d_0 \)-choosable graphs. It is easy to construct a bad \( d_0 \)-assignment on a Gallai tree—hence (1) implies (2). Now if a graph is not a Gallai tree, then some block is neither complete nor an odd cycle. But then, by Lemma 5.2.1, that block contains an induced even cycle with at most one chord. Hence (2) implies (3).

Now we prove that \( C_4 \) and \( K_4^- \) are \( d_0 \)-choosable. If not, then we have a bad \( d_0 \)-assignment \( L \) on \( C_4 \) or \( K_4^- \). By Lemma 5.2.5, \( |\text{Pot}(L)| \leq 3 \). Hence some nonadjacent pair can be colored the same leaving a \( d_1 \)-assignment on the components which can be easily completed.

Thus, by Lemma 5.2.4, any even cycle with at most one chord is \( d_0 \)-choosable. Combining this with Lemma 5.2.2 proves that (3) implies (1).

5.2.2 Basic properties

We also need a few basic lemmas about how \( d_r \)-choosability behaves with respect to induced subgraphs.

Lemma 5.2.6. Fix \( r \geq 0 \). Let \( G \) be a graph and \( H \leq G \) a \( d_r \)-choosable subgraph. If \( L \) is a \( d_r \)-assignment on \( G \) and \( G - H \) is properly colorable from \( L \), then \( G \) is properly colorable from \( L \).

Proof. Color \( G - H \) from \( L \). Let \( L' \) be the resulting list assignment on \( H \). Since each \( v \in V(H) \) must be adjacent to as many vertices as colors in \( G - H \) we see that \( L' \) is again a \( d_r \)-assignment. The lemma follows.
Lemma 5.2.7. Fix $r \geq 0$. Let $G$ be a graph and $H \subseteq G$ a $d_r$-choosable subgraph. If there exists an ordering $v_1, \ldots, v_n$ of the vertices of $G - H$ such that $v_i$ has degree at least $r + 1$ in $G[V(H) \cup \bigcup_{1 \leq j \leq n} v_j]$ for each $i$, then $G$ is $d_r$-choosable.

Proof. Let $L$ be a $d_r$-assignment on $G$. Go through $G - H$ in order $v_1, \ldots, v_n$ coloring $v_i$ with the smallest available color in $L(v_i)$. Since when we go to color $v_i$, it has at least $r + 1$ uncolored neighbors we succeed in coloring $G - H$. Now the lemma follows from Lemma 5.2.6. 

We will also use the following immediate consequence of the pigeonhole principle.

Lemma 5.2.8. If $S_1, \ldots, S_m$ are nonempty subsets of a finite set $T$ and $\sum_{i \geq 1} |S_i| > (m - 1)|T|$, then $\bigcap_{i \geq 1} S_i \neq \emptyset$.

5.3 Handling joins

The main result of this section is Lemma 5.3.8, which plays a key role in our classification of bad graphs $A \ast B$. Specifically, Lemma 5.3.8 is essential to the proof of Lemma 5.3.18, which considers the case when $|A| \geq 4$ and $B$ is arbitrary.

Lemma 5.3.1. Fix $r \geq 0$. Let $A$ be a graph with $|A| \geq r + 1$ and $B$ a nonempty graph. If $A \ast B$ is $d_r$-choosable, then $A \ast C$ is $d_r$-choosable for any graph $C$ with $B \subseteq C$.

Proof. Assume $A \ast B$ is $d_r$-choosable and let $C$ be a graph with $B \subseteq C$. Put $H = C - B$. For each $v \in V(H)$, $|L(v)| \geq d(v) - r \geq d_H(v) + r + 1 - r = d_H(v) + 1$. Thus we may color $H$ from its lists. By Lemma 5.2.6, we can complete the coloring to all of $A \ast C$. 

Lemma 5.3.2. Fix $r \geq 0$. Let $A$ be a graph with $|A| \geq r$ and $B$ a nonempty graph. If $A \ast B$ is $d_r$-choosable, then $A \ast C$ is $d_r$-choosable for any connected graph $C$ with $B \subseteq C$.

Proof. Assume $A \ast B$ is $d_r$-choosable and let $C$ be a connected graph with $B \subseteq C$. Put $H = C - B$. For each $v \in V(H)$, $|L(v)| \geq d(v) - r \geq d_H(v) + r - r = d_H(v)$. Since $C$ is connected, each component of $H$ has a vertex $v$ that hits a vertex in $B$ and hence has $|L(v)| \geq d_H(v) + 1$. Thus we may color $H$ from its lists. By Lemma 5.2.6, we can complete the coloring to all of $A \ast C$. 

Lemma 5.3.3. Fix $r \geq 0$. Let $G$ be a $d_{r-1}$ choosable graph with at least $2r + 2$ vertices. Then $E_2 \ast G$ is $d_r$-choosable.

Proof. Let $x, y$ be the vertices in the $E_2$. Suppose $E_2 \ast G$ is not $d_r$-choosable. Then by the Small Pot Lemma, we have a $d_r$-assignment $L$ with $|\text{Pot}(L)| < 2 + |G|$. Now $|L(x)| + |L(y)| \geq d(x) + d(y) - 2r \geq 2|G| - 2r \geq 2 + |G| > |\text{Pot}(L)|$, since $|G| \geq 2r + 2$. Thus we can use a single common color on $x$ and $y$, leaving a $d_{r-1}$-assignment on $G$. We may now complete the coloring, giving a contradiction. 

Since every graph is $d_{r-1}$-choosable we get the following immediately.

Corollary 5.3.4. For $r \geq 0$, both $E_2^{r+2}$ and $E_2^{r+1} \ast K_2$ are $d_r$-choosable.

Note that this is equivalent to the following fundamental result of Erdős, Rubin and Taylor [24].

Lemma 5.3.5. For all $n \geq 1$, $E_n^n$ is $n$-choosable.

Lemma 5.3.6. Fix $r \geq 0$. Let $A$ be a graph with $|A| \geq 3r + 2$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_r$-choosable, then $\omega(B) \geq |B| - 2r$. 


Proof. Suppose $G := A \ast B$ is not $d_r$-choosable and let $L$ be a minimal bad $d_r$-assignment. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |G| - 1$. Let $g: S \rightarrow \text{Pot}_S(L)$ be a partial coloring of $B$ from $L$ maximizing $|S| - |\text{im}(g)|$ and then minimizing $|S|$. Color $S$ using $g$ and let $L'$ be the resulting list assignment.

Put $H := G - S$ and $C := B - S$. First suppose that $|S| - |\text{im}(g)| \geq r + 1$. For each $v \in C$ we have $|L'(v)| \geq d_C(v) - r + 3r + 2 > d_C(v)$, so we can complete $g$ to $C$. This leaves each $v \in V(A)$ with a list of size at least $d_A(v) - r + |S| - |\text{im}(g)| > d_A(v)$. Hence, we can complete the coloring to all of $G$. Thus $L$ is not bad after all, giving a contradiction.

So instead we assume that $|S| - |\text{im}(g)| \leq r$. By the minimality condition on $|S|$ we see that $g$ has no singleton color classes. In particular, $|S| \geq 2|\text{im}(g)|$. By combining this inequality with $|S| - |\text{im}(g)| \leq r$, we get $|S| \leq 2r$. Since $|C| = |B| - |S| \geq |B| - 2r$, the conclusion will follow if we can show that $C$ is complete.

By definition, $|\text{Pot}(L')| = |\text{Pot}(L)| - |\text{im}(g)|$. By the maximality condition on $g$, every pair of nonadjacent vertices in $C$ must have disjoint lists under $L'$ (otherwise we could use a common color on nonadjacent vertices in $C$ and increase $|S| - |\text{im}(g)|$). Let $I$ be a maximal independent set in $C$. To reach a contradiction, we assume that $|I| \geq 2$. Then for all the elements of $I$ to have disjoint lists, we must have

$$
\sum_{v \in I} |L'(v)| \leq |\text{Pot}(L')| \leq |\text{Pot}(L)| - |\text{im}(g)| = |\text{Pot}(L')| \leq |\text{Pot}(L)| - |\text{im}(g)|.$$

Thus this final inequality contradicts our assumption that $|S| - |\text{im}(g)| \leq r$. Hence $|I| \leq 1$; that is, $C$ is complete.

Lemma 5.3.7. Fix $r \geq 1$. Let $A$ be a connected graph and $B$ an arbitrary graph such that $A \ast B$ is not $d_r$-choosable. Let $L$ be a minimal bad $d_r$-assignment on $A \ast B$. If $B$ is colorable from $L$ using at most $|B| - r$ colors, then $|\text{Pot}(L)| \leq |A| + |B| - 2$.

Proof. To get a contradiction suppose that $|\text{Pot}(L)| \geq |A| + |B| - 1$ and that $B$ is colorable from $L$ using at most $|B| - r$ colors. If $|\text{Pot}_A(L)| \geq |\text{Pot}(L)| + 1 - r$, then coloring $B$ with at most $|B| - r$ colors leaves at worst a $d_0$-assignment $L'$ on $A$ with $|\text{Pot}(L')| \geq |A|$. Hence the coloring can be completed to $A$ by Lemma 5.2.5, a contradiction.

Thus we may assume that $|\text{Pot}_A(L)| \leq |\text{Pot}(L)| - r$. Put $S := \text{Pot}(L) - \text{Pot}_A(L)$. Let $\pi$ be a coloring of $B$ from $L$ using at most $|B| - r$ colors, say $\pi$ uses colors $C$. Then $|C| = |B| - r$ and $S \cap C = \emptyset$ for otherwise coloring $B$ leaves at worst a $d_{-1}$-assignment on $A$. Also, $\pi^{-1}(c) \subseteq V(G_S)$ for any $c \in C$ since otherwise we could recolor $\pi^{-1}(c)$ with colors from $S$ to get at worst a $d_{-1}$-assignment on $A$. In particular, $|G_S| \leq \sum_{c \in C} (|\pi^{-1}(c)| - 1) = |B| - |C| = r \leq |S|$. But this inequality contradicts Lemma 5.1.3.

We now use Lemma 5.3.7 to strengthen Lemma 5.3.6.
**Lemma 5.3.8.** Fix $r \geq 1$. Let $A$ be a connected graph with $|A| \geq 3r + 1$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_r$-choosable, then $\omega(B) \geq |B| - 2r$.

**Proof.** Suppose $G := A \ast B$ is not $d_r$-choosable and let $L$ be a minimal bad $d_r$-assignment. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |G| - 1$. Let $g: S \rightarrow \text{Pot}_S(L)$ be a partial coloring of $B$ from $L$ maximizing $|S| - |\text{im}(g)|$ and then minimizing $|S|$. Color $S$ using $g$ and let $L'$ be the resulting list assignment.

Put $C := B - S$. Running through the argument in Lemma 5.3.6 again completes the proof. \[\square\]

### 5.3.1 The $r = 1$ case

**Some preliminary tools**

The Small Pot Lemma says that if $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |A| + |B| - 1$. In this section, we study conditions under which $|\text{Pot}(L)| \leq |A| + |B| - 2$. We also prove a key lemma for coloring graphs of the form $K_1 \ast B$. In the following section, our results here help us to find nonadjacent vertices with a common color.

**Lemma 5.3.9.** Let $A$ be a graph with $|A| \geq 2$, $B$ an arbitrary graph and $L$ a $d_1$-assignment on $A \ast B$. If $B$ has an independent set $I$ such that $(|A| - 1)|I| + |E_B(I)| > |\text{Pot}(L)|$, then $B$ can be colored from $L$ using at most $|B| - 1$ colors.

**Proof.** Suppose that $B$ has an independent set $I$ such that $(|A| - 1)|I| + |E(I)| > |\text{Pot}(L)|$. Now

\[
\sum_{v \in I} |L(v)| = \sum_{v \in I} (d(v) - 1) = (|A| - 1)|I| + \sum_{v \in I} d_B(v) = (|A| - 1)|I| + |E_B(I)| > |\text{Pot}(L)|.
\]

Hence we have distinct $x, y \in I$ with a common color $c$ in their lists. So we color $x$ and $y$ with $c$. Since $|A| \geq 2$, this leaves at worst a $d_1$-assignment on the rest of $B$. Completing the coloring to the rest of $B$ gives the desired coloring of $B$ from $L$ using at most $|B| - 1$ colors. \[\square\]

**Lemma 5.3.10.** Let $G$ be a graph and $I$ a maximal independent set in $G$. Then $|E(I)| \geq |G| - |I|$. If $I$ is maximum and $|E(I)| = |G| - |I|$, then $G$ is the disjoint union of $|I|$ complete graphs.

**Proof.** Each vertex in $G - I$ is adjacent to at least one vertex in $I$. Hence $|E(I)| \geq |G| - |I|$. Now assume $I$ is maximum and $|E(I)| = |G| - |I|$. Then $N(x) \cap N(y) = \emptyset$ for every distinct pair $x, y \in I$. Also, $N(x)$ must be a clique for each $x \in I$, since otherwise we could swap $x$ out for a pair of nonadjacent neighbors and get a larger independent set. Since we can swap $x$ with any of its neighbors to get another maximum independent set, we see that $G$ has components $\{G[v] \cup N(v) \mid v \in I\}$. \[\square\]

**Lemma 5.3.11.** Let $A$ be a connected graph with $|A| \geq 4$ and $B$ an incomplete graph. If $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |A| + |B| - 2$.

**Proof.** Suppose $A \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |A| + |B| - 1$. Let $I$ be a maximum independent set in $B$. Since $B$ is incomplete, $|I| = \alpha(B) \geq 2$. By Lemma 5.3.10, $|E_B(I)| \geq |B| - |I| = |B| - \alpha(B)$. As $|A| \geq 4$ we have $(|A| - 1)|I| + |E_B(I)| \geq (|A| - 1)\alpha(B) + |B| - \alpha(B) \geq (|A| - 2)\alpha(B) + |B| \geq 2|A| - 4 + |B| > |A| + |B| - 1 \geq |\text{Pot}(L)|$. Hence by Lemma 5.3.9, $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7. \[\square\]
Lemma 5.3.12. Let $A$ be a connected graph with $|A| = 3$ and $B$ a graph that is not the disjoint union of at most two complete subgraphs. If $A \ast B$ is not $d_1$-choosable, then $A \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |B| + 1$.

Proof. Suppose $A \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |B| + 2$.

Let $I$ be a maximum independent set in $B$. Since $B$ is not the disjoint union of at most two complete subgraphs, Lemma 5.3.10 implies that either $|E(I)| > |B| - |I|$ or $|I| \geq 3$. In the first case, $2|I| + |E(I)| > 2|I| + |B| - |I| \geq 2 + |B| \geq |\text{Pot}(L)|$. In the second case, $2|I| + |E(I)| \geq 2|I| + |B| - |I| \geq 3 + |B| > |\text{Pot}(L)|$.

Thus by Lemma 5.3.9, $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7.

Lemma 5.3.13. Let $B$ be a graph containing an induced claw, $C_4$, $K_4^-$, bull, or $2P_3$. If $K_2 \ast B$ is not $d_1$-choosable, then $K_2 \ast B$ has a minimal bad $d_1$-assignment $L$ such that $|\text{Pot}(L)| \leq |B|$.

Proof. Suppose $K_2 \ast B$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $K_2 \ast B$. Then, by the Small Pot Lemma, $|\text{Pot}(L)| \leq |B| + 1$.

Let $H$ be an induced claw, $C_4$, $K_4^-$, bull, or $2P_3$ in $B$ and $M$ a maximum independent set in $H$. Expand $M$ to a maximal independent set $I$ in $B$. We can easily verify that in each case $|E_B(M)| \geq |H| - |M| + 2$, which implies that $|E_B(I)| \geq |B| - |I| + 2$. Hence we have $(|K_2| - 1)|I| + |E_B(I)| \geq 3|I| + |B| + 2 = |B| + 2 > |\text{Pot}(L)|$. Now by Lemma 5.3.9, $B$ can be colored from $L$ using at most $|B| - 1$ colors. But then we are done by Lemma 5.3.7.

Theorem 5.3.14. Let $A$ and $B$ be graphs such that $G := A \ast B$ is not $d_1$-choosable. If either $|A| \geq 2$ or $B$ is $d_0$-choosable and $L$ is a bad $d_1$-assignment on $G$, then

1. for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$; and

2. for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds

   (a) $L(x_1) \cap L(y_1) = \emptyset$;

   (b) $L(x_2) \cap L(y_2) = \emptyset$;

   (c) $|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.

Proof. First, suppose $|A| \geq 2$. If (1) fails for $I$, then color all vertices in $I$ the same, complete the coloring to the rest of $B$ and then to $A$. If (2) fails for $\{x_1, y_1\}$ and $\{x_2, y_2\}$, color $x_1, y_1$ with $c_1 \in L(x_1) \cap L(y_1)$ and $x_2, y_2$ with $c_2 \in L(x_2) \cap L(y_2) - \{c_1\}$, complete the coloring to the rest of $B$ and then to $A$. The more difficult case is when $|A| = 1$, we handle it as follows.

For both (1) and (2) we prove the contrapositive.

(1) Suppose that $B$ has an independent set $I$ of size 3 such that there exists a color $c$ that appears in the list of each vertex in $I$; let $I = \{v_1, v_2, v_3\}$. Since $B$ is $d_0$-choosable, $B$ has an $L$-coloring. We will modify this coloring to get an $L$-coloring that uses $c$ on at least three vertices.

For each $v_i$ in $I$, if $v_i$ does not have a neighbor with color $c$, we recolor $v_i$ with $c$. If $c$ now appears three or more times in our current coloring, then we are done. Assume that $c$ appears on either a single vertex $w_1$ or on two vertices $w_1$ and $w_2$.

If both $w_1$ and $w_2$ have two neighbors in $I$, then we uncolor $w_1$ and $w_2$ and use color $c$ on all vertices of $I$. Otherwise, there exists a single vertex, say $w_1$, with at least two neighbors
in $I$ for which $w_1$ is their only neighbor with color $c_1$. Uncolor $w_1$ and now use color $c$ on all of its neighbors in $I_1$ that no longer have a neighbor with color $c_1$. Since each uncolored vertex has at least two neighbors with color $c$, we can extend the coloring to all of $B$. Now since color $c$ is used 3 or more times on $B$, at most $|G| - 2$ colors are used on $G$, so we can extend the coloring to $A$.

(2) Suppose that $B$ has two disjoint independent sets $I_1$ and $I_2$ each of size 2 and there exist distinct colors $c_1$ and $c_2$ such that (for each $i \in \{1, 2\}$) color $c_i$ appears in the lists of both vertices of $I_i$. Since $B$ is $d_0$-choosable, $B$ has an $L$-coloring. We will show that $B$ has an $L$-coloring in which colors $c_1$ and $c_2$ each appear twice (or one appears at least three times). We will modify our coloring using recoloring arguments similar to that above, although we may need to recolor repeatedly. (If at any point our coloring of $B$ uses a single color three or more times, then we can stop, since we will be able to extend this coloring to $A$.)

If $c_1$ does not appear in our coloring, then we recolor some vertex of $I_1$ with $c_1$. Suppose that color $c_1$ appears only once in our coloring, say on vertex $u$. Either we can recolor some vertex in $I_1$ with $c_1$ or else both vertices in $I_1$ are adjacent to $u$. In this case, we uncolor $u$ and use $c_1$ on both vertices of $I_1$. Now we have some color available for $u$. Thus, we may assume that our coloring uses $c_1$ on exactly two vertices. If neither of these vertices with $c_1$ are in $I_2$, then we can use the same recoloring trick for color $c_2$. Neither vertex with $c_1$ will get recolored, so afterwards both colors $c_1$ and $c_2$ will appear on two vertices (and we'll be able to extend the coloring to $A$).

Suppose instead that both vertices with color $c_1$ are in $I_2$. If neither vertex in $I_2$ is adjacent to a vertex where color $c_2$ is used, then we can recolor both of them with $c_2$. Next we can again apply the recoloring trick for color $c_1$. Since the vertices in $I_2$ with color $c_2$ will not get recolored, this will yield the desired coloring that uses each of $c_1$ and $c_2$ twice. So suppose that $c_1$ is used on both vertices in $I_2$ and $c_2$ is used on a vertex adjacent to at least one vertex in $I_2$. Since we may assume that $c_2$ appears on only one vertex, when we use the recoloring trick for $c_2$, we will color at least one vertex of $I_2$ with $c_2$. Thus, we may assume (up to symmetry of $I_1$ and $I_2$) that color $c_1$ appears on two vertices and that exactly one of them is in $I_2$; we may also assume that color $c_2$ appears on exactly one vertex.

We will show that after applying the recoloring trick at most three times we will get a coloring of $B$ that uses $c_1$ on two vertices and uses $c_2$ on two vertices. We call a vertex $v \in I_1$ miscolored if it is colored with color $c_{3-i}$. We will see that each time we apply the recoloring trick, either we increase the total number of vertices colored with $c_1$ and $c_2$ or else we decrease the number of miscolored vertices. Since we begin with at most two miscolored vertices, after applying the recoloring trick at most three times, our coloring will use colors $c_1$ and $c_2$ each twice (and we will be done).

Assume that $c_1$ appears on two vertices and exactly one of them is miscolored; assume that $c_2$ appears on exactly one vertex, which may or may not be miscolored. When we apply the recoloring trick for $c_2$, we increase the number of vertices using $c_2$. Thus, we are done unless we decrease the number of vertices using $c_1$. Since we only remove color $c_1$ from vertices in $I_2$, we conclude that we’ve reduced the number of miscolored vertices. We now apply the recoloring trick for $c_1$. Again, we are done unless we’ve recolored a miscolored vertex. So assume that we did. Since we have no remaining miscolored vertices, when we now apply the recoloring trick for $c_2$, we get a coloring that uses each of $c_1$ and $c_2$ twice. Thus, we can extend the coloring of $B$ to $A$. 

A simple variation of the (1) case in the above together with Lemma 5.3.7 gives the following pot-shrinking lemma for $K_1 \ast H$.

**Lemma 5.3.15.** Let $H$ be a $d_0$-choosable graph such that $G := K_1 \ast H$ is not $d_1$-choosable and $L$ a minimal bad $d_1$-assignment on $G$. If some nonadjacent pair in $H$ have intersecting lists, then $|\text{Pot}(L)| \leq |H| - 1$.

With the same proof, we have the following.
Lemma 5.3.16. Let $H$ be a $d_0$-choosable graph such that $G := K_1 \ast H$ is not $f$-choosable where $f(v) \geq d(v)$ for the $v$ in the $K_1$ and $f(v) \geq d(x) - 1$ for $x \in V(H)$. If $L$ is a minimal bad $f$-assignment on $G$, then all nonadjacent pairs in $H$ have disjoint lists.

Lemma 5.3.17. Let $A$ be a connected graph, let $G = A \ast B$, and suppose that either $B$ is $d_0$-choosable or $|A| \geq 2$. (1) Let $L$ be a $d_1$-assignment to $G$. If $B$ contains disjoint independent sets $I_1$ and $I_2$ such that $\sum_{v \in I_1} (d(v) - 1) \geq |\text{Pot}(L)| + 1$ and $\sum_{v \in I_2} (d(v) - 1) \geq |\text{Pot}(L)| + 2$, then $A \ast B$ has an $L$-coloring. (2) In particular, if $B$ contains disjoint independent sets $I_1$ and $I_2$ such that $\sum_{v \in I_1} (d(v) - 1) \geq |G| - 1$ and $\sum_{v \in I_2} (d(v) - 1) \geq |G|$, then $A \ast B$ is $d_1$-choosable.

Proof. Let $L$ be a bad $d_1$-list assignment. We prove (1) and (2) simultaneously. By the Small Pot Lemma, $|\text{Pot}(L)| < |G|$. Thus, since $\sum_{v \in I_2} (d(v) - 1) > |\text{Pot}(L)|$, we see that some color $\alpha$ appears on nonadjacent vertices in $I_2$. Either $B$ is $d_0$-choosable or $|A| \geq 2$, so using either Lemma 5.3.15 or Lemma 5.3.7, we get that $|\text{Pot}(L)| = |G| - 2$, so $|G| - 1 \geq |\text{Pot}(L)| + 1$.

Since $\sum_{v \in I_1} (d(v) - 1) \geq |\text{Pot}(L)| + 1$, we see that two vertices of $I_1$ have a common color $\beta$. If $\beta$ appears 3 times in $I_2$, then we are done by Lemma 5.3.14. Otherwise, we use $\beta$ on the vertices of $I_1$ where it appears. After deleting $\beta$ from the lists of $I_2$, we can find a common color on two vertices of $I_2$. Again we are done, by Lemma 5.3.14.

A classification

In this section we classify the $d_1$-choosable graphs of the form $A \ast B$ where $|A| \geq 2$ and $|B| \geq 2$. When $|A| \geq 4$ and $A$ is connected (or similarly for $B$), the characterizations follows from Lemma 5.3.20 and Corollary 5.3.25. The remainder of the section considers the case when each of $A$ and $B$ is small and/or disconnected.

Definition 5.3.1. A graph $G$ is almost complete if $\omega(G) \geq |G| - 1$.

Lemma 5.3.18. Let $A$ be a connected graph with $|A| \geq 4$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_1$-choosable, then $A \ast B$ is $E_3 \ast K_{|B| - 3}$ or almost complete.

Proof. Suppose $A \ast B$ is not $d_1$-choosable and $B$ is neither $E_3 \ast K_{|B| - 3}$ nor almost complete.

Then, by Lemma 5.3.8, we have $\omega(B) = |B| - 2$.

Let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by Lemma 5.3.11, $|\text{Pot}(L)| \leq |A| + |B| - 2$. Choose $x_1, x_2 \in V(B)$ so that $B - \{x_1, x_2\}$ is complete. Since $B$ is not $E_3 \ast K_{|B| - 3}$, we have $x'_1, x'_2 \in V(B)$ such that $\{x_1, x'_1\}$ and $\{x_2, x'_2\}$ are disjoint pairs of nonadjacent vertices. We have $|L(x_i)| + |L(x'_i)| \geq d(x_i) + d(x'_i) - 2 \geq 2|A| + dB(x_i) + |B| - 5$.

First suppose $dB(x_i) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, suppose $i = 1$. Then $|L(x_i)| + |L(x'_i)| \geq |\text{Pot}(L)| + 2$ and $|L(x_2)| + |L(x'_2)| \geq |\text{Pot}(L)| + 1$. Hence we have different colors $c_1, c_2$ such that $c_1 \in L(x_1) \cap L(x'_1)$ and $c_2 \in L(x_2) \cap L(x'_2)$. Coloring the pairs with these colors leaves a list assignment that is easily completable to all of $A \ast B$.

Hence we must have $dB(x_1) = dB(x_2) = 0$. But then $|L(x_i)| + |L(x'_i)| \geq |\text{Pot}(L)| + 1$ for each $i \in \{1, 2\}$ and thus both $L(x_1) \cap L(x'_1)$ and $L(x_2) \cap L(x'_2)$ are nonempty. If they have different colors in common, we can finish as above. If they have the same color $c$ in common, then coloring $x_1, x_2$, and $x'_1$ with $c$ leaves a list assignment that is easily completable to all of $A \ast B$.

Lemma 5.3.19. Let $A$ be a connected graph with $|A| \geq 6$ and $B$ an arbitrary graph. If $A \ast B$ is not $d_1$-choosable, then $B$ is almost complete.

Proof. Suppose $A \ast B$ is not $d_1$-choosable. By Lemma 5.3.18, $B$ is $E_3 \ast K_{|B| - 3}$ or almost complete. Suppose that $B$ is $E_3 \ast K_{|B| - 3}$ and let $x_1, x_2, x_3$ be the vertices in the $E_3$.

Let $L$ be a minimal bad $d_1$-assignment on $A \ast B$. Then, by Lemma 5.3.11, $|\text{Pot}(L)| \leq |A| + |B| - 2$. We have $\sum_{i=1}^{3} |L(x_i)| \geq \sum_{i=1}^{3} (d(x_i) - 1) = 3(|A| + |B| - 4)$. Since $|B| \geq 3$, we have $|A| + |B| \geq 9$ and hence $3(|A| + |B| - 4) > 2(|A| + |B| - 2) \geq 2|\text{Pot}(L)|$. Thus, by Lemma 5.2.8, we have $c \in \bigcap_{i=1}^{3} L(x_i)$. Coloring $x_1, x_2$, and $x_3$ with $c$ leaves a list assignment...
that is easily completable to the rest of $A*B$. This is a contradiction. Hence $B$ is almost complete.

When $A$ is incomplete we can do much better.

**Lemma 5.3.20.** Let $A$ be a connected incomplete graph with $|A| \geq 4$ and $B$ an arbitrary graph. If $A*B$ is not $d_1$-choosable, then $B$ is complete.

**Proof.** By Lemma 5.3.1 it will suffice to show that $A*E_2$ is $d_1$-choosable. Suppose not and let $L$ be a minimal bad $d_1$-assignment on $A*E_2$. Then, by Lemma 5.3.11, $|\text{Pot}(L)| \leq |A|$.

Let $x_1$ and $x_2$ be the vertices in $E_2$. Then $|L(x_1)| + |L(x_2)| \geq d(x_1) + d(x_2) - 2 = 2|A| - 2 \geq |\text{Pot}(L)| + 2$. Hence we have different $c_1,c_2 \in L(x_1) \cap L(x_2)$.

First, suppose there exists $y \in V(A)$ such that $\{c_1,c_2\} \not\subseteq L(y)$. Without loss of generality, assume $c_1 \not\in L(y)$. Then coloring $x_1$ and $x_2$ with $c_1$ leaves a list assignment $L'$ on $A$ where $|L'(v)| \geq d_A(v)$ for all $v \in V(A)$ and $|L'(y)| > d_A(y)$. Hence the coloring can be completed, a contradiction.

Hence $\{c_1,c_2\} \subseteq L(v)$ for all $v \in V(A)$. If $\alpha(A) \geq 3$, then coloring a maximum independent set all with $c_1$ leaves an easily completable list assignment. Also, if $A$ contains two disjoint pairs of nonadjacent vertices, by coloring one with $c_1$ and one with $c_2$ we get another easily completable list assignment. Hence $A$ is almost complete.

Let $z \in V(A)$ such that $A - z$ is complete. Since $A$ is incomplete, we have $w \in V(A - z)$ nonadjacent to $z$. Also, as $A$ is connected we have $w' \in V(A - z)$ adjacent to $z$. Color $x_1$ and $x_2$ with $c_1$ and $w$ and $z$ with $c_2$ to get a list assignment $L'$ on $D := A - \{w,z\}$ where $|L'(v)| \geq d_D(v)$ for all $v \in V(D)$ and $|L'(w')| > d_D(w')$. Hence the coloring can be completed, a contradiction.  

**Lemma 5.3.21.** $E_2*2P_3$ is $d_1$-choosable.

**Proof.** Suppose otherwise. Let the $E_2$ have vertices $x_1$ and $x_2$ and the two $P_3$’s have vertices $y_1,y_2,y_3$ and $y_4,y_5,y_6$. By the Small Pot Lemma, we have a minimal bad $d_1$-assignment on $E_2*2P_3$ with $|\text{Pot}(L)| \leq 7$. Since $|L(x_1)| + |L(x_2)| = 10 \geq |\text{Pot}(L)| + 3$, we have three different colors $c_1,c_2,c_3 \in L(x_1) \cap L(x_2)$. Coloring both $x_1$ and $x_2$ with any $c_i$ leaves at worst a $d_0$-assignment on the $2P_3$. If $c_i \not\in L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_i \not\in L(y_4) \cap L(y_5) \cap L(y_6)$ for some $i$, then we can complete the coloring. Thus, without loss of generality, we have $\{c_1,c_2\} \subseteq L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_3 \in L(y_4) \cap L(y_5) \cap L(y_6)$. Color $y_1$ and $y_3$ with $c_1$ and $y_4$ and $y_6$ with $c_3$. Then we can easily complete the coloring on the rest of the $2P_3$. We have used at most 4 colors on the $2P_3$ and hence we can complete the coloring.

At this point we have enough information to completely classify the $d_1$-choosable graphs of the form $E_2*B$.

**Lemma 5.3.22.** $E_2*B$ is not $d_1$-choosable iff $B$ is the disjoint union of complete subgraphs and at most one $P_3$.

**Proof.** Suppose we have $B$ such that $E_2*B$ is not $d_1$-choosable. By Lemma 5.3.21, $B$ has at most one incomplete component. Suppose we have an incomplete component $C$ and let $y_1,y_2,y_3$ be an induced $P_3$ in $C$. If $C \neq P_3$, then $|C| \geq 4$ and Lemma 5.3.20 gives a contradiction. Hence $C = P_3$.

For the other direction, it is easy to see that for any $B$ such that $E_2*B$ is not $d_1$-choosable adding a disjoint complete subgraph to $B$ does not make it $d_1$-choosable. To see that $E_2*P_3$ is not $d_1$-choosable, let $x_1,x_2$ denote the vertices of the $E_2$ and let $y_1,y_2,y_3$ denote in order the vertices of the $P_3$. Let $L(x_1) = \{a,b\}$, $L(x_2) = \{c,d\}$, $L(y_1) = \{a,c\}$, $L(y_2) = \{a,b,c\}$, and $L(y_3) = \{b,d\}$. It is easy to verify that the graph is not colorable from these lists. This proves the lemma.

For $t \geq 4$, we know that if $K_t*B$ is not $d_1$-choosable then $B$ is almost complete; or $t = 4$ and $B$ is $E_3$ or a claw; or $t = 5$ and $B$ is $E_3$. The following two lemmas show that this completely characterizes the $d_1$-choosable graphs of this form.
Lemma 5.3.23. Almost complete graphs are not \(d_1\)-choosable.

Proof. Let \(G\) be almost complete and \(x \in V(G)\) such that \(G - x\) is complete. Consider the \(d_1\)-assignment \(L\) given by \(L(v) = |d(v) - 1|\) for each \(v \in V(G)\). Now \(G - x\) is a complete graph of size \(|G| - 1\), but the union of the lists on \(G - x\) is only \(|G| - 2\), so by Hall’s theorem, \(G\) has no coloring from these lists.

Lemma 5.3.24. \(K_t \ast E_3\) is \(d_1\)-choosable iff \(t \geq 6\).

Proof. That if \(t \geq 6\), then \(K_t \ast E_3\) is \(d_1\)-choosable follows from Lemma 5.3.19. For the other direction it is enough to show that \(K_5 \ast E_3\) is not \(d_1\)-choosable. Figure 5.1 shows a bad \(d_1\)-assignment on \(K_5 \ast E_3\).

Corollary 5.3.25. For \(t \geq 4\), \(K_t \ast B\) is not \(d_1\)-choosable iff \(B\) is almost complete; or \(t = 4\) and \(B\) is \(E_3\) or a claw; or \(t = 5\) and \(B\) is \(E_3\).

Lemma 5.3.26. \(P_3 \ast B\) is not \(d_1\)-choosable iff \(B\) is \(E_2\) or complete.

Proof. Moving the center of \(P_3\) to the other side of the join and applying Lemma 5.3.22 proves the lemma.

Lemma 5.3.27. \(K_3 \ast P_4\) is \(d_1\)-choosable.

Proof. Suppose otherwise. Denote the vertices of the \(P_4\) as \(y_1, y_2, y_3, y_4\), in order. Note that \(|L(y_1)| + |L(y_3)| = 4 + 5 \geq |G| + 1\) and \(|L(y_2)| + |L(y_4)| = 5 + 4 \geq |G| + 1\). Now we apply (2) of Lemma 5.3.17 with \(I_1 = \{y_1, y_3\}\) and \(I_2 = \{y_2, y_4\}\).

Lemma 5.3.28. \(K_3 \ast \text{antipaw}\) is \(d_1\)-choosable.
Proof. Suppose not. We use the labeling of the antipaw given in Figure 5.2. Since the antipaw is not a disjoint union of at most two complete graphs, Lemma 5.3.12 gives us a minimal bad $d_1$-assignment $L$ on $K_3 * \text{antipaw}$ with $|\text{Pot}(L)| \leq 5$. Note that $|L(y_1)| + |L(y_2)| \geq 6$ and $|L(y_2)| + |L(y_3)| \geq 6$. Hence, by Lemma 5.3.14, $|L(y_1) \cap L(y_2)| = 1$ and $L(y_1) \cap L(y_4) = L(y_2) \cap L(y_3)$. But then we have $c \in L(y_2) \cap L(y_3) \cap L(y_4)$ and after coloring $y_2, y_3$, and $y_4$ with $c$ we can complete the coloring, getting a contradiction.

Lemma 5.3.29. $K_3 * B$ is not $d_1$-choosable iff $B$ is almost complete, $K_1 + K_{|B|-1}, K_1 + K_{|B|-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 * K_{|B|-3}$.

Proof. Let $K_3 * B$ be a graph that is not $d_1$-choosable and let $B$ be none of the specified graphs. Lemma 5.3.12 gives us a minimal bad $d_1$-assignment $L$ on $K_3 * B$ with $|\text{Pot}(L)| \leq |B| + 1$. Furthermore, the proof of Lemma 5.3.12 shows that we can color $B$ with at most $|B| - 1$ colors. In particular we have nonadjacent $x, y \in V(B)$ and $c \in L(x) \cap L(y)$. Coloring $x$ and $y$ with $c$ leaves a list assignment $L'$ on $D := B - \{x, y\}$. If $c \in L'(z)$ for some $z \in V(D)$, then $\{x, y, z\}$ is independent and we can color $z$ with $c$ and complete the coloring to get a contradiction. Hence $\text{Pot}(L') = \text{Pot}(L) - \{c\}$.

Suppose, for a contradiction, that $D$ is not the disjoint union of at most two complete subgraphs. If $\alpha(D) \geq 3$, let $J$ be a maximum independent set in $D$ and set $\gamma := 0$. Otherwise $D$ contains an induced $P_3$ $abc$ and we let $J \subseteq V(D)$ be a maximal independent set containing $\{a, c\}$ and set $\gamma := 1$. Lemma 5.3.10 implies that $\sum_{v \in J} d_D(v) \geq |D| - |J| + \gamma$. Since $L$ is bad, we must have

\[
\sum_{v \in J} |L'(v)| \leq |\text{Pot}(L')| \\
\sum_{v \in J} |L'(v)| \leq |B| \\
2|J| + \sum_{v \in J} d_D(v) \leq |B| \\
2|J| + |D| - |J| + \gamma \leq |B| \\
|J| + |D| + \gamma \leq |B| \\
|J| + |B| - 2 + \gamma \leq |B|.
\]

Hence $|J| \leq 2 - \gamma$, a contradiction. Therefore $D$ is indeed the disjoint union of at most two complete subgraphs. (Additionally, if $D$ is not complete then $v \in V(D)$ is not adjacent to both $x$ and $y$ since then we would get the same contradictory degree sum as in the case when $\gamma = 1$.) We now consider the case that $D$ is a complete graph and the case that $D$ is the disjoint union of two complete subgraphs.

First, suppose $D$ is a complete graph. Plainly, $|D| \geq 2$. Put $X := N(x) \cap V(D)$ and $Y := N(y) \cap V(D)$. Suppose $X - Y \neq \emptyset$ and pick $z \in X - Y$. We have $|L(y)| + |L(z)| \geq d(y) + d(z) - 2 = d_B(y) + d_B(z) + 4 \geq 0 + |B| - 2 + 4 = |B| + 2 > |\text{Pot}(L)|$. By repeating the argument given above for $B - \{x, y\}$, we see that $B - \{y, z\}$ is also the disjoint union of at most two complete subgraphs. In particular, $x$ is adjacent to all or none of $D - z$. If all, then $B$ is almost complete, if none then $B$ contains an induced $P_3$ or antipaw, and both possibilities give contradictions by Lemmas 5.3.27 and 5.3.28. Hence $X - Y = \emptyset$. Similarly, $Y - X = \emptyset$, so $X = Y$. Since $B$ is not $E_2 + K_{|B|-2}$, $|X| > 0$. If $X = V(D)$, then $B$ is almost complete. If $|V(D) - X| \geq 2$, then pick $w_1, w_2 \in V(D) - X$. Now by considering degrees, we see that $L(x) \cap L(w_1)$ and $L(y) \cap L(w_2)$ are both nonempty. Now we can color $x, y, w_1, w_2$ using only 2 colors, and then complete the coloring. Hence, we must have $|V(D) - X| = 1$, so let $\{w\} = V(D) - X$. Now $x$ and $y$ are joined to $D - w$ and hence $B$ is $E_3 * K_{|B|-3}$, a contradiction.

Thus $D$ must instead be the disjoint union of two complete subgraphs $D_1$ and $D_2$. For each $i \in [2]$, put $X_i := N(x) \cap V(D_i)$ and $Y_i := N(y) \cap V(D_i)$. From our parenthetical
remark above, we know that $X_1 \cap Y_1 = \emptyset$. Suppose we have $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $L(z_1) \cap L(z_2) \neq \emptyset$. Then, by Lemma 5.3.14, $L(z_1) \cap L(z_2) = L(x) \cap L(y)$. Since no independent set of size three can have a color in common, the edges $z_1x$ and $z_2y$ or $z_3y$ and $z_3x$ must be present. Using the same argument as for $B - \{y, x\}$, we see that $B - \{z_1, z_2\}$ is the disjoint union of at most two complete subgraphs. So each of $x$ and $y$ is adjacent to all or none of each of $V(D_1 - z_1)$ and $V(D_2 - z_2)$. Thus, by symmetry, we may assume that $V(D_1 - z_1) \subseteq X_1$ and $V(D_2 - z_2) \subseteq Y_2$. If $|D_1| = |D_2| = 1$, then $B$ is the disjoint union of two cliques, a contradiction. So, by symmetry, we may assume that $|D_1| \geq 2$. Pick $w \in V(D_1 - z_1)$. If $x$ is not adjacent to $z_1$, then $xwz_1$ is an induced $P_4$ in $B$. Since $X_1 \cap Y_1 = \emptyset$, this $P_4$ together with $y$ either induces a $P_4$ or an antipaw, contradicting Lemmas 5.3.27 and 5.3.28. Hence $X_1 = V(D_1)$. Similarly, if $|D_2| \geq 2$, then $Y_2 = V(D_2)$ and $B$ is the disjoint union of two complete subgraphs, a contradiction. Hence $D_2 = \{z_2\}$. But $z_2$ must be adjacent to $y$, so $B$ is again the disjoint union of two cliques, a contradiction.

Thus for every $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ we have $L(z_1) \cap L(z_2) = \emptyset$. Suppose there exist $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $z_1$ and $z_2$ are each adjacent to at least one of $x$ and $y$. Then $|L(z_1)| + |L(z_2)| \geq d(z_1) + d(z_2) - 2 \geq d_B(z_1) + d_B(z_2) + 4 \geq |B| - 4 + 2 + 4 = |B| + 2 > |Pot(L)|$. Hence $L(z_1) \cap L(z_2) \neq \emptyset$, a contradiction.

Thus, by symmetry, we may assume that there are no edges between $D_1$ and $\{x, y\}$. Since no vertex in $D_2$ is adjacent to both $x$ and $y$, only one of $x$ or $y$ can have neighbors in $D_2$ lest $B$ contain an induced $P_4$ contradicting Lemma 5.3.27. Without loss of generality, we may assume that $y$ has no neighbors in $D_2$. Pick $w \in D_1$ and $z \in V(D_2)$.

Suppose that $|D_1| \geq 2$, $|D_2| \geq 2$, and there exists $t \in D_2$ such that $x$ and $t$ are nonadjacent. Now choose $v, u \in V(D_1)$ and $w \in V(D_2) \setminus \{t\}$. Now $\{v, w, y\}$ is independent and $|L(v)| + |L(w)| + |L(y)| \geq d(v) + d(w) + d(y) - 3 \geq d_B(v) + d_B(w) + d_B(y) + 6 \geq |B| + 2 > |Pot(L)|$. Hence either $L(v) \cap L(y) \neq \emptyset$ or $L(w) \cap L(y) \neq \emptyset$. Similarly, either $L(u) \cap L(x) \neq \emptyset$ or $L(t) \cap L(x) \neq \emptyset$. Thus, we can color 4 vertices using only 2 colors, and we can complete the coloring. So now either $|D_1| = 1$, $|D_2| = 1$, or $D_2 \subseteq N(x)$.

If $|D_2| = 1$, then either $B = K_1 + K_2 + K_{[B]-3}$ or else $B = E_3 + K_{[B]-3}$, both of which are forbidden. Similarly, if $|D_1| = 1$ and $x$ is adjacent to all or none of $D_2$, then $B = K_1 + K_1 + K_{|B|-2}$ or $E_3 + K_{|B|-3}$. Finally, if $x$ is adjacent to some, but not all of $D_2$, then $B$ contains an antipaw. By Lemma 5.3.28, this is a contradiction.

It remains to show that $K_3 * B$ is not $d_1$-choosable for any of the specified $B$’s. For $B$ almost complete, this follows from Lemma 5.3.23 and for $E_3 * K_{[B]-3}$, from Lemma 5.3.25.

For all the rest of the options we will give a bad list assignment with lists $[|B| + 1]$ on the $K_3$. Suppose $K_{t} + K_{[B]-t}$. On the $K_t$ the lists $[t+1]$ and on the $K_{[B]-t}$ the lists $[|B| + 1] \setminus [t]$. Then any coloring of $K_3 * B$ from the lists must use three colors on the $K_3$ and hence at least one of the cliques loses at least two colors leaving it uncolorable. Now suppose $B = K_1 + K_1 + K_{|B|-4}$, use the list $\{1, |B| + 1\}$ on the $K_1$, the lists $[t+1]$ on the $K_t$ and the lists $[|B| + 1] \setminus [t+1]$ on the $K_{[B]-4}$. This list assignment is clearly bad on $K_3 * B$.

Finally suppose $B = E_3 + K_{|B|-3}$. Give the three $K_1$’s the lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and the $K_{|B|-3}$ the list $[|B| + 1] \setminus [3]$. Again, this is clearly a bad list assignment on $K_3 * B$.

Lemma 5.3.30. $K_2 * P_3$ is $d_1$-choosable.

Proof. Suppose otherwise. By Lemma 5.3.13, we have a minimal bad $d_1$-assignment $L$ on $P_3 * K_2$ with $|Pot(L)| \leq 5$. Let $y_1, y_2, y_3, y_4, y_5$ denote the vertices of the $P_3$ in order. Now $|L(y_2)| + |L(y_3)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_4)| + |L(y_5)| \geq 7 \geq |Pot(L)| + 2$. So $\{y_2, y_4\}$ and $\{y_1, y_3, y_5\}$ satisfy the hypotheses of Lemma 5.3.17, giving a contradiction.

Lemma 5.3.31. $K_2 * \text{chair}$ is $d_1$-choosable.

Proof. Suppose otherwise. We use the labeling of the chair given in Figure 5.3a. Since the chair has an induced claw, Lemma 5.3.13 gives us a minimal bad $d_1$-assignment $L$ on $K_2 * \text{chair}$ with $|Pot(L)| \leq 5$. Now $|L(y_2)| + |L(y_5)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_4)| + |L(y_5)| \geq 7 \geq |Pot(L)| + 2$. Then $\{y_2, y_5\}$ and $\{y_1, y_3, y_4\}$ satisfy the hypotheses of Lemma 5.3.17, giving a contradiction.
Lemma 5.3.32. \( K_2 \ast \text{antichair} \) is \( d_1 \)-choosable.

Proof. Suppose otherwise. We use the labeling of the antichair given in Figure 5.3b. Since the antichair has an induced \( K^-_4 \), Lemma 5.3.13 gives us a minimal bad \( d_1 \)-assignment \( L \) on \( K_2 \ast \text{antichair} \) with \( |\text{Pot}(L)| \leq 5 \). We have \( |L(y_2)| + |L(y_5)| \geq 7 \) and hence \( |L(y_2) \cap L(y_5)| \geq 2 \). But then, by Lemma 5.3.14, we have the contradiction \( |L(y_1)| + |L(y_3)| \leq 3 \).

Lemma 5.3.33. \( K_2 \ast C_5 \) is \( d_1 \)-choosable.

Proof. Suppose otherwise. By the Small Pot Lemma, we have a minimal bad \( d_1 \)-assignment \( L \) on \( C_5 \ast K_2 \) with \( |\text{Pot}(L)| \leq 6 \). Let \( y_0, y_1, y_2, y_3, y_4, y_6 \) denote in order the vertices of the \( C_5 \). Then for \( 0 \leq i < j \leq 4 \) with \( i-j \not\equiv 1(\text{mod} \ 5) \) we have \( |L(y_i)|+|L(y_j)| \geq d(y_i)+d(y_j)-2 = 6 \).

First suppose \( |\text{Pot}(L)| \leq 5 \). Then each nonadjacent pair has a color in common and by applying Lemma 5.3.14 multiple times we see that there must exist \( c \in \bigcap_{0 \leq i \leq 4} L(y_i) \) and no nonadjacent pair can have a color other than \( c \) in common. Put \( S_i = L(y_i) - \{c\} \) and \( T = \text{Pot}(L) - \{c\} \). Then we must have \( S_0 = T - S_3, S_1 = T - S_3 = T - S_4 \) and \( S_2 = T - S_4 \).

Hence \( S_0 = S_1 = S_2 \) contradicting \( S_0 \cap S_2 = \emptyset \).

Therefore we must have \( |\text{Pot}(L)| = 6 \). Thus for nonadjacent \( y_i \) and \( y_j \), \( L(y_i) = \text{Pot}(L) - L(y_j) \). We have \( L(y_6) = \text{Pot}(L) - L(y_3), L(y_1) = \text{Pot}(L) - L(y_3) = \text{Pot}(L) - L(y_4) \) and \( L(y_2) = \text{Pot}(L) - L(y_4) \). Hence \( L(y_6) = L(y_1) = L(y_2) \). Thus we may color \( y_0 \) and \( y_2 \) the same and complete this coloring to the rest of \( B \) contradicting Lemma 5.3.7.

Lemma 5.3.34. \( K_2 \ast 2P_3 \) is \( d_1 \)-choosable.

Proof. Suppose otherwise. Let \( y_1, y_2, y_3 \) and \( y_4, y_5, y_6 \) denote in order the vertices of the two \( P_3 \)’s. Lemma 5.3.13 gives us a minimal bad \( d_1 \)-assignment \( L \) on \( K_2 \ast 2P_3 \) with \( |\text{Pot}(L)| \leq 6 \).

Since \( |L(y_1)| + |L(y_3)| + |L(y_4)| + |L(y_6)| = 8 \geq |\text{Pot}(L)| + 2 \), either three of these vertices share a common color, or else two pairs of them share distinct common colors. Thus, if \( L(y_2) \cap L(y_5) \neq \emptyset \), then we can color \( G \) by Lemma 5.3.14. Hence \( L(y_2) \cap L(y_5) = \emptyset \).
By summing list sizes, we see that some pair among each of \( \{y_1, y_3, y_5\} \) and \( \{y_2, y_4, y_6\} \) must have a color in common. Since there are no edges between \( \{y_1, y_3\} \) and \( \{y_4, y_6\} \), if \( L(y_1) \cap L(y_3) \neq \emptyset \) and \( L(y_4) \cap L(y_6) \neq \emptyset \), then we get a contradiction. By symmetry, we may assume that the other two options are either \( L(y_1) \cap L(y_3) \neq \emptyset \) and \( L(y_2) \cap L(y_4) \neq \emptyset \) or else \( L(y_1) \cap L(y_5) \neq \emptyset \) and \( L(y_2) \cap L(y_4) \neq \emptyset \). In the former case, by Lemma 5.3.14, we must have \( L(y_1) \cap L(y_3) \neq L(y_2) \cap L(y_4) \) since \( L(y_2) \cap L(y_5) = \emptyset \), contradicting Lemma 5.3.14.

\[
\begin{array}{c}
\text{(a) The anticlaw.} \\
\text{(b) The antidiomand.}
\end{array}
\]

Figure 5.5: Labelings of the anticlaw and the antidiomand.

Note that if \( L \) is a bad \( d_1 \) assignment on \( E_3 \ast B \) where the \( E_3 \) is \( \{x_1, x_2, x_3\} \), then \( L(x_1) \cap L(x_2) \cap L(x_3) = \emptyset \).

**Lemma 5.3.35.** \( E_3 \ast \text{anticlaw} \) is \( d_1 \)-choosable.

**Proof.** Suppose otherwise. The Small Pot Lemma gives us a minimal bad \( d_1 \)-assignment \( L \) on \( E_3 \ast \text{anticlaw} \) with \( |\text{Pot}(L)| \leq 6 \). Let the \( E_3 \) have vertices \( x_1, x_2, x_3 \), and let the anticlaw have vertices \( y_1, y_2, y_3, y_4 \), with \( y_2, y_3, y_4 \) mutually adjacent. Then \( \sum_{i} |L(x_i)| = 9 \) and hence there are three colors \( c_1, c_2, c_3 \) such that for each \( t \in [3], c_t \in L(x_i) \cap L(x_j) \) for some \( 1 \leq i < j \leq 3 \).

Suppose there exists \( i \in \{2, 3, 4\} \), say \( i = 2 \), such that \( y_1 \) and \( y_3 \) have a common color \( c \). We use \( c \) on \( y_1 \) and \( y_2 \), and let \( L'(v) = L(v) - c \) for each uncolored \( v \); note that \( c \) must be absent from some \( x_1 \), say \( x_1 \). Now since \( |L'(x_3)| + |L'(x_3)| \geq 4 \), we can color \( x_2 \) and \( x_3 \) such that at least two colors remain available on \( y_3 \). Finally, we greedily color \( y_1, y_2, y_3, y_4 \).

Otherwise, since \( |\text{Pot}(L)| \leq 6 \), we may assume that \( L(y_1) = \{a, b\} \) and \( L(y_2) = L(y_3) = L(y_4) = \{c, d, e, f\} \). Now we can color \( x_1, x_2 \), and \( x_3 \) using only two colors, exactly one of which is in \( \{a, b\} \). Finally, we greedily color \( y_1, y_2, y_3, y_4 \).

**Lemma 5.3.36.** \( E_3 \ast 2K_2 \) is \( d_1 \)-choosable.

**Proof.** Suppose otherwise. The Small Pot Lemma gives us a minimal bad \( d_1 \)-assignment \( L \) on \( E_3 \ast 2K_2 \) with \( |\text{Pot}(L)| \leq 6 \). Let the \( E_3 \) have vertices \( x_1, x_2, x_3 \), and let the \( 2K_2 \) have vertices \( y_1 \) adjacent to \( y_2 \) and \( y_3 \) adjacent to \( y_4 \). Then \( \sum_{i} |L(x_i)| = 9 \) and hence there are three colors \( c_1, c_2, c_3 \) such that for each \( t \in [3], c_t \in L(x_i) \cap L(x_j) \) for some \( 1 \leq i < j \leq 3 \).

If all three \( c_i \) appear on all four \( y_i \), then we can 2-color the \( 2K_2 \), and extend the coloring to the \( E_3 \). So we may assume instead without loss of generality that \( c_1 \) appears on \( x_1 \) and \( x_2 \), but not \( y_1 \). Now use \( c_1 \) on \( x_1 \) and \( x_2 \), then color greedily in the order \( y_3, y_4, x_3, y_2, y_1 \).

**Lemma 5.3.37.** \( E_3 \ast E_4 \) is \( d_1 \)-choosable.

**Proof.** Suppose otherwise. Let the \( E_3 \) have vertices \( x_1, x_2, x_3 \) and let the \( E_4 \) have vertices \( y_1, y_2, y_3, y_4 \). If there exists \( c \in \cap_{i=1}^{3} L(x_i) \), then we use \( c \) on all \( x_i \), and we can finish the coloring, so assume not. By the Small Pot Lemma, \( |\text{Pot}(L)| \leq 6 \), so there exist two \( y_i \), say \( y_1 \) and \( y_2 \), with a common color \( c \); use \( c \) on \( y_1 \) and \( y_2 \). Now there exists some \( x_i \), say \( x_3 \), with \( c \notin L(x_i) \). The 4-cycle induced by \( x_1, x_2, y_3, \) and \( y_4 \) is 2-choosable; then we can extend the coloring to \( x_3 \).

**Lemma 5.3.38.** \( E_3 \ast \text{antidiomand} \) is \( d_1 \)-choosable.
Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad $d_1$-assignment $L$ on $E_3$ * antidiamond with $|Pot(L)| \leq 6$. Let the $E_3$ have vertices $x_1, x_2, x_3$, and let the antidiamond have vertices $y_1, y_2, y_3, y_4$, with $y_3$ adjacent to $y_4$. We can assume that $\cap_{E_3} L(x_i) = \emptyset$ (since otherwise we use a common color on the $x_i$ and then greedily complete the coloring). If $y_3$ or $y_4$ has a common color $c$ with $y_1$ or $y_2$, then we can use $c$ on those two vertices and proceed as in the case of $E_3 * E_4$, so assume not. Again $\sum |L(x_i)| = 9$ and hence there are three colors $c_1, c_2, c_3$ such that for each $t \in \{3\}$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. So assume that $c_1$ appears on $x_1$ and $x_2$, and use it there. If $c_1$ appears on neither $y_1$ or $y_2$, then we greedily color in the order $y_3, y_4, x_3, y_1, y_2$. Otherwise $c_1$ appears on neither $y_3$ or $y_4$, so we greedily color in the order $y_1, y_2, x_3, y_3, y_4$. \hfill \square

Lemma 5.3.39. $E_3 * B$ is not $d_1$-choosable iff $B \in \{K_1, K_2, E_2, E_3, \overline{F_3}, K_3, K_4, K_5\}$.

Proof. Suppose we have $B$ such that $E_3 * B$ is not $d_1$-choosable. By Lemma 5.3.22, $B$ is the disjoint union of complete subgraphs and at most one $P_3$. If $B$ contained a $P_3$, then moving its middle vertex to the other side of the join would violate Lemma 5.3.20. By Lemma 5.3.37, $B$ has at most three components. By Lemma 5.3.38, if $B$ has three components, then $B = E_3$. By Lemma 5.3.36 and Lemma 5.3.35, if $B$ has two components then $B = E_2$ or $B = \overline{F_3}$. Otherwise $B$ is complete and Lemma 5.3.24 shows that $|B| \leq 5$. This proves the forward implication.

For the other direction, it is easy to verify that $E_3 * B$ is not $d_1$-choosable for the listed graphs. The cases $B \in \{K_1, K_2, E_2\}$ are nearly trivial. For $B = E_3$, we are simply recalling that $K_{3,3}$ is not 2-choosable. For $B \in \{K_3, K_4, K_5\}$, see Figure 5.1. Finally, suppose that $B = \overline{F_3}$. Let $x_1, x_2, x_3$ denote the vertices of the $E_3$ and let $y_1, y_2, y_4$ denote the vertices of the $\overline{F_3}$, where $y_2$ and $y_3$ are adjacent. Assign the lists $L(x_1) = \{1, 2\}, L(x_2) = \{1, 3\}, L(x_3) = \{2, 3\}, L(y_1) = \{1, 2\},$ and $L(y_2) = L(y_3) = \{1, 2, 3\}$. To color the $\overline{F_3}$, we clearly use at least two colors, but now some vertex of the $E_3$ has no remaining colors. \hfill \square

Lemma 5.3.40. $\overline{F_3} * 2K_2$ is $d_1$-choosable.

Proof. Let $x_1, x_2, x_3$ be the vertices of $\overline{F_3}$, with $x_2$ adjacent to $x_3$, and let $y_1, y_2, y_3, y_4$ be the vertices of $2K_2$, with $y_1$ adjacent to $y_2$ and $y_3$ adjacent to $y_4$. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so $x_1$ and $x_2$ have a common color $c_1$. If $c_1$ is absent from the list of some $y_i$, say $y_1$, then we can use $c_1$ on $x_1$ and $x_2$, then greedily color in the order $y_3, y_3, x_3, y_4, y_1$. Hence $c_1$ appears on all $y_i$. If $|Pot(L)| \leq 5$, then $x_1$ and $x_2$ have a second common color $c_2$. Since $c_1$ and $c_2$ must appear on all $y_i$, we can 2-color the $2K_2$, then greedily color $x_1, x_2,$ and $x_3$. So we can conclude that $L(x_1) \cap L(x_2) = c_1$ and $L(x_1) \cap L(x_3) = c_1$. Similarly, we can 2-color the $2K_2$ if $y_1$ and $y_4$ have any common color other than $c_1$.

Now we use $c_1$ on $y_2$ and $y_4$, and let $L'(v) = L(v) - c_1$ for all uncolored $v$. Now $|Pot(L')| = |Pot(L)| - 1 = 5$. Let $S = \{x_1, x_2, x_3, y_1, y_3\}$. To show that we can finish the coloring, we use Hall’s Theorem. We only need to consider subsets $T \subset S$ of size 3 or 4. If $|T| = 3$, then either $\{y_1, y_3\} \subset T$, so $|\cup_{v \in T} L'(v)| \geq |L'(y_1)| + |L'(y_3)| \geq 4$, or else $T$ contains $x_2$ or $x_3$. Since $|L'(x_2)| = |L'(x_3)| = 3$, we are done. If $|T| = 4$, then either $\{y_1, y_3\} \subset T$ or $\{x_1, x_2\} \subset T$ or $x_1, x_3 \subset T$. In each case $|\cup_{v \in T} L'(v)| \geq 4$. \hfill \square

Lemma 5.3.41. $\overline{F_3} * \text{antidiamond}$ is $d_1$-choosable.

Proof. Let $x_1, x_2, x_3$ be the vertices of $\overline{F_3}$, with $x_2$ adjacent to $x_3$, and let $y_1, y_2, y_3, y_4$ be the vertices of the antidiamond, with $y_3$ adjacent to $y_4$. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so $x_1$ and $x_2$ have a common color $c$. If $c$ is absent from $y_4$, then we use $c$ on $x_1$ and $x_2$, then greedily color $y_1, y_2, x_3, y_4$. Similarly, if $c$ is absent from $y_1$ and $y_2$, then we use $c$ on $x_1$ and $x_2$, then greedily color $y_1, y_4, x_3, y_2$. So $c$ must appear on $y_1$ (or $y_2$) and $y_3$, and we use it there. Let $L'(v) = L(v) - c$ for all uncolored vertices. Now if there exists $c_2 \in L'(y_2) \setminus L'(x_2)$, then we can use $c_2$ on $y_2$ and greedily color $x_1, y_3, x_3, x_2$. The same argument holds if there exists $c_2 \in L'(y_4) \setminus L'(x_2)$. Thus, we must have $(L'(y_2) \cup L'(y_4)) \subseteq L'(x_2)$, so $y_2$ and $y_4$ have a common color $c_2$. We use it on them and greedily color $x_1, x_2, x_3$. \hfill \square
**Lemma 5.3.42.** \( P_3^*E_4 \) is \( d_1 \)-choosable.

*Proof.* Let \( x_1, x_2, x_3 \) be the vertices of \( P_3^* \), with \( x_2 \) adjacent to \( x_1 \), and let \( y_1, y_2, y_3, y_4 \) be the vertices of \( E_4 \). If three of the \( y_i \)'s (say \( y_1, y_2, \) and \( y_3 \)) have a common color \( c \), then use \( c \) on them, and now greedily color in the order \( y_4, x_1, x_2, x_3 \). By the Small Pot Lemma, \( x_1 \) and \( x_2 \) have a common color \( c \), which we use on them. Now \( c \) appears on at most two \( y_i \), say \( y_1 \) and \( y_2 \), so we can greedily color in the order \( y_1, y_2, x_3, y_3, y_4 \). \( \square \)

**Lemma 5.3.43.** \( P_3^*B \) is not \( d_1 \)-choosable iff \( B \) is \( E_3, K_{|B|}, \) or \( K_1 + K_{|B|-1} \).

*Proof.* Since \( P_3^* \) contains an \( E_2 \), Lemma 5.3.22 shows that \( B \) is the disjoint union of complete subgraphs and at most one \( P_3 \). If \( B \) contained a \( P_3 \), then moving its middle vertex to the other side of the join would violate Lemma 5.3.20. By Lemma 5.3.40 at most one component of \( B \) has more than one vertex. If \( B \) has more than two components, then Lemma 5.3.41 shows that \( B \) is independent and thus Lemma 5.3.42 shows that \( B = E_3 \). If \( B \) has two components then it is \( K_1 + K_{|B|-1} \). \( B \) is complete. This proves the forward implication.

The reverse implication is easily checked. For \( B = E_3 \), see Lemma 5.3.39. If \( B = K_{|B|} \), then \( G \) is almost complete. Suppose that \( B = K_{|B|-1} \). Now \( \Delta(G) = \omega(G) = |B| + 1 \), so \( G \) is not \( d_1 \)-choosable. \( \square \)

**Lemma 5.3.44.** Let \( A \) and \( B \) be graphs with \( |A| \geq 4 \) and \( |B| \geq 4 \). The graph \( A \ast B \) is not \( d_1 \)-choosable iff \( A \ast B \) is almost complete, \( K_5 \ast E_3 \), or \( (K_1 + K_{|A|-1}) \ast (K_1 + K_{|B|-1}) \).

*Proof.* Suppose \( A \) and \( B \) are graphs with \( |A| \geq |B| \geq 4 \) such that \( A \ast B \) is not \( d_1 \)-choosable and not one of the specified graphs.

First suppose \( A \) is connected. If \( A \) is complete then by Corollary 5.3.25, \( |A| = 4 \) and \( B \) is a claw or \( B \) is almost complete. But this implies that \( G = K_5 \ast E_3 \) or \( G \) is almost complete. Hence \( A \) is incomplete. Now Lemma 5.3.20 shows that \( B \) is complete. By reversing the roles of \( A \) and \( B \) in this argument, we get a contradiction; so \( A \) is disconnected. The same argument shows that \( B \) is also disconnected.

Suppose \( \alpha(A) \geq 3 \). Then Lemma 5.3.39 shows that \( B = K_4 \) or \( K_5 \), both impossible as above. Thus \( \alpha(A) = 2 \) and hence \( A \) is the disjoint union of two complete graphs. The same goes for \( B \). Now Lemma 5.3.43 shows that \( A = K_1 + K_{|A|-1} \) and \( B = K_1 + K_{|B|-1} \). The reverse implication is easily checked. If \( A \ast B \) is almost complete, then clearly it is not \( d_1 \)-choosable. For \( A \ast B = K_5 \ast E_3 \), see Figure 5.1. So suppose that \( A \ast B = (K_1 + K_{|A|-1}) \ast (K_1 + K_{|B|-1}) \). Now \( \Delta(A \ast B) = \omega(A \ast B) = |A| + |B| - 2 \), so \( A \ast B \) is not \( d_1 \)-choosable. \( \square \)

**Joins with \( K_2 \)**

**Definition 5.3.2.** The *net* is formed by adding one edge incident to each vertex of \( K_3 \). The *bowtie* is formed by identifying one vertex in each of two copies of \( K_3 \). The \( M \) is formed from the bowtie by adding an edge incident to a vertex of degree 2.

**Lemma 5.3.45.** The graph \( K_2 \ast A \) is \( d_1 \)-choosable for all

\[
A \in \{2P_3, C_4, C_5, P_5, chair, antichair, K_1 \ast antipaw, K_1 \ast P_4, net, M\}.
\]

*Proof.* For eight of these ten choices of \( A \), we have already proved that \( K_2 \ast A \) is \( d_1 \)-choosable. Specifically, we have proved this for \( 2P_3 \) (Lemma 5.3.34), \( C_5 \) (Lemma 5.3.33), \( P_5 \) (Lemma 5.3.30), chair (Lemma 5.3.31), antichair (Lemma 5.3.32), \( K_1 \ast antipaw \) (Lemma 5.3.28), \( K_1 \ast P_4 \) (Lemma 5.3.27), and \( C_4 \) (since \( C_4 = E_3^2 \), this is the case \( r = 1 \) in Corollary 5.3.4).

Now we consider the remaining two cases: net and \( M \).

Let \( G = K_2 \ast net \). Let \( x_1, x_2 \) denote the vertices of the \( K_2 \), let \( y_1, y_2, y_3 \) denote the degree-3 vertices in the net, and let \( z_1, z_2, z_3 \), denote the leaves of the net, with \( z_i \) adjacent
to $y_i$. We consider three cases. (1) If there exists $c_1 \in \cap_{i=1}^3 L(z_i)$, then we first use $c_1$ on all three $z_i$ and afterwards color $y_1, y_2, y_3, x_1, x_2$ greedily. (2) Suppose there exist $y_1$ and $z_j$, with $i \neq j$, such that there exists $c_1 \in L(y_1) \cap L(z_j)$; by symmetry we assume this is $y_1$ and $z_2$. We use $c_1$ on $y_1$ and $z_2$ and let $L'(v) = L(v) - c_1$ for each uncolored vertex $v$. Now we have $|\text{Pot}(L')| < |G\{y_1, z_2\}| = 6$. Since we have $|L(z_1)| + |L'(y_2)| + |L'(z_3)| \geq 1 + 3 + 2 = 6$, we must have a common color $c_2$ (different from $c_1$) on two of $z_1, y_2$, and $z_3$. We use this color on these two vertices, then greedily color the remaining vertices of the net before coloring $x_1$ and $x_2$. (3) Observe that if $L(z_1)$ and $L(z_2)$ are disjoint, then (since $|\text{Pot}(L)| \leq 7$) either $L(z_1) \cap L(y_3) \neq \emptyset$ or $L(z_2) \cap L(y_3) \neq \emptyset$; in each case, we are in (2). Thus, if we are not in (1) or (2) above, then (again, since $|\text{Pot}(L)| \leq 7$) by symmetry we have $L(z_1) = \{a, b\}$, $L(z_2) = \{a, c\}$, $L(z_3) = \{b, c\}$, and $L(y_1) = L(y_2) = L(y_3) = \{d, e, f, g\}$. By symmetry, either $a \notin L(x_1)$ or $d \notin L(x_1)$. Thus, we use $a$ on $z_1$ and $z_2$ and we use $d$ on $y_3$. Now we greedily color $z_3, y_1, y_2, x_2, x_1$.

Let $G = K_2 \ast M$ and let $x_1, x_2$ denote the vertices of the $K_2$; for the $M$, let $y_1$ denote the 1-vertex, $y_2$ the 3-vertex, $y_3$ the 2-vertex adjacent to $y_2$, $y_4$ the 4-vertex, and $y_5$ and $y_6$ the remaining 2-vertices. By the Small Pot Lemma, $|\text{Pot}(L)| \leq 7$. Since $|L(y_1)| + |L(y_3)| + |L(y_6)| = 8$, two of them must have a common color $c$. If all three of $y_1, y_3, y_6$ have $c$, then we use $c$ on all three, and afterward we color greedily $y_2, y_4, y_5, x_1, x_2$. So now we consider three cases. (1) If $c$ appears in $L(y_3) \cap L(y_6)$, then we use $c$ on $y_3$ and $y_6$, and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. By the Small Pot Lemma, $|\text{Pot}(L')| \leq 5$. Since $|L(y_1)| + |L'(y_4)| \geq 2 + 4 > 5$, we have a common color $d$ (different from $c$) on $y_1$ and $y_4$. After we use $d$ on $y_1$ and $y_4$, we color greedily $y_2, y_5, x_1, x_2$. (2) If $c$ appears in $L(y_1) \cap L(y_3)$, then we use $c$ on $y_1$ and $y_3$ and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. Again we have $|\text{Pot}(L')| \leq 5$ and $|L(y_2)| + |L'(y_5)| \geq 3 + 3 > 5$. After using a common color on $y_2$ and $y_5$, we greedily color $y_4, y_6, x_1, x_2$. (3) Now suppose that $c$ appears in $L(y_1) \cap L(y_6)$. If $c \in L(y_2)$, then we use $c$ on $y_2$ and $y_6$, and let $L'(v) = L(v) - c$ for each uncolored vertex $v$. Again we have $|\text{Pot}(L')| \leq 5$ and $|L(y_1)| + |L'(y_4)| + |L'(y_5)| \geq 1 + 3 + 2$ (since $c \notin L(y_3)$). So again we use a common color on two of $y_1, y_3$, and $y_5$, then greedily color the remaining vertices of the $M$ before coloring $x_1$ and $x_2$. Suppose instead that $c \notin L(y_2)$. Now we use $c$ on $y_1$ and $y_6$, and then use a common color on $y_4$ and $y_5$ (since $|\text{Pot}(L')| \leq 5 < 6 = 4 + 2 \leq |L'(y_2)| + |L'(y_5)|$). Finally, we greedily color $y_3, y_4, x_1, x_2$.

\textbf{Lemma 5.3.46.} The graph $K_2 \ast (B + K_t)$ is not $d_1$-choosable iff $K_2 \ast B$ is not $d_1$-choosable.

\textbf{Proof.} Suppose $K_2 \ast B$ is not $d_1$-choosable and let $L$ be a bad list assignment (not using the colors in $[t]$). To form a list assignment for $K_2 \ast (B + K_t)$, we start with $L$, then assign $[t]$ to each vertex in the $K_t$ and add $[t]$ to the lists for the vertices in the $K_2$. Clearly $K_2 \ast (B + K_t)$ has no coloring from these lists.

Conversely, suppose $K_2 \ast B$ is $d_1$-choosable. Given a list assignment for $K_2 \ast (B + K_t)$, we greedily color the $K_t$; what remains is a list assignment for $K_2 \ast B$; thus, we can finish the coloring.

Since $K_2 \ast 2P_3$ is $d_1$-choosable (Lemma 5.3.34) we see that any graph $B$ such that $K_2 \ast B$ is not $d_1$-choosable must have at most one incomplete component.

\textbf{Lemma 5.3.47.} If $K_2 \ast B$ is not $d_1$-choosable, then $B$ consists of a disjoint union of complete subgraphs, together with at most one incomplete component $H$. If $H$ has a dominating vertex $v$, then $K_2 \ast H = K_3 \ast (H - v)$, so by Lemma 5.3.29 we can completely describe $H$. Otherwise $H$ is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not $d_1$-choosable.

\textbf{Proof.} Let $B$ be a graph such that $K_2 \ast B$ is not $d_1$-choosable, and let $H$ be the unique incomplete component of $B$. Suppose that $H$ does not contain a dominating vertex. We first show that $H$ is a tree of edge-disjoint cliques (clique tree), i.e., every cycle has an edge
between every pair of its vertices. Since $K_2 * C_4$, $K_2 * C_5$, and $K_2 * P_5$ are $d_1$-choosable, we get that $H$ has no induced $C_4$, $C_5$, or $P_5$; thus $H$ is chordal. So if $H$ is not a clique tree, then $H$ contains an induced copy of $K_4^-$; call it $D$.

Let $w$ denote a vertex adjacent to $D$. Each vertex adjacent to $D$ can attach to the vertices of $D$ in 8 possible ways (up to isomorphism); it can attach to 0, 1, or 2 of the vertices of degree 2, and also to 0, 1, or 2 of the vertices of degree 3 (but it must attach to at least one vertex), thus $3 + 3 - 1 = 8$ possibilities. Five of these possibilities yield a graph $J$ such that $K_2 * J$ is $d_1$-choosable (since $J$ contains an induced copy of either the antichair, $K_1 * $antipaw, $K_1 * P_4$, or $C_4$). So we consider the other three possibilities (these are the three possibilities when $w$ is adjacent to both vertices of degree 3 in $D$).

If $D$ is not dominating, then some vertex $x$ is distance 2 from $D$, via $w$. In each case, the subgraph induced by $D$, $w$, and $x$ contains an induced $d_1$-choosable subgraph (in two cases this is a antichair, and in the third case it is $K_1 * $antipaw). Hence, $D$ is dominating, and all of its neighbors are adjacent to both vertices of degree 3 in $D$. But now $H$ has two dominating vertices. This contradicts our assumption that $H$ has no dominating vertex. Hence, $H$ is a clique tree.

Since $H$ has no dominating vertex, it must contain an induced $P_4$, call it $P$. Since $H$ has neither a $P_5$ nor a “chair” as an induced subgraph, each vertex adjacent to $P$ must be adjacent to at least two vertices of $P$. Since $C_4$ and the antichair and $K_1 * P_4$ are all forbidden, each vertex adjacent to $P$ is adjacent to exactly two consecutive vertices of $P$. Since both $P_3$ and the net are forbidden, every vertex in $H$ is adjacent to $P$. Since $P_1 * $antipaw is forbidden, every pair of vertices that are adjacent to the same two vertices of $P$ are also adjacent to each other. Finally, since $M$ is forbidden, $H$ must be formed in one of two ways. Either (a) begin with two disjoint cliques and add an edge between them, or else (b) begin with a clique and add exactly one edge incident to exactly two vertices of the clique. Furthermore, all graphs $H$ formed by either (a) or (b) are such that $K_2 * H$ is not $d_1$-choosable. In (a), suppose that we begin with a $K_r$ and a $K_s$. We assign lists as follows: the $K_r$ gets $[r]$, the $K_s$ gets $\{r + 1, \ldots, r + s\}$, the dominating vertices (on the other side of the join) get $[r + t]$; finally, the two endpoints of the additional edge also get $a$ added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one or $[r + t]$ must be used on $H$.

In (b), suppose that we begin with a $K_r$. We assign lists as follows: the $K_r$ gets $[r]$, the two degree 1 vertices get $\{r + 1, r + 2\}$, the dominating vertices (on the other side of the join) get $[r + 2]$; finally, the two vertices in the $K_r$ that are endpoints of the pendant edges also get $r + 1$ added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one of $[r + 2]$ must be used on $H$.

Mixed list assignments

Lemma 5.3.48. Let $A$ be a graph with $|A| \geq 4$. Let $L$ be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component $D$ of $A$ has a vertex $v$ such that $|L(v)| \geq d(v)$. Then $L$ is good on $G$.

Proof. By the Small Pot Lemma, $|Pot(L)| \leq |A| + 1$. Say the $E_2$ has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \geq 2 |A| - 2 > |A| + 1$ since $|A| \geq 4$. Coloring $x$ and $y$ the same leaves at worst a $d_0$ assignment $L'$ on $A$ where each component $D$ has a vertex $v$ with $|L'(v)| > d_D(v)$.

Hence we can complete the coloring.

Lemma 5.3.49. Let $A$ be a graph with $|A| \geq 3$. Let $L$ be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$. $|L(v)| \geq d(v)$ for some $v$ in the $E_2$ and each component $D$ of $A$ has a vertex $v$ such that $|L(v)| \geq d(v)$. Then $L$ is good on $G$.

Proof. By the Small Pot Lemma, $|Pot(L)| \leq |A| + 1$. Say the $E_2$ has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \geq 2 |A| - 1 > |A| + 1$ since $|A| \geq 3$. Coloring $x$ and $y$ the same leaves at worst a $d_0$ assignment $L'$ on $A$ where each component $D$ has a vertex $v$ with $|L'(v)| > d_D(v)$.

Hence we can complete the coloring.  

Joins with $K_1$

Let $G$ be a $d_0$-choosable graph. If $K_1 \ast G$ is not $d_1$-choosable, then we call $G$ bad; otherwise we call $G$ good. Adding a leaf to a graph does not change whether it is bad, so we focus on bad $G$ such that $\delta(G) \geq 2$. We will also restrict our attention to connected bad graphs.

In this section, we apply Lemma 5.3.14 to characterize all bad triangle-free graphs. An easy special case of this classification for triangle-free graphs is the following lemma. We frequently use the idea of an independent set with a common color, so we call an independent set of size $k$ with a common color an independent $k$-set.

**Lemma 5.3.50.** If $G$ is a connected bipartite graph with more edges than vertices, then $K_1 \ast G$ is $d_1$-choosable.

**Proof.** Let $A$ and $B$ be the parts of $G$. Let $L$ be a minimal bad $d_1$-assignment for $K_1 \ast G$. Since $G$ has more edges than vertices, $G$ has a cycle. Since $G$ is also bipartite, $G$ is $d_0$-choosable (by the classification of $d_0$-choosable graphs at the start of Section 5.2). By the Small Pot Lemma, $\text{Pot}(L) \leq |G|$. Note that $\sum_{v \in A} d(v) = |E(G)| > |V(G)| \geq |\text{Pot}(L)|$. Similarly $\sum_{v \in B} d(v) > |\text{Pot}(L)|$. Now we apply Lemma 5.3.17 with $I_1 = A$ and $I_2 = B$. This proves the lemma.

**Lemma 5.3.51.** Let $C$ be a collection of sets $I_1, \ldots, I_k$, each of size 2. If for all $i \neq j$, we have $I_i \cap I_j \neq \emptyset$, then either there exists $v \in \cap_{i=1}^k I_i$ or there exist $v_1, v_2$, and $v_3$ such that each $I_i$ equals either $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$.

**Proof.** Suppose that $\cap_{i=1}^k I_i = \emptyset$. Consider distinct sets $I_1$ and $I_2$. Let $\{v_1\} = I_1 \cap I_2$, and let $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$. Since $\cap_{i=1}^k I_i = \emptyset$, there exists $I_3$ such that $v_1 \notin I_3$. So we must have $I_3 = \{v_2, v_3\}$. Now for all $k \geq 4$, we must have $|I_i \cap \{v_1, v_2, v_3\}| = 2$.

Using Lemmas 5.3.14 and 5.3.51, we can prove the following classification.

**Lemma 5.3.52.** If a $d_0$-choosable graph $G$ is bad, then $K_1 \ast G$ has a $d_1$-list assignment $L$ such that one of the following 5 conditions holds.

1. $L$ is a $d$-clique cover of $G$ of size at most $|G|$.

2. There exists $v \in V(G)$ such that $L$ is a $d$-clique cover of $G - v$ of size at most $|G| - 1$.

3. There exists a color $c$ such that the union of all independent 2-sets in $c$ induces $P_4$ and all other independent 2-sets are the end vertices of the $P_4$.

4. The union of all independent 2-sets is $E_3$ or $E_2$.

5. All independent 2-sets in $L$ are the same color.

**Proof.** Let $z$ denote the $K_1$. We consider the possible ways for a bad list assignment $L$ to satisfy Lemma 5.3.14. Clearly $L$ has no independent $k$-sets, for $k \geq 3$. If $L$ has no independent 2-sets, then Condition 1 holds. If all independent 2-sets in $L$ are the same color, then Condition 5 holds. If $L$ has only the same independent 2-set in multiple colors, then the 2-sets induce $E_2$, so Condition 4 holds. So instead $L$ must have distinct independent 2-sets in distinct colors.

Assume that additionally all independent 2-sets intersect in a common vertex $v$. If $|\text{Pot}_{G - v}(L)| \leq |G| - 1$, then Condition 2 holds. So instead $|\text{Pot}_{G - v}(L)| \geq |G|$. So there exist some $w \in G - v$ and some color $c \in L(w)$ such that $c \notin L(z)$. By Lemma 5.3.14, $G$ has an $L$-coloring that uses $c$ on $w$ and uses some other common color on two vertices of $G - w$. Now we can extend the coloring to $z$.

Now suppose that no vertex $v$ lies in all independent 2-sets. If all independent 2-sets are distinct colors, then Lemma 5.3.51 implies that Condition 4 holds. Suppose we have two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$ in the same color $c$. Since $L$ has no independent 3-set, $v_2$ is adjacent to $v_3$. Recall that $L$ has an independent 2-set $I_3$ of another
color $c'$. If $v_1 \notin I_3$, then $I_3$ is disjoint from either $I_1$ or $I_2$, so we can finish the coloring, by (2) in Lemma 5.3.14. Hence $v_1 \in I_3$. So the only independent 2-sets not containing $v_1$ must be of color $c$, say $\{v_2, v_4\}$. Since $L$ has no independent 3-sets, we must have $v_1$ adjacent to $v_4$. Now we see that every independent 2-set in a color other than $c$ must be $\{v_1, v_2\}$. This implies that $v_2$ and $v_3$ must be adjacent. Now Condition 3 holds.

Finally, suppose that $L$ has two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_3, v_4\}$ in a common color. If we are not in the case above, then $G[v_1, v_2, v_3, v_4] = C_4$. Now every independent 2-set $I_3$ of another color can intersect at most one of $I_1$ and $I_2$, so we can color the graph by (2) in Lemma 5.3.14.

\[ \square \]

### 5.4 Connectivity of complements

As a basic application of our list coloring lemmas, we prove that for $k \geq 5$ any $G \in \mathcal{C}_k$ has maximally connected complement.

**Lemma 5.4.1.** Fix $k \geq 5$. If $G \in \mathcal{C}_k$ and $A \ast B \not\subseteq G$ for graphs $A$ and $B$ with $1 \leq |A| \leq |B|$, then $|A \ast B| \leq \Delta(G) + 1$.

**Proof.** Let $G \in \mathcal{C}_k$ and $A \ast B \not\subseteq G$ for graphs $A$ and $B$ with $1 \leq |A| \leq |B|$. Assume $|A \ast B| > \Delta(G) + 1$. To avoid a vertex with degree larger than $\Delta(G)$, we must have $\Delta(A) \leq |A| - 2$ and $\Delta(B) \leq |B| - 2$. In particular, both $A$ and $B$ are incomplete, so $2 \leq |A| \leq |B|$ and both $A$ and $B$ contain an induced $E_2$. Hence, by Lemma 5.3.22, both $A$ and $B$ are the disjoint union of complete subgraphs and at most one $P_3$.

First, assume $|A| = 2$, say $A = \{x_1, x_2\}$. Since $|B| \geq \Delta(G)$, we conclude that $N(x_1) = N(x_2)$. Thus $x_1$ and $x_2$ are nonadjacent twins in a vertex critical graph which is impossible.

Thus we may assume that $|A| \geq 3$. If $A$ contained an induced $P_3$, then $G$ would have an induced $E_2 \ast (K_1 \ast B)$. For $K_1 \ast B$ to be the disjoint union of complete subgraphs and at most one $P_3$, $B$ must either be $E_2$ or complete, both of which are impossible. Hence $A$ is a disjoint union of at least two complete subgraphs. The same goes for $B$.

Assume that $A$ is edgeless. Then, by Lemma 5.3.39, $B$ must be $E_3$ or $P_3$. Hence $\Delta(G) + 1 < |A| + |B| = 6$, giving the contradiction $\Delta(G) \leq 4$.

Since $A$ is the disjoint union of at least two complete subgraphs and contains an edge, it contains $P_3$. By Lemma 5.3.43, $B$ must be either $E_3$ or the disjoint union of a vertex and a complete subgraph. As above, $B = E_3$ is impossible. In particular $B$ contains $P_3$ and using Lemma 5.3.43 again, we conclude that $A$ is the disjoint union of a vertex and a complete subgraph giving the final contradiction $\omega(G) \geq \omega(A \ast B) \geq \omega(A) + \omega(B) \geq |A| + |B| - 2 \geq \Delta(G)$.

\[ \square \]

**Lemma 5.4.2.** Fix $k \geq 5$. If $G \in \mathcal{C}_k$, then $\overline{G}$ is maximally connected; that is, $\kappa(\overline{G}) = \delta(\overline{G})$.

**Proof.** Let $G \in \mathcal{C}_k$ and let $S$ be a cutset in $\overline{G}$ with $|S| = \kappa(\overline{G})$. To get a contradiction, assume that $|S| < \delta(\overline{G}) = |G| - (\Delta(G) + 1)$. Since $\overline{G} - S$ is disconnected, $G - S = A \ast B$ for some graphs $A$ and $B$ with $1 \leq |A| \leq |B|$. We have $|A| + |B| = |\overline{G} - S| = |G| - |S| > |G| - (|G| - (\Delta(G) + 1)) = \Delta(G) + 1$, a contradiction by Lemma 5.4.1.

\[ \square \]
Some of the material in this chapter appeared in [19] and is joint work with Dan Cranston.

In [21], Dhurandhar proved the Borodin-Kostochka Conjecture for a superset of line graphs of simple graphs defined by excluding the claw, $K_5 - e$ and another graph $D$ as induced subgraphs. Kierstead and Schmerl [39] improved this by removing the need to exclude $D$. The aim of this chapter is to remove the need to exclude $K_5 - e$; that is, to prove the Borodin-Kostochka Conjecture for claw-free graphs.

**Theorem 6.4.5.** Every claw-free graph satisfying $\chi \geq \Delta \geq 9$ contains a $K_{\Delta}$.

This also generalizes the result of Beutelspacher and Hering [5] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. The value of 9 in Theorem 6.4.5 is best possible since the counterexample for $\Delta = 8$ in Figure 1.1 is claw-free.

Theorem 6.4.5 is also optimal in the following sense. We can reformulate the statement as: every claw-free graph with $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. Consider a similar statement with $\Delta - 1$ replaced by $f(\Delta)$ for some $f: \mathbb{N} \to \mathbb{N}$ and 9 replaced by $\Delta_0$. We show that $f(x) \geq x - 1$ for $x \geq \Delta_0$. Consider $G_t := K_t \ast C_5$. We have $\chi(G_t) = t + 3$, $\omega(G_t) = t + 2$ and $\Delta(G_t) = t + 4$ and $G_t$ is claw-free. Hence for $t \geq \Delta_0 - 4$ we have $t + 3 \leq \max\{t + 2, f(t + 4)\} \leq f(t + 4)$ giving $f(x) \geq x - 1$ for $x \geq \Delta_0$.

As shown in [60] (also Section 6.1) the situation is very different for line graphs of multigraphs which satisfy $\chi \leq \max\{\omega, \frac{7\Delta+10}{8}\}$. There it was conjectured that $f(x) := \frac{5x + 8}{6}$ works for line graphs of multigraphs; this would be best possible. The example of $K_t \ast C_5$ is claw-free, but it isn’t quasi-line.

**Question.** What is the situation for quasi-line graphs? That is, what is the optimal $f$ such that every quasi-line graph with large enough maximum degree satisfies $\chi \leq \max\{\omega, f(\Delta)\}$.

### 6.1 Line graphs

The material in this section appeared in [60].

In this section we prove the Borodin-Kostochka Conjecture for line graphs of multigraphs. Moreover, we prove a strengthening of Brooks’ theorem for line graphs of multigraphs and conjecture the best possible such bound.

**Lemma 6.1.1.** Fix $k \geq 0$. Let $H$ be a multigraph and put $G = L(H)$. Suppose $\chi(G) = \Delta(G) + 1 - k$. If $xy \in E(H)$ is critical and $u(xy) \geq 2k + 2$, then $xy$ is contained in a $\chi(G)$-clique in $G$. 
Proof. Let \( xy \in E(H) \) be a critical edge with \( \mu(xy) \geq 2k + 2 \). Let \( A \) be the set of all edges incident with both \( x \) and \( y \). Let \( B \) be the set of edges incident with either \( x \) or \( y \) but not both. Then, in \( G \), \( A \) is a clique joined to \( B \) and \( B \) is the complement of a bipartite graph. Put \( F = G[A \cup B] \). Since \( xy \) is critical, we have a \( \chi(G) - 1 \)-coloring of \( G - F \). Viewed as a partial \( \chi(G) - 1 \)-coloring of \( G \) this leaves a list assignment \( L \) on \( F \) with \( |L(v)| = \chi(G) - 1 - (d_G(v) - d_F(v)) = d_F(v) - k + \Delta(G) - d_G(v) \) for each \( v \in V(F) \). Put \( j = k + d_G(xy) - \Delta(G) \).

Let \( M \) be a maximum matching in the complement of \( B \). First suppose \( |M| \leq j \). Then, since \( B \) is perfect, \( \omega(B) = \chi(B) \) and we have

\[
\omega(F) = \omega(A) + \omega(B) = |A| + \chi(B) \\
\geq |A| + |B| - j = d_G(xy) + 1 - j \\
= \Delta(G) + 1 - k = \chi(G).
\]

Thus \( xy \) is contained in a \( \chi(G) \)-clique in \( G \).

Hence we may assume that \( |M| \geq j + 1 \). Let \( \{ \{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\} \} \) be a matching in the complement of \( B \). Then, for each \( 1 \leq i \leq j + 1 \) we have

\[
|L(x_i)| + |L(y_i)| \geq d_F(x_i) + d_F(y_i) - 2k \\
\geq |B| - 2 |A| - 2k \\
= d_G(xy) + |A| - 2k - 1 \\
\geq d_G(xy) + 1.
\]

Here the second inequality follows since \( \alpha(B) \leq 2 \) and the last since \( |A| = \mu(xy) \geq 2k + 2 \). Since the lists together contain at most \( \chi(G) - 1 = \Delta(G) - k \) colors we see that for each \( i \),

\[
|L(x_i) \cap L(y_i)| \geq |L(x_i)| + |L(y_i)| - (\Delta(G) - k) \\
\geq d_G(xy) + 1 - \Delta(G) + k \\
= j + 1.
\]

Thus we may color the vertices in the pairs \( \{x_1, y_1\}, \ldots, \{x_{j+1}, y_{j+1}\} \) from \( L \) using one color for each pair. Since \( |A| \geq k + 1 \) we can extend this to a coloring of \( B \) from \( L \) by coloring greedily. But each vertex in \( A \) has \( j + 1 \) colors used twice on its neighborhood, thus each vertex in \( A \) is left with a list of size at least \( d_A(v) - k + \Delta(G) - d_G(v) + 1 = d_A(v) + 1 \). Hence we can complete the \( (\chi(G) - 1) \)-coloring to all of \( F \) by coloring greedily. This contradiction completes the proof.

\[ \square \]

Theorem 6.1.2. If \( G \) is the line graph of a multigraph \( H \) and \( G \) is vertex critical, then

\[
\chi(G) \leq \max \left\{ \omega(G), \Delta(G) + 1 - \frac{\mu(H) - 1}{2} \right\}.
\]

Proof. Let \( G \) be the line graph of a multigraph \( H \) such that \( G \) is vertex critical. Say \( \chi(G) = \Delta(G) + 1 - k \). Suppose \( \chi(G) > \omega(G) \). Since \( G \) is vertex critical, every edge in \( H \) is critical. Hence, by Lemma 6.1.1, \( \mu(H) \leq 2k + 1 \). That is, \( \mu(H) \leq 2(\Delta(G) + 1 - \chi(G)) + 1 \). The theorem follows.

\[ \square \]

This upper bound is tight. To see this, let \( H_t = t \cdot C_5 \) (i.e. \( C_5 \) where each edge has multiplicity \( t \)) and put \( G_t = L(H_t) \). As Catlin [16] showed, for odd \( t \) we have \( \chi(G_t) = \frac{5t + 1}{2} \), \( \Delta(G_t) = 3t - 1 \), and \( \omega(G_t) = 2t \). Since \( \mu(H_t) = t \), the upper bound is achieved.

We need the following lemma which is a consequence of the fan equation (see [4, 13, 25, 27]).
Lemma 6.1.3. Let $G$ be the line graph of a multigraph $H$. Suppose $G$ is vertex critical with $\chi(G) > \Delta(H)$. Then, for any $x \in V(H)$ there exist $z_1, z_2 \in N_H(x)$ such that $z_1 \neq z_2$ and

- $\chi(G) \leq d_H(z_1) + \mu(xz_1)$,
- $2\chi(G) \leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2)$.

Lemma 6.1.4. Let $G$ be the line graph of a multigraph $H$. If $G$ is vertex critical with $\chi(G) > \Delta(H)$, then

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$  

Proof. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. By Lemma 6.1.3 we have $z \in N_H(x)$ such that $\chi(G) \leq d_H(z) + \mu(xz)$. Hence

$$\Delta(G) + 1 \geq d_H(x) + d_H(z) - \mu(xz) \geq d_H(x) + \chi(G) - 2\mu(xz).$$

Which gives

$$\chi(G) \leq \Delta(G) + 1 - \Delta(H) + 2\mu(H).$$

Adding Vizing’s inequality $\chi(G) \leq \Delta(H) + \mu(H)$ gives the desired result. \qed

Combining this with Theorem 6.1.2 we get the following upper bound.

Theorem 6.1.5. If $G$ is the line graph of a multigraph, then

$$\chi(G) \leq \max \left\{ \omega(G), \frac{7\Delta(G) + 10}{8} \right\}.$$  

Proof. Suppose not and choose a counterexample $G$ with the minimum number of vertices. Say $G = L(H)$. Plainly, $G$ is vertex critical. Suppose $\chi(G) > \omega(G)$. By Theorem 6.1.2 we have

$$\chi(G) \leq \Delta(G) + 1 - \frac{\mu(H) - 1}{2}.$$  

By Lemma 6.1.4 we have

$$\chi(G) \leq \frac{3\mu(H) + \Delta(G) + 1}{2}.$$  

Adding three times the first inequality to the second gives

$$4\chi(G) \leq \frac{7}{2}(\Delta(G) + 1) + \frac{3}{2}.$$  

The theorem follows. \qed

Corollary 6.1.6. If $G$ is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 11$, then $G$ contains a $K_{\Delta(G)}$.

With a little more care we can get the 11 down to 9. Using Lemma 4.2.2, we can inductively reduce to the $\Delta = 9$ case.

Theorem 6.1.7. If $G$ is the line graph of a multigraph with $\chi(G) \geq \Delta(G) \geq 9$, then $G$ contains a $K_{\Delta(G)}$. 
Proof. Suppose the theorem is false and choose a counterexample $G$ minimizing $\Delta(G)$. Then $G$ is vertex critical. By Lemma 4.2.2, $\Delta(G) = 9$.

Let $H$ be such that $G = L(H)$. Then by Lemma 6.1.1 and Lemma 6.1.4 we know that $\mu(H) = 3$. Let $x \in V(H)$ with $d_H(x) = \Delta(H)$. Then we have $z_1, z_2 \in N_H(x)$ as in Lemma 6.1.3. This gives

$$9 \leq d_H(z_1) + \mu(xz_1),$$

$$18 \leq d_H(z_1) + \mu(xz_1) + d_H(z_2) + \mu(xz_2).$$

In addition, we have for $i = 1, 2$,

$$9 \geq d_H(x) + d_H(z_i) - \mu(xz_i) - 1 = \Delta(H) + d_H(z_i) - \mu(xz_i) - 1.$$

Thus,

$$\Delta(H) \leq 2\mu(xz_1) + 1 \leq 7,$$

$$\Delta(H) \leq \mu(xz_1) + \mu(xz_2) + 1.$$ (6.3)

(6.4)

Now, let $ab \in E(H)$ with $\mu(ab) = 3$. Then, since $G$ is vertex critical, we have $8 = \Delta(G) - 1 \leq d_H(a) + d_H(b) - \mu(ab) - 1 \leq 2\Delta(H) - 4$. Thus $\Delta(H) \geq 6$. Hence we have $6 \leq \Delta(H) \leq 7$. Thus, by (3), we must have $\mu(xz_1) = 3$.

First, suppose $\Delta(H) = 7$. Then, by (4) we have $\mu(xz_2) = 3$. Let $y$ be the other neighbor of $x$. Then $\mu(xy) = 1$ and thus $d_H(x) + d_H(y) - 2 \leq 9$. That gives $d_H(y) \leq 4$. Then we have vertices $w_1, w_2 \in N_H(y)$ guaranteed by Lemma 6.1.3. Note that $x \notin \{w_1, w_2\}$. Now $4 \geq d_H(y) \geq 1 + \mu(yw_1) + \mu(yw_2)$. Thus $\mu(yw_1) + \mu(yw_2) \leq 3$. This gives $d_H(w_1) + d_H(w_2) \geq 2\Delta(G) - 3 = 15$ contradicting $\Delta(H) \leq 7$.

Thus we must have $\Delta(H) = 6$. By (1) we have $d_H(z_1) = 6$. Then, applying (2) gives $\mu(xz_2) = 3$ and $d_H(z_2) = 6$. Since $x$ was an arbitrary vertex of maximum degree and $H$ is connected we conclude that $G = L(3 \cdot C_n)$ for some $n \geq 4$. But no such graph is 9-chromatic by Brooks’ theorem.

The graphs $G_t = L(t \cdot C_3)$ discussed above show that the following upper bound would be tight. Creating a counterexample would require some new construction technique that might lead to more counterexamples to Borodin-Kostochka for $\Delta = 8$.

Conjecture 6.1.8. If $G$ is the line graph of a multigraph, then

$$\chi(G) \leq \max \left\{ \omega(G), \frac{5\Delta(G) + 8}{6} \right\}.$$ (6.2)

6.2 Circular interval graphs

A representation of a graph $G$ in a graph $H$ consists of:

- an injection $f: V(G) \rightarrow V(H)$;

- for each $xy \in E(G)$, a choice of path $p_{xy} \subseteq H$ from $f(x)$ to $f(y)$ such that $f^{-1}(V(p_{xy}))$ is a clique in $G$.

A graph is a circular interval graph if it has a representation in a cycle. We note that this class coincides with the class of proper circular arc graphs. A graph is a linear interval graph if it has a representation in a path.

Lemma 6.2.1. Every circular interval graph satisfying $\chi_l \geq \Delta \geq 9$ contains a $K_\Delta$. 
Proof. Suppose the contrary and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$. Then $\chi(G) = \Delta$, $\omega(G) \leq \Delta - 1$, $\delta(G) \geq \Delta - 1$ and $\chi(G-v) \leq \Delta - 1$ for all $v \in V(G)$. Since $G$ is a circular interval graph, by definition $G$ has a representation in a cycle $v_1v_2\ldots v_n$. Let $K$ be a maximum clique in $G$. By symmetry we may assume that $V(K) = \{v_1, v_2, \ldots, v_t\}$ for some $t \leq \Delta - 1$; further, if possible we label the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$ and the edge goes through $v_{t-2}, v_{t-1}, v_t$.

Claim 1. $v_1 \not\leftrightarrow v_{t+1}$ and $v_2 \not\leftrightarrow v_{t+2}$ and $v_1 \not\leftrightarrow v_{t+2}$. Suppose the contrary. Clearly we can’t have $v_1 \leftrightarrow v_{t+1}$ and have the edge go through $v_2, v_3, \ldots, v_t$ (since then we get a clique of size $t+1$). Similarly, we can’t have $v_2 \leftrightarrow v_{t+2}$ and have the edge go through $v_3, v_4, \ldots, v_{t+1}$. So assume the edge $v_1v_{t+2}$ exists and goes around the other way. If $v_1 \leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1\}$ and if $v_1 \not\leftrightarrow v_{t+1}$, then let $G' = G \setminus \{v_1, v_{t+1}\}$. Now let $V_1 = \{v_2, v_3, \ldots, v_t\}$ and $V_2 = V(G')\setminus V_1$. Let $K' = G[V_1]$ and $L' = G[V_2]$; note that $K'$ and $L'$ are each cliques of size at most $\Delta - 2$. Now for each $S \subseteq V_2$, we have $|N_G(S) \cap V_1| \geq |S|$ (otherwise we get a clique of size $t$ in $G'$ and a clique of size $t+1$ in $G$). Now by Hall’s Theorem, we have a matching in $\overline{G}$ between $V_1$ and $V_2$ that saturates $V_2$. This implies that $G' \subseteq E^{\Delta-2}_2$, which in turn gives $G \subseteq E^{\Delta-1}_2$. By Lemma 5.3.5, $G$ is $(\Delta - 1)$-choosable, which is a contradiction.

Claim 2. $v_{t-3} \leftrightarrow v_{t+2}$ and the edge passes through $v_{t-2}, v_{t-1}, v_t$. Suppose the contrary. If $t \geq 7$, then since $t \leq \Delta - 1$, $v_4$ has some neighbor outside of $K$; by (reflectional) symmetry we could have labeled the vertices so that $v_{t-3} \leftrightarrow v_{t+1}$. So we must have $t \leq 6$. Each vertex $v$ that is high has either at least $\left\lceil \frac{\Delta}{2} \right\rceil$ clockwise neighbors or at least $\left\lfloor \frac{\Delta}{2} \right\rfloor$ counterclockwise neighbors. This gives a clique of size $1 + \left\lceil \frac{\Delta}{2} \right\rceil \geq 6$. If $v_3$ is high, then either $v_3$ has at least 4 clockwise neighbors, so $v_3 \leftrightarrow v_7$, or else $v_3$ has at least 6 counterclockwise neighbors, so $|K| \geq 7$. Thus, we may assume that $v_3$ is low; by symmetry (and our choice of labeling prior to Claim 1) $v_4$ is also low. Now since $v_4$ has only 3 counterclockwise neighbors, we get $v_4 \leftrightarrow v_7$ (in fact, we get $v_7 \leftrightarrow v_9$). Thus, $\{v_3, v_4, v_5, v_6, v_7\}$ induces $K_3 \ast E_2$ with a low degree vertex in both the $K_3$ and the $E_2$, which contradicts Lemma 5.3.49.

Claim 3. $v_{t-2} \not\leftrightarrow v_{t+2}$. Suppose the contrary. By Claim 1 the edge goes through $v_{t-1}, v_t, v_1$. If $v_{t-3} \leftrightarrow v_{t+2}$, then $\{v_1, v_2, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_3 \ast B$, where $B$ is not almost complete; this contradicts Lemma 5.3.18. If $v_{t-3} \not\leftrightarrow v_{t+2}$, then we get a $K_3 \ast P_4$ induced by $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$, which contradicts Lemma 5.3.27.

Claim 4. $v_{t-1} \not\leftrightarrow v_{t+2}$. If not, then $\{v_1, v_{t-3}, v_{t-2}, v_{t-1}, v_t, v_{t+1}, v_{t+2}\}$ induces $K_2 \ast$ antichair (with $v_{t-1}, v_t$ in the $K_2$), which contradicts Lemma 5.3.32.

Claim 5. $G$ is $(\Delta - 1)$-choosable. Let $S = \{v_{t-3}, v_{t-2}, v_{t-1}, v_t\}$. If any vertex of $S$ is low, then $S \cup \{v_1, v_{t+1}\}$ induces $K_4 \ast E_2$ with a low vertex in the $K_4$, which contradicts Lemma 5.3.48. So all of $S$ is high. If $v_1 \not\leftrightarrow v_{t+2}$, then $\{v_1, v_{t-1}, \ldots, v_{t-\Delta+1}\}$ (subscripts are modulo $n$) induces $K_\Delta$; So $v_1 \leftrightarrow v_{t+2}$. Since $v_{t-1} \not\leftrightarrow v_{t+2}$ and all of $S$ is high, we get $v_n \in (\cap_{v \in (S \setminus \{v_1\})} N(v)) \setminus N(v_1)$. Now we must have $v_n \not\leftrightarrow v_{t+1}$ (for otherwise $G$ is $(\Delta - 1)$-choosable, as in Claim 1). So we get $K_3 \ast P_4$ induced by $\{v_{t+1}, v_t, v_{t-1}, v_{t-2}, v_{t-3}, v_1, v_n\}$, which contradicts Lemma 5.3.29.

6.3 Quasi-line graphs

A graph is quasi-line if every vertex is bisimplicial (its neighborhood can be covered by two cliques). We apply a version of Chudnovsky and Seymour’s structure theorem for quasi-line graphs from King’s thesis [41]. The undefined terms will be defined after the statement.

Lemma 6.3.1. Every connected skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.

A homogeneous pair of cliques $(A_1, A_2)$ in a graph $G$ is a pair of disjoint nonempty cliques such that for each $i \in \{2\}$, every vertex in $G - (A_1 \cup A_2)$ is either joined to $A_i$ or misses all of $A_i$ and $|A_1| + |A_2| \geq 3$. A homogeneous pair of cliques $(A_1, A_2)$ is skeletal if for any $e \in E(A, B)$ we have $\omega(G[A \cup B] - e) < \omega(G[A \cup B])$. A graph is skeletal if it contains no nonskeletal homogeneous pair of cliques.
Generalizing a lemma of Chudnovsky and Fradkin [17], King proved a lemma allowing us to handle nonskeletal homogeneous pairs of cliques.

**Lemma 6.3.2 (King [41]).** If $G$ is a nonskeletal graph, then there is a proper subgraph $G'$ of $G$ such that:

1. $G'$ is skeletal;
2. $\chi(G') = \chi(G)$;
3. If $G$ is claw-free, then so is $G'$;
4. If $G$ is quasi-line, then so is $G'$.

It remains to define the generalization of line graphs introduced by Chudnovsky and Seymour [18]; this is the notion of compositions of strips (for a more detailed introduction, see Chapter 5 of [41]). We use the modified definition from King and Reed [43]. A strip $(H, A_1, A_2)$ is a claw-free graph $H$ containing two cliques $A_1$ and $A_2$ such that for each $i \in [2]$ and $v \in A_i$, $N_H(v) - A_i$ is a clique. If $H$ is a linear interval graph, then $(H, A_1, A_2)$ is a linear interval strip. Now let $H$ be a directed multigraph (possibly with loops) and suppose for each edge $e$ of $H$ we have a strip $(H_e, X_e, Y_e)$. For each $v \in V(H)$ define

$$C_v := \left( \bigcup \{X_e \mid e \text{ is directed out of } v\} \right) \cup \left( \bigcup \{Y_e \mid e \text{ is directed into } v\} \right)$$

The graph formed by taking the disjoint union of $\{H_e \mid e \in E(H)\}$ and making $C_v$, a clique for each $v \in V(H)$, is the composition of the strips $(H_e, X_e, Y_e)$. Any graph formed in such a manner is called a composition of strips. It is easy to see that if for each strip $(H_e, X_e, Y_e)$ in the composition we have $V(H_e) = X_e = Y_e$, then the constructed graph is just the line graph of the multigraph formed by replacing each $e \in E(H)$ with $|H_e|$ copies of $e$.

It will be convenient to have notation and terminology for a strip together with how it attaches to the graph. An interval 2-join in a graph $G$ is an induced subgraph $H$ such that:

1. $H$ is a (nonempty) linear interval graph,
2. The ends of $H$ are (not necessarily disjoint) cliques $A_1$, $A_2$,
3. $G - H$ contains cliques $B_1$, $B_2$ (not necessarily disjoint) such that $A_1$ is joined to $B_1$ and $A_2$ is joined to $B_2$,
4. there are no other edges between $H$ and $G - H$.

Note that $A_1, A_2, B_1, B_2$ are uniquely determined by $H$, so we are justified in calling both $H$ and the quintuple $(H, A_1, A_2, B_1, B_2)$ the interval 2-join. An interval 2-join $(H, A_1, A_2, B_1, B_2)$ is trivial if $V(H) = A_1 = A_2$ and canonical if $A_1 \cap A_2 = \emptyset$. A canonical interval 2-join $(H, A_1, A_2, B_1, B_2)$ with leftmost vertex $v_1$ and rightmost vertex $v_2$ is reducible if $H$ is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$ or $N_H(A_2) \setminus A_2 = N_H(v_1) \setminus A_2$. We call such a canonical interval 2-join reducible because we can reduce it as follows. Suppose $H$ is incomplete and $N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1$. Put $C := N_H(v_1) \setminus A_1$ and then $A_1' := C \setminus A_2$ and $A_2' := A_2 \setminus C$. Since $H$ is not complete $v_1 \in A_2'$ and hence $H' := G[A_1' \cup A_2']$ is a nonempty linear interval graph that gives the reduced canonical interval 2-join $(H', A_1', A_2', C \cap A_2', B_2 \cup (C \cap A_2))$.

**Lemma 6.3.3.** If $(H, A_1, A_2, B_1, B_2)$ is an irreducible canonical interval 2-join in a vertex critical graph $G$ with $\chi(G) = \Delta(G) \geq 9$, then $B_1 \cap B_2 = \emptyset$ and $|A_1|, |A_2| \leq 3$. Moreover, if $G$ is skeletal, then $H$ is complete.
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Proof. Let \((H, A_1, A_2, B_1, B_2)\) be an irreducible canonical interval 2-join in a vertex critical graph \(G\) with \(\chi(G) = \Delta(G) \geq 9\). Put \(\Delta := \Delta(G)\).

Note that, since it is vertex critical, \(G\) contains no \(K_\Delta\) and in particular \(G\) has no simplicial vertices. Label the vertices of \(H\) left-to-right as \(v_1, \ldots, v_t\). Say \(A_1 = \{v_1, \ldots, v_{\ell}\}\) and \(A_2 = \{v_{\ell+1}, \ldots, v_{t}\}\). For \(v \in V(H)\), define \(r(v) := \max \{i \in [t] \mid v \leftrightarrow v_i\}\) and \(l(v) := \min \{i \in [t] \mid v \leftrightarrow v_i\}\). These are well-defined since \(|H| \geq 2\) and \(H\) is connected by the following claim.

Claim 1. \(A_1, A_2, B_1, B_2 \neq \emptyset, B_1 \not\subseteq B_2, B_2 \not\subseteq B_1\) and \(H\) is connected. Otherwise \(G\) would contain a clique cutset.

Claim 2. If \(H\) is complete, then \(R - L = 1\). Suppose \(V(H) \neq A_1 \cup A_2\). Then any \(v \in V(H) \setminus A_1 \cup A_2\) would be simplicial in \(G\), which is impossible. Hence \(R - L = 1\).

Claim 3. If \(H\) is not complete, then \(r(v_L) = r(v_1) + 1\) and \(l(v_R) = l(v_1) - 1\). In particular, \(v_1, v_2\) are low and \(|A_1|, |A_2| \geq 2\). Suppose otherwise that \(H\) is not complete and \(r(v_L) \neq r(v_1) + 1\). By definition, \(N_H(v_1) \subseteq N_H(v_L)\) and \(v_1, v_L\) have the same neighbors in \(G \setminus H\). Hence if \(r(v_L) > r(v_1) + 1\), then \(d(v_L) - d(v_1) \geq 2\), impossible. So we must have \(r(v_L) = r(v_1)\) and hence \(N_H(A_1) \setminus A_1 = N_H(v_1) \setminus A_1\). Thus the 2-join is reducible, a contradiction. Therefore \(r(v_L) = r(v_1) + 1\). Similarly, \(l(v_R) = l(v_1) - 1\).

Claim 4. \(|A_1|, |A_2| \leq 3\). Suppose otherwise that \(|A_1| \geq 4\). First, suppose \(H\) is complete. By Claim 2, \(V(H) = A_1 \cup A_2\). If \(v_1\) is low, then for any \(w_1 \in B_1 \setminus B_2\) the vertex set \(\{v_1, \ldots, v_\ell, v_{\ell+1}, w_1\}\) induces a \(K_4 \ast E_2\) violating Lemma 5.3.48. Hence \(v_1\) is high. If \(|A_2| \geq 2\) and \(|B_1 \setminus B_2| \geq 2\), then for any \(w_1, w_2 \in B_1 \setminus B_2\), the vertex set \(\{v_1, \ldots, v_\ell, v_{\ell+1}, w_1, w_2\}\) induces a \(K_4 \ast 2K_2\), which is impossible by Lemma 5.3.18. Hence either \(|A_2| = 1\) or \(|B_1 \setminus B_2| = 1\). Suppose \(|A_2| = 1\). Then, since \(A_1 \cup B_1\) induces a clique and \(|A_1 \cup B_1| = d(v_1)\), \(v_1\) must be low, impossible. Hence we must have \(|B_1 \setminus B_2| = 1\). Thus \(|B_1 \cap B_2| = |B_1| - 1\). Hence \(V(H) \cup B_1 \cap B_2\) induces a clique with \(|A_1| + |A_2| + |B_1| - 1 = d(v_1) = \Delta\) vertices, impossible.

Therefore \(H\) must be incomplete. By Claim 3, \(v_1\) is low. But then as above for any \(w_1 \in B_1 \setminus B_2\) the vertex set \(\{v_1, \ldots, v_\ell, v_{\ell+1}, w_1\}\) induces a \(K_4 \ast E_2\) violating Lemma 5.3.48. Hence we must have \(|A_1| \leq 3\). Similarly, \(|A_2| \leq 3\). Claim 5. \(R - L = 1\). Suppose otherwise that \(R - L \geq 2\). Then by Claim 2, \(H\) is incomplete. Hence by Claim 3, \(r(v_L) = r(v_1) + 1\), \(l(v_R) = l(v_1) - 1\), \(v_1, v_\ell\) are low and \(|A_1|, |A_2| \geq 2\).

Subclaim 5a. \(L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1\). Since \(v_{L+1}\) has exactly \(L\) neighbors to the left, we have \(r(v_{L+1}) \leq L + 1 + \Delta - L = \Delta + 1 \leq L + \Delta - 1\). If \(v_{L+1}\) is high, the previous computation is exact and \(r(v_{L+1}) = \Delta + 1 \geq L + \Delta - 2\). Suppose \(v_{L+1}\) is low. If \(L = 3\), then for some \(w_1 \in B_1\) the vertex set \(\{v_1, v_2, v_3, v_4, w_1\}\) induces a \(K_3 \ast E_2\) violating Lemma 5.3.49. Hence \(L = 2\) and \(r(v_{L+1}) = L + 1 + \Delta - 1 - L = \Delta \geq L + \Delta - 2\).

Subclaim 5b. \(L + \Delta - 2 \leq r(v_{L+2}) \leq L + \Delta\). By Subclaim 5a, \(r(v_{L+2}) \geq L + \Delta - 2\). Since \(H\) contains no \(\Delta\)-clique, \(v_{L+2}\) has at least 2 neighbors to the left if it is high and at least 1 neighbor to the left if it is low. Thus \(r(v_{L+2}) \leq L + 2 + \Delta - 2 = L + \Delta\).

Subclaim 5c. If \(v_{L+4}\) is high, then \(l(v_{L+4}) \leq L\). Suppose otherwise. Recall that \(v_{L+4} \leftrightarrow v_{L+4}\). Then \(v_{L+4}\) has exactly 3 neighbors to the left, so \(r(v_{L+4}) = L + \Delta + 1\). Consider the subgraph induced on \(\{v_{L+1}, v_{L+2}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+7}, v_{L+9}, v_{L+10}\}\).

By Subclaim 5a and Subclaim 5b, this induces a subgraph violating Lemma 5.3.18.

Subclaim 5d. \(l(v_{L+3}) \leq L\). Suppose otherwise. Since \(v_{L+1} \leftrightarrow v_{L+3}\), vertex \(v_{L+3}\) has exactly 2 neighbors to the left, so \(r(v_{L+3}) \geq L + \Delta\). By Subclaim 5c, \(v_{L+4} \leftrightarrow v_{L+4}\) is low. By Subclaim 5a, \(L + \Delta - 2 \leq r(v_{L+1}) \leq L + \Delta - 1\). Therefore \(\{v_{L+1}, v_{L+3}, v_{L+4}, v_{L+5}, v_{L+6}, v_{L+9}\}\) induces a \(K_4 \ast E_2\) violating Lemma 5.3.48.

Subclaim 5e. \(l(v_{L+2}) \leq L - 1\). By Subclaim 5d \(r(v_{L+2}) \geq L + 3\) and hence by Claim 3, \(r(v_1) \geq L + 2\). Hence \(l(v_{L+2}) \leq L - 1\). Subclaim 5f. Claim 5 is true. Let \(\pi\) be a \((\Delta - 1)\)-coloring of \(G \setminus H\) and define a list assignment \(J\) on \(H\) by \(J(v) := [\Delta - 1] - \pi(N_{G \setminus H}(v))\). Then \(|J(v)| \geq d_H(v) - 1\) for all \(v \in V(H)\) and since \(v_1\) is low, \(|J(v_1)| \geq d_H(v_1)\). Pick \(w \in B_1\). Note that \(\pi(w) \not\in J(v_i)\) for \(i \in [L]\). Since \(J(v_{L+1}) = [\Delta - 1]\), we may color \(v_{L+1}\)
with \( \pi(w) \) to get a new list assignment \( J' \) on \( H' := H - v_{L+1} \). Then, since \( \pi(w) \not\in J(v_i) \) for \( i \in [L] \), we have \( |J'(v_i)| \geq d_{H'}(v_i) \) for \( i \in [L] \) and \( |J'(v_i)| \geq d_{H'}(v_i) + 1 \). Now color the vertices of \( H' \) greedily from their lists in the order \( v_1, v_{L-1}, \ldots, v_1 \). Since \( G \) has no \( \Delta \)-clique, we must have \( N(v_i) \not\subseteq A_2 \cup B_2 \), so \( l(v_i) \leq R - 1 \). Since \( l(v_R) = l(v_i) - 1 \), each of \( v_{L-1}, \ldots, v_1 \) have at least two neighbors to the left in \( H' \). For \( L + 4 \leq i < R \), since \( G \) doesn’t contain \( K_\Delta \), every high \( v_i \) has at least two neighbors to the left in \( H' \) and every low \( v_i \) at least one neighbor. By Subclaim 5d, the same holds for \( v_{L+3} \) and by Subclaim 5e, it holds for \( v_{L+2} \).

Hence each vertex will have a color free to use when we encounter it, so we can complete the \((\Delta - 1)\)-coloring to all of \( G \), a contradiction.

**Claim 6.** \( B_1 \cap B_2 = \emptyset \). Suppose otherwise that we have \( w \in B_1 \cap B_2 \).

**Subclaim 6a.** Each \( v \in V(H) \) is low, \( |B_1| = |B_2|, \ |B_1 \setminus B_2| = |B_2 \setminus B_1| = 1, d(v) = |A_1| + |A_2| + |B_1| - 1 \) for each \( v \in V(H) \) and \( H \) is complete. By Claim 5, we have \( d(v) \leq |A_1| + |A_2| + |B_1| - 1 \) for \( v \in A_1 \) and \( d(v) \leq |A_1| + |A_2| + |B_2| - 1 \) for \( v \in A_2 \). Also, as \( B_1 \not\subseteq B_2 \) and \( B_2 \not\subseteq B_1 \), we have \( d(w) \geq \max \{|B_1|, |B_2|\} + |A_1| + |A_2| \). So \( d(w) \geq d(v) + 1 \) for any \( v \in V(H) \). This implies that each \( v \in V(H) \) is low, \( |B_1'| = |B_2'\), \( |B_1 \setminus B_2'| = |B_2 \setminus B_1'| = 1, d(v) = |A_1| + |A_2| + |B_1'| - 1 \) for each \( v \in V(H) \) and hence \( H \) is complete.

**Subclaim 6b.** \( |B_1 \cup B_2| \leq 3 \). Suppose otherwise that \( |B_1 \cap B_2| \geq 4 \). Pick \( w_1 \in B_1 \setminus B_2, w_2 \in B_2 \setminus B_1 \) and \( z_1, z_2, z_3, z_4 \in B_1 \cap B_2 \). Then the set \( \{z_1, z_2, z_3, z_4, w_1, w_2, v_1, v_1\} \) induces a subgraph violating Lemma 5.3.18. Hence \( |B_1 \cup B_2| \leq 3 \).

**Subclaim 6c.** Claim 6 is true. By Subclaim 6a and Subclaim 6b, we have \( 2 \geq |B_1 \cap B_2| = |B_1| - 1 \) and hence \( |B_1| = |B_2| \leq 4 \). Suppose \( |A_1|, |A_2| \leq 2 \). Then \( \Delta - 1 = d(v_1) \leq 3 + |B_1| \leq 7 \), a contradiction. Hence by symmetry we may assume that \( |A_1| \geq 3 \). But then for \( w_1 \in B_1 \setminus B_2 \), the set \( \{v_1, v_2, v_3, v_1\} \) induces a \( K_3 \) \* \( E_2 \) violating Lemma 5.3.49.

**Claim 7.** If \( G \) is skeletal, then \( H \) is complete. Suppose \( G \) is skeletal and \( H \) is incomplete. By Claim 5, \( R - L = 1 \). Then, by Claim 3 \( r(v_L) = r(v_1) + 1 \) and \( l(v_R) = l(v_1) - 1 \). Since \( v_1 \) is not simplicial, \( r(v_1) \geq L + 1 = R \). Hence \( l(v_R) = 1 \) and thus \( l(v_i) = 2 \). Similarly, \( r(v_1) = t - 1 \). So, \( H \) is \( K_t \) less an edge. But \( (A_1, A_2) \) is a homogeneous pair of cliques with \( |A_1|, |A_2| \geq 2 \) and hence there is an edge between \( A_1 \) and \( A_2 \) that we can remove without decreasing \( \omega(G[A_1 \cup A_2]) \). This contradicts the fact that \( G \) is skeletal. \( \square \)

**Lemma 6.3.4.** An interval 2-join in a vertex critical graph satisfying \( \chi = \Delta \geq 9 \) is either trivial or canonical.

**Proof.** Let \((H, A_1, A_2, B_1, B_2)\) be an interval 2-join in a vertex critical graph satisfying \( \chi = \Delta \geq 9 \). Suppose \( H \) is nontrivial; that is, \( A_1 \neq A_2 \). Put \( C := A_1 \cap A_2 \). Then \((H \setminus C, A_1 \setminus C, A_2 \setminus C, C \cup B_1, C \cup B_2)\) is a canonical interval 2-join. Reduce this 2-join until we get an irreducible canonical interval 2-join \((H', A_1', A_2', B_1', B_2')\) with \( H' \leq H \setminus C \). Since \( C \) is joined to \( H - C \), it is also joined to \( H' \). Hence \( C \subseteq B_1' \cap B_2' = \emptyset \) by Lemma 6.3.3. Hence \( A_1 \cap A_2 = C = \emptyset \) showing that \( H \) is canonical. \( \square \)

**Theorem 6.3.5.** Every quasi-line graph satisfying \( \chi \geq \Delta \geq 9 \) contains a \( K_\Delta \).

**Proof.** We will prove the theorem by reducing to the case of line graphs, i.e., for every strip \((H, A_1, A_2)\) we have \( A_1 = A_2 \). Suppose not and choose a counterexample \( G \) minimizing \( |G| \).

Plainly, \( G \) is vertex critical. By Lemma 6.3.2, we may assume that \( G \) is skeletal. By Lemma 6.2.1, \( G \) is not a circular interval graph. Therefore, by Lemma 6.3.1, \( G \) is a composition of linear interval strips. Choose such a composition representation of \( G \) using the maximum number of strips.

Let \((H, A_1, A_2)\) be a strip in the composition. Suppose \( A_1 \neq A_2 \). Put \( B_1 := N_G[H](A_1)\) and \( B_2 := N_G[H](A_2)\). Then \((H, A_1, A_2, B_1, B_2)\) is an interval 2-join. Since \( A_1 \neq A_2 \), \( H \) is canonical by Lemma 6.3.4. Suppose \( H \) is reducible. By symmetry, we may assume that \( N_H[A_1] \setminus A_1 = N_H(v_1) \setminus A_1 \). But then replacing the strip \((H, A_1, A_2)\) with the two strips \((G[A_1], A_1, A_1)\) and \((H \setminus A_1, N_H[A_1] \setminus A_1, A_2)\) gives a composition representation of \( G \) using more strips, a contradiction. Hence \( H \) is irreducible. By Lemma 6.3.3, \( H \) is complete and
thus replacing the strip \((H, A_1, A_2)\) with the two strips \((G[A_1], A_1, A_1)\) and \((G[A_2], A_2, A_2)\) gives another contradiction.

Therefore, for every strip \((H, A_1, A_2)\) in the composition we must have \(V(H) = A_1 = A_2\). Hence \(G\) is a line graph of a multigraph. But this is impossible by Lemma 6.1.7.

\[
\Box
\]

\section{Claw-free graphs}

In this section we reduce the Borodin-Kostochka conjecture for claw-free graphs to the case of quasi-line graphs. We first show that a certain graph cannot appear in the neighborhood of any vertex in our counterexample.

![Figure 6.1: The graph \(N_6\).](image)

\begin{lemma}
The graph \(K_1 \ast N_6\) where \(N_6\) is the graph in Figure 6.1 is \(d_1\)-choosable.
\end{lemma}

\begin{proof}
Suppose not and let \(L\) be a minimal bad \(d_1\)-assignment on \(K_1 \ast N_6\). Then, by the Small Pot Lemma, \(|\text{Pot}(L)| \leq 6\). Let \(v\) be the vertex in the \(K_1\). Note that \(|L(v)| = 5\), \(|L(y)| = 4\), \(|L(x_2)| = 2\), and \(|L(x_i)| = 3\) for all \(i \in [4]\). Since \(\sum_{i=1}^{5} |L(x_i)| = 14 > |\text{Pot}(L)|\omega(C_5)\), we see that two nonadjacent \(x_i\)'s have a common color. Hence, by Lemma 5.3.15, we have \(|\text{Pot}(L)| \leq 5\). Thus we have \(c \in L(y) \cap L(x_3)\). Also, \(L(x_1) \cap L(x_4) \neq \emptyset\), \(L(x_1) \cap L(x_3) \neq \emptyset\) and \(L(x_2) \cap L(x_4) \neq \emptyset\). By Lemma 5.3.14, the common color in all of these sets must be \(c\). Hence \(c\) is in all the lists.

Now consider the list assignment \(L'\) where \(L'(z) = L(z) - c\) for all \(z \in N_6\). Then \(|\text{Pot}(L')| = 4\) and since \(\sum_{i=1}^{5} |L'(x_i)| = 9 > |\text{Pot}(L')|\omega(C_5)\), we see that that nonadjacent \(x_i\)'s have a common color different than \(c\). Now applying Lemma 5.3.14 gives a final contradiction.
\end{proof}

By a thickening of a graph \(G\), we just mean a graph formed by replacing each \(x \in V(G)\) by a complete graph \(T_x\) such that \(|T_x| \geq 1\) and for \(x, y \in V(G)\), \(T_x\) is joined to \(T_y\) if \(x \leftrightarrow y\).

\begin{lemma}
Any graph \(H\) with \(\alpha(H) \leq 2\) such that every induced subgraph of \(K_1 \ast H\) is not \(d_1\)-choosable can either be covered by two cliques or is a thickening of \(C_5\).
\end{lemma}

\begin{proof}
Suppose not and let \(H\) be a counterexample.

\textbf{Claim 1.} \(K_1 \ast H\) is \(d_0\)-choosable. Otherwise \(K_1 \ast H\) is a Gallai tree with a universal vertex. Since \(\alpha(H) \leq 2\), \(K_1 \ast H\) has at most two blocks and they must be complete. Hence \(H\) can be covered by two cliques, a contradiction. Claim 1 will allow us to apply Lemma 5.3.14 below.

\textbf{Claim 2.} \(H\) contains an induced \(C_4\) or an induced \(C_5\). Suppose not. Then \(H\) must be chordal since \(\alpha(H) \leq 2\). In particular, \(H\) contains a simplicial vertex \(x\). But then \(\{x\} \cup N_H(x)\) and \(V(H) - N_H(x) - \{x\}\) are two cliques covering \(H\), a contradiction.

\textbf{Claim 3.} \(H\) does not contain an induced \(C_5\) together with a vertex joined to at least 4 vertices in the \(C_5\). Suppose the contrary. If the vertex is joined to all of the \(C_5\), then we have a induced \(K_2 \ast C_5\), which is \(d_1\)-choosable by Lemma 5.3.47. If the vertex is joined to only four vertices in the \(C_5\), we have an induced \(K_1 \ast N_6\), impossible by Lemma 6.4.1.

\textbf{Claim 4.} \(H\) contains no induced \(C_4\). Suppose otherwise that \(H\) contains an induced \(C_4\), say \(x_1x_2x_3x_4x_1\). Put \(R := V(H) - \{x_1, x_2, x_3, x_4\}\). Let \(y \in R\). As \(\alpha(H) \leq 2\), \(y\)
has a neighbor in \( \{x_1, x_3\} \) and a neighbor in \( \{x_2, x_4\} \). If \( y \) is adjacent to all of \( x_1, \ldots, x_4 \), then \( K_1 \ast H \) contains \( K_2 \ast C_4 \) which is \( d_1 \)-choosable, impossible. If \( y \) is adjacent to three of \( x_1, \ldots, x_4 \), then \( K_1 \ast H \) contains \( E_2 \ast \text{paw} \) which is \( d_1 \)-choosable, impossible.

Thus every \( y \in R \) is adjacent to all and only the vertices on one side of the \( C_4 \). We show that any two vertices in \( R \) must be adjacent to the same or opposite side and this gives the desired covering by two cliques. If this doesn’t happen, then by symmetry we may suppose we have \( y_1, y_2 \in R \) such that \( y_1 \leftrightarrow x_1, x_2 \) and \( y_2 \leftrightarrow x_2, x_3 \). We must have \( y_1 \leftrightarrow y_2 \) for otherwise \( \{y_1, y_2, x_4\} \) is an independent set. But now \( x_1 y_1 y_2 x_3 x_4 x_1 \) is an induced \( C_5 \) in which \( x_2 \) has 4 neighbors, impossible by Claim 3.

**Claim 5.** \( H \) does not exist. By Claim 2 and Claim 4, \( H \) contains an induced \( C_5 \). That \( H \) is a thickening of this \( C_5 \) is now immediate from \( \alpha(H) \leq 2 \) and Claim 3. This final contradiction completes the proof.

![Figure 6.2: The graph \( D_8 \).](image)

**Lemma 6.4.3.** The graph \( D_8 \) is \( d_1 \)-choosable.

**Proof.** Suppose not and let \( L \) be a minimal bad \( d_1 \)-assignment on \( G := D_8 \).

**Claim 1.** \( |\text{Pot}(L)| \leq 6 \). By the Small Pot Lemma, we know that \( |\text{Pot}(L)| \leq 7 \). Suppose \( |\text{Pot}(L)| = 7 \). Say \( \text{Pot}(L) - L(w) = \{a, b\} \).

We must have \( L(y_3) = \{a, b\} \). Otherwise we could color \( y_3 \) from \( L(y_3) - \{a, b\} \) and note that \( G - y_3 - w \) is \( d_0 \)-choosable and hence has a coloring from its lists. Then we can easily modify this coloring to use both \( a \) and \( b \) at least once. But now we can color \( w \).

If there exist distinct vertices \( u, v \in V(G) - y_3 \) such that \( a \in L(u) \), \( b \in L(v) \) and \( \{u, v\} \not\subseteq \{x_2, x_3, x_4\} \), then we can color \( G \) as follows. Color \( y_3 \) arbitrarily to leave \( a \) available on \( u \) and \( b \) available on \( v \). Again, \( G - y_3 - w \) has a coloring. We can modify it to use \( a \) and \( b \), then color \( w \). Thus, \( a \) and \( b \) each appear only on some subset of \( \{y_3, x_2, x_3, x_4\} \).

If \( a \in L(x_2) \cap L(x_4) \), then we use \( a \) on \( x_2 \) and \( x_4 \) and color greedily \( y_3, x_3, y_4, x_1, x_5, w \) (actually any order will work if \( y_3 \) is first and \( w \) is last). If \( a \) appears only on \( y_3 \) and exactly one neighbor \( x_i \), then we violate Lemma 5.1.2 since \( |\text{Pot}_{y_3, x_i}(L)| < 7 \). So now \( a \) appears precisely on either \( y_3, x_2, x_3 \) or \( y_3, x_3, x_4 \). Similarly \( b \) appears precisely on either \( y_3, x_2, x_3 \) or \( y_3, x_3, x_4 \).

If \( \{a, b\} \cap L(x_2) = \emptyset \), then we use \( a \) on \( y_3 \) and \( b \) on \( x_3 \), then greedily color \( y_4, x_1, x_5, x_1, w, x_2 \). By symmetry, we may assume that \( a \in L(x_2) \). But then since \( \{a, b\} \subseteq L(x_3) \) we have \( |\text{Pot}_{y_3, x_2, x_3}(L)| < 7 \) violating Lemma 5.1.2. Hence \( |\text{Pot}(L)| \leq 6 \).

**Claim 2.** \( |\text{Pot}(L)| \leq 5 \). Suppose \( |\text{Pot}(L)| = 6 \). Choose \( a \in \text{Pot}(L) - L(w) \) and \( b \in L(w) \cap L(y_3) \). Put \( H := G - y_3 - w \).

First we show that \( b \in L(x_2) \cap L(x_3) \cap L(x_4) \). If not, we use \( b \) on \( y_3 \) and \( w \), then greedily color \( x_1, x_5, y_4 \). Now we can finish by coloring last the \( x_i \) such that \( b \notin L(x_i) \).

We must have \( a \in L(y_3) \) or else we color \( x_2, x_4 \) with \( b \) and something else in \( H \) with \( a \) (since \( G_a \) contains an edge by Lemma 5.1.2) and finish. Now \( a \notin L(x_1), L(x_3), L(y_4) \), for otherwise we color \( x_2, x_4 \) with \( b, y_3 \) with \( a \) and then color \( x_1, x_5, y_4, x_3 \) in order using \( a \) when we can, then color \( w \). Now \( a \) is on \( y_3 \) and at least two of \( x_2, x_3, x_4 \) or else we
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violate Lemma 5.1.2. Now \( a \notin L(x_2) \cap L(x_4) \) since otherwise we color \( x_2, x_4 \) with \( a \), then \( y_3 \) with \( b \), then greedily color \( x_1, x_5, y_4, x_3, w \). Also, \( a \notin L(x_2) \cap L(x_3) \) since then \( \{a, b\} \subseteq L(y_3) \cap L(x_2) \cap L(x_3) \) and hence \( |\text{Pot}_{y_3, x_2, x_3}(L)| < 6 \) violating Lemma 5.1.2. Therefore \( V(G_s) = \{y_3, x_3, x_4\} \).

Now \( |\text{Pot}_{y_3, x_3, x_4}(L)| \leq 6 \) and hence \( L(x_3) \cap L(x_4) = \{a, b\} \) for otherwise we violate Lemma 5.1.2. Say \( L(x_3) = \{a, b, c, d\} \) and \( L(x_4) = \{a, b, e, f\} \). Then by symmetry \( L(x_1) \) contains either \( c \) or \( e \). If \( c \in L(x_1) \), color \( x_1, x_3 \) with \( c, x_4 \) with \( a \) and \( y_3 \) with \( b \). Now we can greedily finish. If \( e \in L(x_1), color \ x_1, x_4 \ with \ e, \ x_3 \ with \ a \ and \ y_3 \ with \ b, again we can greedily finish. Hence \( |\text{Pot}(L)| \leq 5 \).

Claim 3. \( L \) does not exist. Since \( |\text{Pot}(L)| \leq 5 \) we see that \( x_3, x_4 \) have two colors in common and \( x_2, x_4 \) have two colors in common as well. In fact, these sets of common colors must be the same and equal \( L(y_3) = \{a, b\} \) or we can finish the coloring. Similarly, we may assume that \( a \in L(y_4) \) (if \( \{a, b\} \cap L(y_4) = \emptyset \), then we have \( L(x_2) \cap L(y_3) \cap (\text{Pot}(L) \setminus \{a, b\}) \neq \emptyset \) and color \( a \) on \( x_3, x_5 \), so we can color \( y_3 \) with \( b \) and then finish by Lemma 5.3.14). Similarly, \( L(x_1) \) contains \( a \) or \( b \). But it can’t contain \( a \) for then we could color \( y_3, y_4, x_1 \) with \( a \), and \( x_2, x_4 \) with \( b \), and then greedily finish. Say \( L(x_4) = \{a, b, c, d\} \). Then as no nonadjacent pair has a color in common that is in \( \text{Pot}(L) \setminus \{a, b\} \) we have \( L(x_2) = \{a, b, e\} \), then by symmetry of \( c \) and \( d \) we have \( L(x_5) = \{a, b, e\} \). Then \( L(x_3) = \{a, b, e\} \) and hence \( L(x_1) = \{a, b\} \), which contradicts that \( a \notin L(x_1) \). We conclude that \( L \) cannot exist.

Lemma 6.4.4. Let \( H \) be a thickening of \( C_5 \) such that \( |H| \geq 6 \). Then \( K_1 \ast H \) is \( f \)-choosable where \( f(v) \geq d(v) \) for the \( v \) in the \( K_1 \) and \( f(x) \geq d(x) - 1 \) for \( x \in V(H) \).

Proof. Suppose not and let \( L \) be a minimal bad \( f \)-assignment on \( K_1 \ast H \). By the Small Pot Lemma, \( |\text{Pot}(L)| \leq |H| \). Note that \( H \) is \( d_0 \)-choosable since it contains an induced diamond. Let \( x_1, \ldots, x_5 \) be the vertices of an induced \( C_5 \) in \( H \). Then \( \sum_{x_i} |L(x_i)| = \sum_i d_H(x_i) = 3|H| - 5 > 2|H| \geq \omega(H[x_1, \ldots, x_5]) |\text{Pot}(L)| \) and hence some nonadjacent pair in \( \{x_1, \ldots, x_5\} \) have a color in common. Now applying Lemma 5.3.16 gives a contradiction.

We are now in a position to finish the proof of Borodin-Kostochka for claw-free graphs.

Theorem 6.4.5. Every claw-free graph satisfying \( \chi \geq \Delta \geq 9 \) contains a \( K_\Delta \).

Proof. Suppose not and choose a counterexample \( G \) minimizing \( |G| \). Then \( G \) is vertex critical and not quasi-line by Lemma 6.3.5. Hence \( G \) contains a vertex \( v \) that is not bisimplicial. By Lemma 6.4.2, \( G_v := G[N(v)] \) is a thickening of a \( C_5 \). Also, by Lemma 6.4.4, \( v \) is high. Pick a \( C_5 \) in \( G_v \) and label its vertices \( x_1, \ldots, x_5 \) in clockwise order. For \( i \in [5] \), let \( T_i \) be the thickening clique containing \( x_i \). Also, let \( S \) be those vertices in \( V(G) - N(v) - \{v\} \) that have a neighbor in \( \{x_1, \ldots, x_5\} \). First we establish a few properties of vertices in \( S \).

Claim 1. For each \( z \in S \) there is \( i \in [5] \) such that \( N(z) \cap \{x_1, \ldots, x_5\} \subseteq \{x_i, x_{i+1}, x_{i+2}\} \). Let \( z \in S \) and put \( N := N(z) \cap \{x_1, \ldots, x_5\} \). If \( |N| \geq 4 \), then some subset of \( \{v, z\} \cup N \) induces the \( d_1 \)-choosable graph \( E_2 \ast P_4 \). Hence \( |N| \leq 3 \). Since \( G \) is claw-free, the vertices in \( N \) must be contiguous.

Claim 2. If \( z \in S \) is adjacent to \( x_1, x_{i+1}, x_{i+2} \), then \( |T_i| = |T_{i+1}| = |T_{i+2}| = 1 \). Suppose not. First, lets deal with the case when \( |T_{i+1}| \geq 2 \). Pick \( y \in T_{i+1} - x_{i+1} \). If \( y \nleftrightarrow z \), then \( \{x_i, y, z, x_{i+1}\} \) induces a claw, impossible. Thus \( y \leftrightarrow z \) and \( \{v, z, x_i, x_{i+1}, x_{i+2}, y\} \) induces the \( d_1 \)-choosable graph \( E_2 \ast \text{paw} \).

Hence, by symmetry, we may assume that \( |T_i| \geq 2 \). Now, if \( y \nleftrightarrow z \), then \( \{v, x_1, \ldots, x_5, y, z\} \) induces a \( D_8 \) contradicting Lemma 6.4.3. Hence \( y \leftrightarrow z \) and \( \{v, z, x_i, x_{i+1}, x_{i+2}, y\} \) induces the \( d_1 \)-choosable graph \( E_2 \ast \text{paw} \), a contradiction.

Claim 3. For \( i \in [5] \), let \( B_i \) be the \( z \in S \) with \( N(z) \cap \{x_1, \ldots, x_5\} = \{x_i, x_{i+1}\} \). Then \( B_i \cup B_{i+1} \) and \( B_i \cup T_i \cup T_{i+1} \) both induce cliques for any \( i \in [5] \). Otherwise there would be a claw.

Claim 4. \( |T_i| \leq 2 \) for all \( i \in [5] \). Suppose otherwise that we have \( i \) such that \( |T_i| \geq 3 \). Put \( A_i := N(x_i) \cap S \). By Claim 2, \( A_i \subseteq B_{i-1} \cup B_i \) and \( A_i \) is joined to \( T_i \). Thus \( T_i \) is joined to \( F_i := \{v\} \cup A_i \cup T_{i-1} \cup T_{i+1} \). If \( A_i \neq \emptyset \), then \( F_i \) induces a graph that is connected and not
almost complete, so this is impossible by Lemma 5.3.29. If $A_i = \emptyset$, then $x_i$ must have at least $\Delta - 2$ neighbors in $T_{i-1} \cup T_i \cup T_{i+1}$. But that leaves at most one vertex for $T_{i-2} \cup T_{i+2}$, impossible.

**Claim 5.** $G$ does not exist. Since $d(v) = \Delta \geq 9$, by symmetry we may assume that $|T_i| = 2$ for all $i \in [4]$. As in the proof of Claim 4, we get that $T_2$ is joined to $F_2$. Since $|T_i| \leq 2$ for all $i$, we must have $A_i \neq \emptyset$ (for all $i$, but in particular for $A_2$). Since $A_i \subseteq B_{i-1} \cup B_i$, by symmetry, we may assume that $A_2 \cap B_2 \neq \emptyset$. Pick $z \in A_2 \cap B_2$ and $y_i \in T_i - x_i$ for $i \in [3]$. Then $F_2$ has the graph in Figure 6.3 as an induced subgraph, but this is impossible by Lemma 5.3.47.

![Figure 6.3: $K_2$ joined to this graph is $d_1$-choosable](image)

We note that this reduction to the quasi-line case also works for the Borodin-Kostochka conjecture for list coloring; that is, we have the following result.

**Theorem 6.4.6.** If every quasi-line graph satisfying $\chi_l \geq \Delta \geq 9$ contains a $K_\Delta$, then the same statement holds for every claw-free graph.
The material in this chapter appeared in [20] and is joint work with Dan Cranston.

In this chapter we carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification, in Chapter 5, of graph joins $A \ast B$ with $|A| \geq 2$, $|B| \geq 2$ which are $f$-choosable, where $f(v) := d(v) - 1$ for each vertex $v$. Since such a join cannot be an induced subgraph of a vertex critical graph with $\chi = \Delta$, we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture. In Section 7.2, we exploit this information and minimality to improve Reed’s Lemma 1.1.11 as follows (see Corollary 7.2.10).

**Lemma 7.0.7.** Let $G$ be a vertex critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If $H$ is a $K_{\Delta - 1}$ in $G$, then any vertex in $G - H$ has at most 1 neighbor in $H$.

Moreover, we lift the result out of the context of a minimum counterexample to the Borodin-Kostochka conjecture, to the more general context of graphs satisfying a certain criticality condition—we call such graphs mules. This allows us to prove meaningful results for values of $\Delta$ less than 9.

Since a graph containing $K_{\Delta}$ as a subgraph also contains $K_{t,\Delta - t}$ as a subgraph for any $t \in [\Delta - 1]$, the Borodin-Kostochka conjecture implies the following conjecture. Our main result in this chapter is that the two conjectures are equivalent.

**Conjecture 7.0.8.** Any graph with $\chi = \Delta \geq 9$ contains some $A_1 \ast A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = \Delta$ and $A_i \neq K_1 + K_{|A_i| - 1}$ for some $i \in [2]$.

In fact, using Kostochka’s reduction (Lemma 4.2.2) to the case $\Delta = 9$, the following conjecture is also equivalent.

**Conjecture 7.0.9.** Any graph with $\chi = \Delta = 9$ contains some $A_1 \ast A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = 9$ and $A_i \neq K_1 + K_{|A_i| - 1}$ for some $i \in [2]$.

As a special case, we get a couple more palatable equivalent conjectures (see Lemma 7.2.17 and the comment following it).

**Conjecture 7.0.10.** Any graph with $\chi = \Delta \geq 9$ contains $K_3 \ast E_{\Delta - 3}$ as a subgraph.

**Conjecture 7.0.11.** Any graph with $\chi = \Delta = 9$ contains $K_3 \ast E_6$ as a subgraph.

The condition $A_i \neq K_1 + K_{|A_i| - 1}$ is unnatural and by removing it we get a (possibly) weaker conjecture than the Borodin-Kostochka conjecture which has more aesthetic appeal.

**Conjecture 7.0.12.** Let $G$ be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $3 \leq t \leq k - 3$, then $G$ can be $(k - 1)$-colored.
Conjecture 7.0.13. Conjecture 7.0.12 is equivalent to the Borodin-Kostochka conjecture.

Perhaps it would be easier to attack Conjecture 7.0.12 with $2 \leq t \leq k-2$ replaced by $2 \leq t \leq k-3$. We are unable to prove even this conjecture. Making this change and bringing $k$ down to 5 gives the following conjecture, which, if true, would imply the remaining two cases of Grünbaum’s girth problem for graphs with girth at least five.

Conjecture 7.0.14. Let $G$ be a graph with $\Delta(G) = k \geq 5$. If $K_{t,k-t} \not\subseteq G$ for all $2 \leq t \leq k-2$, then $G$ can be $(k-1)$-colored.

If $G$ is a graph with $\Delta(G) = k \geq 5$ and girth at least five, then it contains no $K_{t,k-t}$ for all $2 \leq t \leq k-2$ and hence Conjecture 7.0.14 would give a $(k-1)$-coloring. This conjecture would be tight since the Grünbaum graph and the Brinkmann graph are examples with $\chi = \Delta = 4$ and girth at least five.

Finally, we prove that the following conjecture is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6 (see Theorem 7.2.23).

Conjecture 7.0.15. Every graph satisfying $\chi = \Delta = 9$ and $\alpha \leq 6$ contains a $K_8$.

7.1 What is a mule?

Definition 7.1.1. If $G$ and $H$ are graphs, an epimorphism is a graph homomorphism $f: G \rightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\rightarrow$.

Definition 7.1.2. Let $G$ be a graph. A graph $A$ is called a child of $G$ if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f: H \rightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $G$. We call this the child order on $G$ and denote it by $\prec$. By definition, if $H \prec G$ then $H \prec G$.

Lemma 7.1.1. The ordering $\prec$ is well-founded on $G$; that is, every nonempty subset of $G$ has a minimal element under $\prec$.

Proof. Let $T$ be a nonempty subset of $G$. Pick $G \in T$ minimizing $|G|$ and then maximizing $\|G\|$. Since any child of $G$ must have fewer vertices or more edges (or both), we see that $G$ is minimal in $T$ with respect to $\prec$.

Definition 7.1.3. Let $T$ be a collection of graphs. A minimal graph in $T$ under the child order is called a $T$-mule.

With the definition of mule we have captured the important properties (for coloring) of a counterexample first minimizing the number of vertices and then maximizing the number of edges. Viewing $T$ as a set of counterexamples, we can add edges to or contract independent sets in induced subgraphs of a $T$-mule and get a non-counterexample. We could do the same with a minimal counterexample, but with mules we have more minimal objects to work with. One striking consequence of this is that many of our proofs naturally construct multiple counterexamples to Borodin-Kostochka for small $\Delta$.

7.2 Excluding induced subgraphs in mules

Our main goal in this section is to prove Lemma 7.2.11, which says that (with only one exception) for $k \geq 7$, no $k$-mule contains $K_3 \ast E_{k-4}$ as a subgraph. This result immediately implies that the Borodin-Kostochka conjecture is equivalent to Conjecture 7.2.12. This equivalence is a major step toward our main result. Our approach is based on Lemma 5.3.25, which implies that if $G$ is a counterexample to Lemma 7.2.11, then the vertices of the $E_{k-4}$ induce either $E_3$, a claw, a clique, or an almost complete graph. Our job in this section
For $k \in \mathbb{N}$, by a $k$-mule we mean a $C_k$-mule.

**Lemma 7.2.1.** Let $G$ be a $k$-mule with $k \geq 4$. If $A$ is a child of $G$ with $\Delta(A) \leq k$ then either

- $A$ is $(k-1)$-colorable; or,
- $A$ contains a $K_k$.

**Proof.** Let $A$ be a child of $G$ with $\Delta(A) \leq k$, $H \subseteq G$ and $f : H \rightarrow A$ an epimorphism. Without loss of generality, $A$ is vertex critical. Suppose $A$ is not $(k-1)$-colorable. Then $\chi(A) \geq k \geq \Delta(A)$. Since $A < G$ and $G$ is a mule, $A \notin C_k$. Thus we have $\chi(A) > \Delta(A) \geq 3$, so Brooks’ theorem implies that $A = K_k$. \qed

Note that adding edges to a graph yields an epimorphism.

**Lemma 7.2.2.** Let $G$ be a $k$-mule with $k \geq 4$ and $H \subseteq G$. Assume $x,y \in V(H)$, $xy \notin E(H)$ and both $d_H(x) \leq k - 1$ and $d_H(y) \leq k - 1$. If for every $(k-1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$, then $H$ contains $\{x,y\} \ast K_{k-2}$.

**Proof.** Suppose that for every $(k-1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$. Using the inclusion epimorphism $f_{xy} : H \rightarrow H + xy$ in Lemma 7.2.1 shows that either $H + xy$ is $(k-1)$-colorable or $H + xy$ contains a $K_k$. Since a $(k-1)$-coloring of $H + xy$ would induce a $(k-1)$-coloring of $H$ with $x$ and $y$ colored differently, we conclude that $H + xy$ contains a $K_k$. But then $H$ contains $\{x,y\} \ast K_{k-2}$ and the proof is complete. \qed

We will often begin by coloring some subgraph $H$ of our graph $G$, and work to extend this partial coloring. More formally, let $G$ be a graph and $H \subset G$. For $t \geq \chi(H)$, let $\pi$ be a proper $t$-coloring of $H$. For each $x \in V(G - H)$, put $L_\pi(x) := \{1,\ldots,t\} - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$. Then $\pi$ is completable to a $t$-coloring of $G$ iff $L_\pi$ admits a coloring of $G - H$. We will use this fact repeatedly in the proofs that follow. The following generalizes a lemma due to Reed [67], the proof is essentially the same.

**Lemma 7.2.3.** For $k \geq 6$, if a $k$-mule $G$ contains an induced $E_2 \ast K_{k-2}$, then $G$ contains an induced $E_3 \ast K_{k-2}$.

**Proof.** Suppose $G$ is a $k$-mule containing an induced $E_2 \ast K_{k-2}$, call it $F$. Let $x,y$ be the vertices of degree $k-2$ in $F$ and $C := \{w_1,\ldots,w_{k-2}\}$ the vertices of degree $k-1$ in $F$. Put $H := G - F$. Since $G$ is vertex critical, we may $k-1$ color $H$. Doing so leaves a list assignment $L$ on $F$ with $|L(z)| \geq d_F(z) - 1$ for each $z \in V(F)$. Now $|L(x)| + |L(y)| \geq d_F(x) + d_F(y) - 2 = 2k - 6 > k - 1$ since $k \geq 6$. Hence we have $c \in L(x) \cap L(y)$. Coloring both $x$ and $y$ with $c$ leaves a list assignment $L'$ on $C$ with $|L'(w_i)| \geq k - 3$ for each $1 \leq i \leq k - 2$. Now, if $|L'(w_i)| \geq k - 2$ or $L'(w_i) \neq L'(w_j)$ for some $i,j$, then we can complete the partial $(k-1)$-coloring to all of $G$ using Hall’s Theorem. Hence we must have $d(w_i) = k$ and $L'(w_i) = L'(w_j)$ for all $i,j$. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that $N$ is an independent set since it is contained in a single color class in every $(k-1)$-coloring of $H$. Also, each $w \in C$ has exactly one neighbor in $N$.

Proving that $|N| = 1$ will give the desired $E_3 \ast K_{k-2}$ in $G$. Thus, to reach a contradiction, suppose that $|N| \geq 2$.

We know that $H$ has no $(k-1)$-coloring in which two vertices of $N$ get different colors since then we could complete the partial coloring as above. Let $v_1,v_2 \in N$ be different. Since both $v_1$ and $v_2$ have a neighbor in $F$, we may apply Lemma 7.2.2 to conclude that $\{v_1,v_2\} \ast K_{v_1,v_2}$ is in $H$, where $K_{v_1,v_2}$ is a $K_{k-2}$. \qed
First, suppose $|N| \geq 3$, say $N = \{v_1, v_2, v_3\}$. We have $z \in K_{v_1,v_2} \cap K_{v_1,v_3}$ for otherwise $d(v_1) \geq 2(k-2) > k$. Since $z$ already has $k$ neighbors among $K_{v_1,v_2} - \{z\}$ and $v_1, v_2, v_3$, we must have $K_{v_1,v_3} = K_{v_1,v_2}$. But then $\{v_1, v_2, v_3\} + K_{v_1,v_2}$ is our desired $E_3 \ast K_{k-2}$ in $G$.

Hence we must have $|N| = 2$, say $N = \{v_1, v_2\}$. For $i \in [2]$, $v_i$ has $k - 2$ neighbors in $K_{v_i,v_2}$ and thus at most two neighbors in $C$. Hence $|C| \leq 4$. Thus we must have $k = 6$.

We may apply the same reasoning to $\{v_1, v_2\} \ast K_{v_1,v_2}$ that we did to $F$ to get vertices $v_{2,1}, v_{2,2}$ such that $\{v_{2,1}, v_{2,2}\} \ast K_{v_{2,1},v_{2,2}}$ is in $G$. But then we may do it again with $\{v_{2,1}, v_{2,2}\} \ast K_{v_{2,1},v_{2,2}}$ and so on. Since $G$ is finite, at some point this process must terminate. But the only way to terminate is to come back around and use $x$ and $y$. This graph is 5-colorable since we may color all the $E_3$’s with the same color and then 4-color the remaining $K_4$ components. This final contradiction completes the proof.

\[\]

**Figure 7.1:** The mule $M_{6,1}$.

**Figure 7.2:** The mule $M_{7,1}$.

**Lemma 7.2.4.** For $k \geq 6$, the only $k$-mules containing an induced $E_2 \ast K_{k-2}$ are $M_{6,1}$ and $M_{7,1}$.

**Proof.** Suppose we have a $k$-mule $G$ that contains an induced $E_2 \ast K_{k-2}$. Then by Lemma 7.2.3, $G$ contains an induced $E_3 \ast K_{k-2}$, call it $F$.

Let $x, y, z$ be the vertices of degree $k-2$ in $F$ and let $C := \{w_1, \ldots, w_{k-2}\}$ be the vertices of degree $k$ in $F$. Put $H := G - C$. Since each of $x, y, z$ have degree at most 2 in $H$ and $G$ is a mule, the homomorphism from $H$ sending $x, y$, and $z$ to the same vertex must produce a $K_k$. Thus we must have $k \leq 7$ and $H$ contains a $K_{k-1}$ (call it $D$) such that $V(D) \subseteq N(x) \cup N(y) \cup N(z)$. Put $A := G[V(F) \cup V(D)]$. Then $A$ is $k$-chromatic and as $G$ is a mule, we must have $G = A$. If $k = 7$, then $G = M_{7,1}$. Suppose $k = 6$ and $G \neq M_{6,1}$. Then one of $x, y$, or $z$ has only one neighbor in $D$. By symmetry we may assume it is $x$. But we can add an edge from $x$ to a vertex in $D$ to form $M_{6,1}$ and hence $G$ has a proper child, which is impossible. \[\]
Lemma 7.2.5. Let $G$ be a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $H \not\preceq G$. If $x, y \in V(H)$ and both $d_H(x) \leq k - 1$ and $d_H(y) \leq k - 1$, then there exists a $(k - 1)$-coloring $\pi$ of $H$ such that $\pi(x) \neq \pi(y)$.

Proof. Suppose $x, y \in V(H)$ and both $d_H(x) \leq k - 1$ and $d_H(y) \leq k - 1$. First, if $xy \in E(H)$ then any $(k - 1)$-coloring of $H$ will do. Otherwise, if for every $(k - 1)$-coloring $\pi$ of $H$ we have $\pi(x) = \pi(y)$, then by Lemma 7.2.2, $H$ contains $\{x, y\} \ast K_{k-2}$. The lemma follows since this is impossible by Lemma 7.2.4.

Lemma 7.2.6. Let $G$ be a $k$-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $F \not\preceq G$. Put $C := \{v \in V(F) \mid d(v) - d_F(v) \leq 1\}$. At least one of the following holds:

- $G - F$ has a $(k - 1)$-coloring $\pi$ such that for some $x, y \in C$ we have $L_\pi(x) \neq L_\pi(y)$; or,
- $G - F$ has a $(k - 1)$-coloring $\pi$ such that for some $x \in C$ we have $|L_\pi(x)| = k - 1$; or,
- there exists $z \in V(G - F)$ such that $C \subseteq N(z)$.

Proof. Put $H := G - F$. Suppose that for every $(k - 1)$-coloring $\pi$ of $H$ we have $L_\pi(x) = L_\pi(y)$ for every $x, y \in C$. By assumption, the vertices in $C$ have at most one neighbor in $H$. If some $v \in C$ has no neighbors in $H$, then for any $(k - 1)$-coloring $\pi$ of $H$ we have $|L_\pi(v)| = k - 1$. Thus we may assume that every $v \in C$ has exactly one neighbor in $H$.

Let $N := \bigcup_{w \in C} N(w) \cap V(H)$. Suppose $|N| \geq 2$. Pick different $z_1, z_2 \in N$. Then, by Lemma 7.2.5, there is a $(k - 1)$-coloring $\pi$ of $H$ for which $\pi(z_1) \neq \pi(z_2)$. But then $L_{\pi}(x) \neq L_{\pi}(y)$ for some $x, y \in C$ giving a contradiction. Hence $N = \{z\}$ and thus $C \subseteq N(z)$.

By Lemma 5.3.19, no graph in $C_k$ contains an induced $E_3 \ast K_{k-3}$ for $k \geq 9$. For mules, we can improve this as follows.

Lemma 7.2.7. For $k \geq 7$, the only $k$-mule containing an induced $E_3 \ast K_{k-3}$ is $M_{7,1}$.

Proof. The lemma is false and suppose $G$ be a $k$-mule, other than $M_{7,1}$, containing such an induced subgraph $F$. Let $z_1, z_2, z_3 \in F$ be the vertices with degree $k - 3$ in $F$ and $C$ the rest of the vertices in $F$ (all of degree $k - 1$ in $F$). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of $C$. Let $\pi$ be a $(k - 1)$-coloring of $H$ guaranteed by Lemma 7.2.6 and put $L := L_\pi$. Since $|L(z_1)| + |L(z_2)| + |L(z_3)| \geq 3(k - 4) > k - 1$ we have $1 \leq i < j \leq 3$ such that $L(z_i) \cap L(z_j) \neq \emptyset$. Without loss of generality, $i = 1$ and $j = 2$. Pick $c \in L(z_1) \cap L(z_2)$ and color both $z_1$ and $z_2$ with $c$. Let $L'$ be the resulting list assignment on $F - \{z_1, z_2\}$. Now $|L'(z_3)| \geq k - 4$ and $|L'(v)| \geq k - 3$ for each $v \in C$. By our choice of $\pi$, either two of the lists in $C$ differ or for some $v \in C$ we have $|L'(v)| \geq k - 2$. In either case, we can complete the $(k - 1)$-coloring to all of $G$ by Hall’s Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of $C$. Thus $G$ contains the induced subgraph $K_{k-3} \ast G[z_1, z_2, z_3, x]$. Therefore $k = 7$ and $x$ is adjacent to each of $z_1, z_2, z_3$ by Lemma 5.3.25. Hence $G$ contains the induced subgraph $K_5 \ast E_3$ contradicting Lemma 7.2.4.

Lemma 7.2.8. For $k \geq 7$, no $k$-mule contains an induced $T_3 \ast K_{k-3}$.

Proof. Suppose the lemma is false and let $G$ be a $k$-mule containing such an induced subgraph $F$. Note that $M_{7,1}$ has no induced $T_3 \ast K_{k-3}$, so $G \not\preceq M_{7,1}$. Let $z \in V(F)$ be the vertex with degree $k - 3$ in $F$, $v_1, v_2 \in F$ the vertices of degree $k - 2$ in $F$ and $C$ the rest of the vertices in $F$ (all of degree $k - 1$ in $F$). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of $C$. Let $\pi$ be a $(k - 1)$-coloring of $H$ guaranteed by Lemma 7.2.6 and put $L := L_\pi$. Then, we have $|L(z)| \geq k - 4$ and $|L(v_1)| \geq k - 3$. Since $k \geq 7$, $|L(z)| + |L(v_1)| \geq 2k - 7 > k - 1$. Hence, by Lemma 5.2.8, we may color $z$ and $v_1$ the same. Let $L'$ be the resulting list assignment
on $F - \{z, v_1\}$. Now $|L'(v_2)| \geq k - 4$ and $|L'(v)| \geq k - 3$ for each $v \in C$. By our choice of $\pi$, either two of the lists in $C$ differ or for some $v \in C$ we have $|L'(v)| \geq k - 2$. In either case, we can complete the $(k-1)$-coloring to all of $G$ by Hall’s Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of $C$. Thus $G$ contains the induced subgraph $K_4 \ast G[z, v_1, v_2, x]$. By Lemma 5.3.25, $G[z, v_1, v_2, x]$ must be almost complete and hence $x$ must be adjacent to both $v_1$ and $v_2$. But then $G[v_1, v_2, x] \ast C$ is a $K_k$ in $G$, giving a contradiction. \hfill $\square$

Reed proved that for $k \geq 9$, a vertex outside a $(k - 1)$-clique $H$ in a $k$-mule can have at most 4 neighbors in $H$. We improve this to at most one neighbor.

![Figure 7.3: The mule $M_{7,2}$.](image)

**Lemma 7.2.9.** For $k \geq 7$ and $r \geq 2$, no $k$-mule except $M_{7,1}$ and $M_{7,2}$ contains an induced $K_r \ast (K_1 + K_{k-(r+1)})$.

**Proof.** Suppose the lemma is false and let $G$ be a $k$-mule, other than $M_{7,1}$ and $M_{7,2}$, containing such an induced subgraph $F$ with $r$ maximal. By Lemma 7.2.4 and Lemma 7.2.8, the lemma holds for $r \geq k - 3$. So we have $r \leq k - 4$. Now, let $z \in V(F)$ be the vertex with degree $r$ in $F$, $v_1, v_2, \ldots, v_{k-(r+1)} \in V(F)$ the vertices of degree $k - 2$ in $F$ and $C$ the rest of the vertices in $F$ (all of degree $k - 1$ in $F$). Put $H := G - F$.

Let $Z_1 := \{za \mid a \in N(v_1) \cap V(H)\}$. Consider the graph $D := H + z + Z_1$. Since $v_1$ has at most two neighbors in $H$, $|Z_1| \leq 2$ and thus to form $D$ from $H + z$, we added $E(A)$ where $A \in \{K_1, K_2, P_3\}$. Since $|C| \geq 2$, $\Delta(D) \leq k$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$. Suppose $\chi(D) \leq k - 1$. If $A = K_1$, $A = K_2$, or $A = P_3$, then we have a contradiction by the fact that $\omega(G) < k$, Lemma 7.2.4, and Lemma 7.2.8, respectively. Thus we must have $\chi(D) \leq k - 1$, which gives a $(k - 1)$-coloring of $H + z$ in which $z$ receives a color $c$ which is not received by any of the neighbors of $v_1$ in $H$. Thus $c$ remains in the list of $v_1$ and we may color $v_1$ with $c$. After doing so, each vertex in $C$ has a list of size at least $k - 3$ and $v_1$ for $i > 1$ has a list of size at least $k - 4$. If any pair of vertices in $C$ had different lists, then we could complete the partial coloring by Hall’s Theorem. Let $N := \bigcup_{u \in C} N(u) \cap V(H)$ and note that $N$ is an independent set since it is contained in a single color class in the $(k-1)$-coloring of $H$ just constructed.

Suppose $|N| \geq 2$. Pick $a_1, a_2 \in N$. Consider the graph $D := H + z + Z_1 + a_1a_2$. Plainly, $\Delta(D) \leq k$. To form $D$ from $H + z$ we added $E(A)$, where $A \in \{K_1, K_2, P_3, P_4, K_2 + P_3\}$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$. If $\chi(D) \geq k$, then we have a contradiction since $A = K_1$, $A = K_2$, and $A = P_3$ are impossible as above. To show that $A = K_1$, $A = P_4$, and $A = K_2 + P_3$ are impossible, we apply Lemma 7.2.7 (this is where we use the fact that $G \neq M_{7,1}$), Lemma 5.3.27 (since $K_1 - E(P_4) = P_4 * K_{t-4}$), and Lemma 5.3.22, respectively.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k - 1)$-coloring of $H + z$ in which $a_1$ and $a_2$ are in different color classes and $z$ receives a color not received by any neighbor of $v_1$ in $H$. As above we can complete this partial coloring to all of $G$ by first coloring $z$ and $v_1$ the same and then using Hall’s Theorem.

Hence there is a vertex $x \in V(H)$ which is adjacent to all of $C$. Note that $x$ is not adjacent to any of $v_1, v_2, \ldots, v_{k-(r+1)}$ by the maximality of $r$. Let $Z_2 := \{xa \mid a \in N(v_2) \cap V(H)\}$. 


Consider the graph $D := H + z + Z_1 + Z_2$. As above, both $Z_1$ and $Z_2$ have cardinality at most 2. Since $|C| \geq 2$, both $x$ and $z$ have degree at most $k$ in $D$. Since both $xa$ and $za$ were added only if $a$ was a neighbor of both $v_1$ and $v_2$, all the neighbors of $v_1$ in $H$ have degree at most $k$ in $D$. Similarly for $v_2$’s neighbors. Hence $\Delta(D) \leq k$. To form $D$ from $H + z$ we added $E(A)$ where $A \in \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3, 2K_2, P_5, 2P_3, C_4\}$. Hence Lemma 7.2.1 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$.

Suppose $\chi(D) \geq k$. Then $A = K_1$, $A = K_2$, $A = P_3$, $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible as above. Applying Lemma 5.3.22 shows that $A = 2K_2$, $A = P_5$, and $A = 2P_3$ are impossible. Thus we must have $A = C_4$. If $k \geq 8$, then Lemma 5.3.18 gives a contradiction. Hence we must have $k = 7$. Since $H + z$ contains an induced $K_3 * 2K_2$, we must have $N(v_1) \cap V(H) = N(v_2) \cap V(H)$, say $N(v_1) \cap V(H) = \{w_1, w_2\}$. Moreover, $xz \in E(G)$, $w_1w_2 \in E(G)$ and there are no edges between $\{w_1, w_2\}$ and $\{x, z\}$ in $G$.

Put $Q := \{v_1, \ldots, v_{k-(r+1)}\}$. Then for $v \in Q$, by the same argument as above, we must have $N(v) \cap V(H) = \{w_1, w_2\}$. Hence $Q$ is joined to $\{w_1, w_2\}$, $C$ is joined to $Q$, and $\{x, z\}$ and both $\{x, z\}$ and $\{w_1, w_2\}$ are joined to the same $K_3$ in $H$. We must have $r = 3$ for otherwise one of $x, z, w_1, w_2$ has degree larger than 7. Thus we have an $M_{7,2}$ in $G$ and therefore $G$ is $M_{7,2}$, a contradiction.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k - 1)$-coloring of $H + z$ in which $z$ receives a color $c_1$ which is not received by any of the neighbors of $v_1$ in $H$ and $x$ receives a color $c_2$ which is not received by any of the neighbors of $v_2$ in $H$. Thus $c_1$ is in $v_1$’s list and $c_2$ is in $v_2$’s list. Note that if $x$ and $z$ are adjacent then $c_1 \neq c_2$. Hence, we can 2-color $G[x, z, v_1, v_2]$ from the lists. This leaves $k - 3$ vertices. The vertices in $C$ have lists of size at least $k - 3$ and the rest have lists of size at least $k - 5$. Since the union of any $k - 4$ of the lists contains one list of size $k - 3$, we can complete the partial coloring by Hall’s Theorem.

**Corollary 7.2.10.** For $k \geq 7$, if $H$ is a $(k - 1)$-clique in a $k$-mule $G$ other than $M_{7,1}$ and $M_{7,2}$, then any vertex in $G - H$ has at most one neighbor in $H$.

**Proof.** Let $v \notin H$ be adjacent to $r$ vertices in $H$. Now $G[H \cup \{v\}] = K_r \ast (K_1 + K_{k-(r+1)})$. If $r \geq 2$, then $G[H \cup \{v\}]$ is forbidden by Lemma 7.2.9.

**Lemma 7.2.11.** For $k \geq 7$, no $k$-mule except $M_{7,1}$ contains $K_4 \ast E_{k-4}$ as a subgraph.

**Proof.** Let $G$ be a $k$-mule other than $M_{7,1}$ and suppose $G$ contains an induced $K_4 \ast D$ where $|D| = k - 4$. Then $G$ is not $M_{7,2}$. By Lemma 5.3.25, $D$ is $E_3$, a claw, a clique, or almost complete. If $D$ is a clique then $G$ contains $K_k$, a contradiction. Now Corollary 7.2.10 shows that $D$ being almost complete is impossible. Finally, Lemma 7.2.7 shows that $D$ cannot be $E_3$ or a claw. This contradiction completes the proof.

Since $K_4 \ast E_{\Delta-4} \subseteq K_{\Delta}$, Lemma 7.2.11 shows that the following conjecture is equivalent to the Borodin-Kostochka conjecture.

**Conjecture 7.2.12.** Any graph with $\chi \geq \Delta \geq 9$ contains $K_4 \ast E_{\Delta-4}$ as a subgraph.

**Lemma 7.2.13.** Let $G$ be a $k$-mule with $k \geq 8$. Let $A$ and $B$ be graphs with $4 \leq |A| \leq k - 4$ and $|B| = k - |A|$ such that $A \ast B \subseteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 \ast K_{|B|-1}$.

**Proof.** Note that $|B| \geq 4$. By Lemma 5.3.44, $A \ast B$ is almost complete, $K_5 \ast E_3$ or our desired conclusion holds. The first and second cases are impossible by Corollary 7.2.10 and Lemma 7.2.7.

This shows that the following conjecture is a natural weakening of Borodin-Kostochka.

**Conjecture 7.2.14.** Let $G$ be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $4 \leq t \leq k - 4$, then $G$ can be $(k - 1)$-colored.

In the next section we create the tools needed to reduce the 4 in these lemmata to 3.
7.2.1 Tooling up

For an independent set $I$ in a graph $G$, we write $\frac{G}{[I]}$ for the graph formed by collapsing $I$ to a single vertex and discarding duplicate edges. We write $[I]$ for the resulting vertex in the new graph. If more than one independent set $I_1, I_2, \ldots, I_m$ are collapsed in succession we indicate the resulting graph by $\frac{G}{[I_1][I_2]\cdots[I_m]}$.

**Lemma 7.2.15.** Let $G$ be a $k$-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \not\subseteq G$. If $x, y \in V(H)$, $xy \not\in E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$, then there exists a $(k-1)$-coloring $\pi$ of $H$ such that $\pi(x) = \pi(y)$.

**Proof.** Suppose $x, y \in V(H)$, $xy \not\in E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$. Put $H' := \frac{H}{x,y}$. Then $H' \prec H$ via the natural epimorphism $f : H \to H'$. By applying Lemma 7.2.1 we either get the desired $(k-1)$-coloring $\pi$ of $H$ or a $K_{k-1}$ in $H$ with $V(K_{k-1}) \subseteq N(x) \cup N(y)$. But $k - 1 \geq 6$, so one of $x$ or $y$ has at least three neighbors in $K_{k-1}$ violating Corollary 7.2.10.

**Lemma 7.2.16.** Let $G$ be a $k$-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \not\subseteq G$. Suppose there are disjoint nonadjacent pairs $\{x_1, y_1\}, \{x_2, y_2\} \subseteq V(H)$ with $d_H(x_1), d_H(y_1) \leq k-1$ and $|N_H(x_2) \cup N_H(y_2)| \leq k$. Then there exists a $(k-1)$-coloring $\pi$ of $H$ such that $\pi(x_1) \neq \pi(y_1)$ and $\pi(x_2) = \pi(y_2)$.

**Proof.** Put $H' := \frac{H}{x_1,y_1} + x_1y_1$. Then $H'$ $\prec$ $H$ via the natural epimorphism $f : H \to H'$. Suppose the desired $(k-1)$-coloring $\pi$ of $H$ doesn’t exist. Apply Lemma 7.2.1 to get a $K_k$ in $H'$. Put $z := [x_2, y_2]$. By Lemma 7.2.4 the $K_k$ must contain $z$ and by Lemma 7.2.9, the $K_k$ must contain $x_1y_1$; hence the $K_k$ contains $x_1, y_1, z$. Thus $H$ contains an induced subgraph $A := \{x_1, y_1\} \ast K_{k-3}$ where $V(A) \subseteq N_H(x_2) \cup N_H(y_2)$. Then $x_2$ and $y_2$ each have at most two neighbors in the $K_{k-3}$ by Lemma 7.2.11 and Lemma 5.3.29. Thus $k = 7$ and both $x_2$ and $y_2$ have exactly two neighbors in the $K_4$. One of $x_2$ or $y_2$ has at least one neighbor in $\{x_1, y_1\}$, so by symmetry we may assume that $x_2$ is adjacent to $x_1$. But then $\{x_2\} \cup V(A)$ induces either a $K_2 \ast$ antichair (if $x_2 \not\leftrightarrow y_1$) or a graph containing $K_2 \ast C_4$ (if $x_2 \leftrightarrow y_1$), and both are impossible by Lemma 5.3.45.

7.2.2 Using our new tools

![Figure 7.4: The mule $M_8$.](image)

**Lemma 7.2.17.** For $k \geq 7$, the only $k$-mules containing $K_3 \ast E_{k-3}$ as a subgraph are $M_{7,1}$, $M_{7,2}$ and $M_8$.

**Proof.** Suppose not and let $G$ be a $k$-mule other than $M_{7,1}$, $M_{7,2}$ and $M_8$ containing $F := C \ast B$ as an induced subgraph where $C = K_3$ and $B$ is an arbitrary graph with $|B| = k-3$. By Lemma 5.3.29, $B$ is: $E_3 \ast K_{|B|-3}$, almost complete, $K_i + K_{|B|-i}$, $K_1 + K_i + K_{|B|-i}$, or $E_3 + K_{|B|-3}$. The first two options are impossible by Lemma 7.2.11.

First, suppose there is no $z \in V(G - F)$ with $C \subseteq N(z)$. Let $\pi$ be the $(k-1)$-coloring of $G - F$ guaranteed by Lemma 7.2.6. Put $L := L_{\pi}$. Let $I$ be a maximal independent set
in $B$. If there are $x, y \in I$ and $c \in L(x) \cap L(y)$, then we may color $x$ and $y$ with $c$ and then greedily complete the coloring to the rest of $F$ giving a contradiction. Thus we must have

$$k - 1 \geq \sum_{v \in I} |L(v)|$$

$$\geq \sum_{v \in I} (d_F(v) - 1)$$

$$= \sum_{v \in I} (d_B(v) + 3 - 1)$$

$$= 2 |I| + \sum_{v \in I} d_B(v)$$

$$= |B| + |I|$$

$$= k - 3 + |I|.$$ 

Therefore $|I| \leq 2$ and hence $B$ is $K_4 + K_{1,1}$. Put $N := \bigcup_{w \in C} N(w) \cap V(G - F)$. Then $|N| \geq 2$ by assumption. Pick $x_1, y_1 \in N$ and nonadjacent $x_2, y_2 \in V(B)$ and put $H := G[V(G - F) \cup \{x_2, y_2\}]$. Plainly, the conditions of Lemma 7.2.16 are satisfied and hence we have a $(k - 1)$-coloring $\gamma$ of $H$ such that $\gamma(x_1) \neq \gamma(y_1)$ and $\gamma(x_2) = \gamma(y_2)$. But then we can greedily complete this coloring to all of $G$, a contradiction.

Thus we have $z \in V(G - F)$ with $C \subseteq N(z)$. Put $B' := G[V(B) \cup \{z\}]$ and $F' := G[V(F) \cup \{z\}]$. As above, using Lemma 5.3.29 and Lemma 7.2.11, we see that $B'$ is $K_1 + K_{|B'|-t}, K_1 + K_{|B'|-t-1} \text{ or } E_3 + K_{|B'|-3}$.

Suppose $B'$ is $E_3 + K_{|B'|-3}$, say $E_3 \leq \{z_1, z_2, z_3\}$. Since $k \geq 7$, we have $w_1, w_2 \in V(B') - \{z_1, z_2, z_3\}$. Then $d_{B'}(z_3) + d_{B'}(w_1) = k$ and hence we may apply Lemma 7.2.15 to get a $(k - 1)$-coloring $\zeta$ of $G - F'$ such that there is some $c \in L_{\zeta}(z_3) \cap L_{\zeta}(w_1)$. Now $|L_{\zeta}(z_1)| + |L_{\zeta}(z_2)| + |L_{\zeta}(w_2)| \geq 2 + k - 4 = k$ and hence there is a color $c_1$ that is in at least two of $L_{\zeta}(z_1), L_{\zeta}(z_2)$ and $L_{\zeta}(w_2)$. If $c_1 = c$, then $c$ appears on an independent set of size 3 in $B'$ and we may color this set with $c$ and greedily complete the coloring. Otherwise, $B'$ contains two disjoint nonadjacent pairs which we can color with different colors and again complete the coloring greedily, a contradiction.

Now suppose $B'$ is $K_1 + K_{|B'|-t}$. By Lemma 7.2.9, we have $2 \leq t \leq |B'|-3$. Let $x$ be the vertex in the $K_1, w_1, w_2 \in V(K_1)$ and $z_1, z_2 \in V(K_{|B'|-t})$. Then $d_{B'}(w_1) + d_{B'}(z_3) = k$ and hence we may apply Lemma 7.2.15 to get a $(k - 1)$-coloring $\zeta$ of $G - F'$ such that there is some $c \in L_{\zeta}(w_1) \cap L_{\zeta}(z_1)$. Now $|L_{\zeta}(x)| + |L_{\zeta}(w_2)| + |L_{\zeta}(w_2)| \geq 2 + k - 1 = k + 1$ and hence there are at least two colors $c_1, c_2$ that are each in at least two of $L_{\zeta}(x), L_{\zeta}(w_2)$ and $L_{\zeta}(z_2)$. If $c_1 \neq c$ or $c_2 \neq c$, then $B'$ contains two disjoint nonadjacent pairs which we can color with different colors and then complete the coloring greedily. Otherwise $c$ appears on an independent set of size 3 in $B'$ and we may color this set with $c$ and greedily complete the coloring, a contradiction.

Therefore $B'$ must be $K_1 + K_{|B'|-t}$. By Lemma 7.2.9, we have $3 \leq t \leq |B'|-3$. Thus $k \geq 8$. Let $X$ and $Y$ be the two cliques covering $B'$. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Put $H := G[V(G - F') \cup \{x_1, x_2, y_1, y_2\}]$ and $H' := \frac{H}{x_1,y_1\cup x_2,y_2}$. For $i \in [2], d_{H'}(x_i) + d_{H'}(y_i) = k + 2$ and thus $\Delta(H') \leq k$. If $\chi(H') \leq k - 1$, then we have a $(k - 1)$-coloring of $H'$ which can be greedily completed to all of $G$, a contradiction. Hence, by Lemma 7.2.1, $H'$ contains $K_k$. Hence $H - \{x_1, x_2, y_1\}$ contains a $K_{k-2}$, call it $A$, such that $V(A) \subseteq N(x_i) \cup N(y_i)$ for $i \in [2]$. Since $d_{H'}(x_i) + d_{H'}(y_i) = k + 2$, we see that $N_H(x_i) \cap N_H(y_i) = \emptyset$ for $i \in [2]$. But we can play the same game with the pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$. We conclude that $N(x_1) \cap V(A) = N(x_2) \cap V(A)$ and $N(y_1) \cap V(A) = N(y_2) \cap V(A).$ In fact we can extend this equality to all of $X$ and $Y$. Put $Q := N(x_1) \cap V(A)$ and $P := N(y_1) \cap V(A)$. Then we conclude that $X$ is joined to $Q$ and $Y$ is joined to $P$. Moreover, we already know that $X$ and $Y$ are joined to the same $K_3$. The edges in these joins exhaust the degrees
of all the vertices, hence \( G \) is a 5-cycle with vertices blown up to cliques. If \( k = 8 \), then \(|X| = |Y| = 3\) and thus \(|Q| = |P| = 3\), but then \( G = M_8 \), a contradiction. So \( k \geq 9\). Since \(|X| + |Y| = k - 2 \geq 7\), we have either \(|X| \geq 4\) or \(|Y| \geq 4\). If \(|X| \geq 4\), then for each \( q \in Q\), we have \( d(q) \geq (k - 2) - 1 + |X| \geq k + 1\), contradiction. If \(|Y| \geq 4\), then for each \( p \in P\), we have \( d(p) \geq (k - 2) - 1 + |Y| \geq k + 1\), contradiction.

Since \( K_3 \ast E_{\Delta-3} \subseteq K_\Delta \), Lemma 7.2.17 shows that Conjecture 7.0.10 is equivalent to the Borodin-Kostochka conjecture.

**Lemma 7.2.18.** Let \( G \) be a \( k \)-mule with \( k \geq 7 \) other than \( M_{7,1} \), \( M_{7,2} \) and \( M_8 \). Let \( A \) and \( B \) be graphs with \( 3 \leq |A| \leq k - 3 \) and \(|B| = k - |A|\) such that \( A \ast B \subseteq G\). Then \( A = K_1 + K_{|A| - 1} \) and \( B = K_1 + K_{|B| - 1} \).

**Proof.** Suppose the lemma is false and let \( A \ast B \not\subseteq G \) be a counterexample.

First suppose \(|A|, |B| \geq 4\). Then, by Lemma 5.3.44, \( A \ast B \) is almost complete or \( K_5 \ast E_3 \). The first and second cases are impossible by Corollary 7.2.10 and Lemma 7.2.7 respectively.

Thus we may assume \(|A| = 3\). By Lemma 7.2.17, \( A \in \{E_3, P_3, K_1 + K_2\}\). If \( A = E_3\), then \( B \) is complete by Lemma 5.3.39, but this is impossible by Lemma 7.2.7. If \( A = P_3\), then \( B \) is complete by Lemma 5.3.20, but this is impossible by Lemma 7.2.4. Hence \( A = K_1 + K_2\).

By Lemma 5.3.43, \( B \) is complete or \( K_1 + K_{|B| - 1} \). The former is impossible by Lemma 7.2.8 and the latter by supposition.

Lemma 7.2.18 proves our main result, that Conjecture 7.0.8 is equivalent to the Borodin-Kostochka conjecture.

### 7.2.3 The low vertex subgraph of a mule

In this section we show that if a mule is not regular, then the subgraph of non-maximum-degree vertices is severely restricted. For a vertex critical graph \( G \) we write \( \mathcal{L}(G) \) for the subgraph induced on the vertices of degree \( \chi(G) - 1 \) in \( G \) and \( \mathcal{H}(G) \) for the subgraph induced on the rest of the vertices. We call \( v \in V(G) \) low if \( v \in V(\mathcal{L}(G)) \) and high otherwise.

**Lemma 7.2.19.** For \( k \geq 6\), no \( k \)-mule contains an induced \( E_2 \ast K_{k-2} \) with some vertex low.

**Proof.** Since \( M_{6,1} \) and \( M_{7,1} \) contain no such induced subgraph, the lemma follows from Lemma 7.2.4.

**Lemma 7.2.20.** If \( G \) is a \( k \)-mule with \( k \geq 6 \), then \( \mathcal{L}(G) \) is complete.

**Proof.** Let \( G \) be a \( k \)-mule with \( k \geq 6 \) and suppose \( G \) has nonadjacent low vertices \( x \) and \( y \). Then \( G + xy \not\sim G \) and hence, by Lemma 7.2.1, \( G + xy \) contains a \( K_k \). But then \( G \) contains an \( E_2 \ast K_{k-2} \) with some vertex low, contradicting Lemma 7.2.19. Hence \( \mathcal{L}(G) \) is complete.

**Lemma 7.2.21.** If \( G \) is a \( k \)-mule with \( k \geq 6 \) other than \( M_{6,1} \) and \( M_{7,1} \), then \(|\mathcal{L}(G)| \leq k - 2\).

**Proof.** Let \( G \) be a \( k \)-mule with \( k \geq 6 \) other than \( M_{6,1} \) and \( M_{7,1} \). By Lemma 7.2.20, \( \mathcal{L}(G) \) is complete and hence \(|\mathcal{L}(G)| \leq k - 1\). Suppose \(|\mathcal{L}(G)| = k - 1\). Since \( G \) doesn’t contain \( K_k \), no high \( z \) is adjacent to all of \( \mathcal{L}(G) \). Hence, by Lemma 7.2.6, there is a \((k - 1)\)-coloring of \( \mathcal{H}(G) \) that we can complete to all of \( G \) using Hall’s Theorem. This contradiction completes the proof.

**Lemma 7.2.22.** Let \( G \) be a \( k \)-mule with \( k \geq 6 \). If a high \( x \in V(G) \) has at least three low neighbors, then \( x \) is adjacent to all low vertices in \( G \).

**Proof.** Assume the lemma is false. Let \( x \) be a high degree vertex with at least three neighbors in \( V(\mathcal{L}(G)) \). If \(|V(\mathcal{L}(G))| = 3\), then the claim holds. So assume that \(|V(\mathcal{L}(G))| \geq 4\) and choose \( y \in V(\mathcal{L}(G)) \setminus N(x) \). Let \( A = V(\mathcal{L}(G)) \cap N(x) \). By Lemma 7.2.20, \( \mathcal{L}(G) \) is complete.

Thus, \( G[\{x, y\} \cup A] = E_2 \ast K_{|A|} \). Since \( L(v) = d(v) \) for all \( v \in (A \cup \{y\}) \), Lemma 5.3.49 implies that \( E_2 \ast K_{|A|} \) cannot appear in \( G \). This contradiction implies the lemma.
7.2.4 Restrictions on the independence number

The Borodin-Kostochka conjecture has been proven for graphs with independence number at most two [5]. Here we prove that if we wish to prove the Borodin-Kostochka conjecture for graphs with independence number at most $a$ for any $a \leq 6$, it suffices to construct a $K_{\Delta - 1}$.

For $a \geq 2$, let $C^a_k$ be those $G \in C_k$ with $\alpha(G) \leq a$. By a $(k,a)$-mule we mean a $C^a_k$-mule. Note that if $G \in C^a_k$ and for some $H \in C_k$ we have $H \prec G$, then $H \in C^a_k$ as well. Therefore any $(k,a)$-mule is also a $k$-mule.

**Theorem 7.2.23.** For $k \geq 7$ and $2 \leq a \leq k - 3$, no $(k,a)$-mule except $M_{7,1}$ contains a $K_{k-1}$.

**Proof.** Suppose otherwise and let $G$ be such a $(k,a)$-mule containing a $K_{k-1}$, call it $H$. By Corollary 7.2.10, each vertex in $G - H$ has at most one neighbor in $H$. Let $\pi$ be a $(k - 1)$-coloring of $G - H$. Then $|L_\pi(v)| \geq k - 3$ for all $v \in V(H)$. Since $H$ cannot be colored from $L_\pi$, applying Hall’s Theorem shows that either $|Pot(L_\pi)| \leq k - 2$ or there is some $x \in V(H)$ such that $|Pot_{H-x}(L_\pi)| \leq k - 3$. In the former case, $\pi$ must have some color class to which each vertex of $H$ is adjacent and hence $\alpha(G) \geq k - 1$, a contradiction. In the latter case, $\pi$ must have two color classes to which each vertex of $H - x$ is adjacent and hence $G$ has two disjoint independent sets of size $k - 2$. Again we have a contradiction since $\alpha(G) \geq k - 2$.

It follows that Conjecture 7.0.15 is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6.
Using ideas developed for strong coloring by Haxell [31] and by Aharoni, Berger and Ziv [1],
we make explicit a recoloring technique and apply it to the Borodin-Kostochka conjecture.

8.1 Strong coloring

For a positive integer \( r \), a graph \( G \) with \( |G| = rk \) is called strongly \( r \)-colorable if for every
partition of \( V(G) \) into parts of size \( r \) there is a proper coloring of \( G \) that uses all \( r \) colors on each part. If \( |G| \) is not a multiple of \( r \), then \( G \) is strongly \( r \)-colorable iff the graph formed
by adding \( r \left\lceil \frac{|G|}{r} \right\rceil - |G| \) isolated vertices to \( G \) is strongly \( r \)-colorable. The strong chromatic
number \( s\chi(G) \) is the smallest \( r \) for which \( G \) is strongly \( r \)-colorable.

Note that a strong \( r \)-coloring of \( G \) with respect to a partition \( V_1, \ldots, V_k \) of \( V(G) \) with
\( |V_i| = r \) must partition \( V(G) \) into \( r \) independent transversals of \( V_1, \ldots, V_k \). In [72], Szabó
and Tardos constructed partitioned graphs with part sizes \( 2\Delta - 1 \) that have no independent
transversal. So we must have \( s\chi(G) \geq 2\Delta(G) \). It is conjectured that this bound is tight.

Haxell [31] proved that \( s\chi(G) \leq 3\Delta(G) - 1 \). Aharoni, Berger and Ziv [1] gave a simple
proof that \( s\chi(G) \leq 3\Delta(G) \). It is this latter proof whose recoloring technique we use. First
we need a lemma allowing us to pick an independent transversal when one of the sets has
only one element.

Lemma 8.1.1. Let \( H \) be a graph and \( V_1 \cup \cdots \cup V_r \) a partition of \( V(H) \). Suppose that
\( |V_i| \geq 2\Delta(H) \) for each \( i \in [r] \). If a graph \( G \) is formed by attaching a new vertex \( x \) to fewer
than \( 2\Delta(H) \) vertices of \( H \), then \( G \) has an independent set \( \{x, v_1, \ldots, v_r\} \) where \( v_i \in V_i \) for each \( i \in [r] \).

Proof. Suppose not. Remove \( \{x\} \cup \overline{N(x)} \) from \( G \) to form \( H' \) with induced partition \( V_1', V_2', \ldots, V_r' \). Then \( V_1', V_2', \ldots, V_r' \) has no independent transversal since we could combine one with \( x \) to
get our desired independent set in \( G \). Note that \( |V_i'| \geq 1 \). Create a graph \( Q \) by removing edges from \( H' \) until it is edge minimal without an independent transversal. Pick \( yz \in E(Q) \) and apply Lemma 4.1.9 on \( yz \) with the induced partition to get the guaran-
teed \( J \subseteq [r] \) and the totally dominating induced matching \( M \) with \( |M| = |J| - 1 \). Now
\( \bigcup_{i \in J} V_i' > 2\Delta(H) |J| - 2\Delta(H) = 2(|J| - 1)\Delta(H) \) and hence \( M \) cannot dominate, a con-
tradiction.

Theorem 8.1.2. Every graph satisfies \( s\chi \leq 3\Delta \).

Proof. We only need to prove that graphs with \( n := 3\Delta k \) vertices have a \( 3\Delta \) coloring for
each \( k \geq 1 \). Suppose not and choose a counterexample \( G \) minimizing \( \|G\| \). Put \( r := 3\Delta(G) \)
and let \( V_1, \ldots, V_k \) be a partition of \( G \) for which there is no acceptable coloring. Then the
CHAPTER 8. STRONG COLORING

$V_i$ are independent by minimality of $\|G\|$. By symmetry we may assume there are adjacent
vertices $x \in V_1$ and $y \in V_2$. Apply minimality of $\|G\|$ to get an $r$-coloring $\pi$ of $G - xy$ with
$\pi(V_i) = [r]$ for each $i \in [k]$. We will modify $\pi$ to get such a coloring of $G$.

By symmetry, we may assume that $\pi(x) = \pi(y) = 1$. For $2 \leq i \leq k$, let $z_i$ be the unique
element of $\pi^{-1}(1) \cap V_i$ and put

$$W_i := V_i - \{v \in V_i \mid \pi(v) = \pi(w) \text{ for some } w \in N(z_i)\}.$$  

Then $|W_i| \geq 2\Delta(G)$ and we may apply Lemma 8.1.1 to get a $G$-independent transversal
$w_1, w_2, \ldots, w_k$ of $\{x\}, W_2, W_3, \ldots, W_k$. Define a new coloring $\zeta$ of $G$ by

$$\zeta(v) := \begin{cases} 
1 & \text{if } v = w_i \\
\pi(w_i) & \text{if } v = z_i \\
\pi(v) & \text{otherwise.}
\end{cases}$$

Then $\zeta$ is a proper coloring of $G$ with $\zeta(V_i) = [r]$ for each $i \in [k]$, a contradiction.  \qed

For our application we will need a lopsided version of Lemma 8.1.1 Lemma 4.1.4.

Lemma 8.1.3. Let $H$ be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose there
exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min \{t, |V_i| - t\}$.
For any $S \subseteq V(H)$ with $|S| < \min \{|V_1|, \ldots, |V_r|\}$, there is an independent transversal $I$ of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$.

Proof. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let $V'_1, \ldots, V'_r$
be the induced partition of $H'$. Then there is no independent transversal of $V'_1, \ldots, V'_r$ and
$|V'_i| \geq 1$ for each $i \in [r]$. Create a graph $Q$ by removing edges from $H'$ until it is edge
minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 4.1.9 on
$yz$ with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree $T$ with vertex set
$J$ and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V'_a$ and $v \in V'_b$. By our condition,
for each $uv \in E(V_i, V_j)$, we have $|N_H(u) \cup N_H(v)| \leq \min \{|V_i|, |V_j|\}$.

Choose a root $c$ of $T$. Traversing $T$ in leaf-first order and for each leaf $a$ with parent $b$ picking $|V_a|$ from min $\{|V_a|, |V_b|\}$ we get that the vertices in $M$ together dominate at most $\sum_{i \in J-\{c\}} |V_i|$ vertices in $H$. Since $|S| < |V_c|$, $M$ cannot totally dominate $\bigcup_{i \in J} V'_i$, a
contradiction.  \qed

We note that the condition on $S$ can be weakened slightly. Suppose we have ordered the $V_i$ so that $|V_1| \leq |V_2| \leq \cdots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that
$V_1 \not\subseteq S$, there is an independent transversal $I$ of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$. The proof is the same except when we choose our root $c$, choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

8.2 The recoloring technique

We can extract the idea in the proof of Theorem 8.1.2 to get a general recoloring technique.
Suppose $G$ is a $k$-vertex-critical graph and pick $x \in V(G)$ and $(k - 1)$-coloring $\pi$ of $H := G - x$. Let $Z$ be a color class of $\pi$, say $Z = \pi^{-1}(1)$. For each $z \in Z$, let $O_z$ be the
neighbors of $z$ which get a color that no other neighbor of $z$ gets; that is, put $O_z := \{v \in N_H(z) \mid \pi(v) \notin \pi(N_H(z) - v)\}$. Suppose the $O_z$ are pairwise disjoint. If we could find
an independent transversal $\{x\} \cup \{v_z\}_{z \in Z}$ of $\{x\}$ together with the $O_z$, then recoloring each
$z \in Z$ with $\pi(v_z)$ and coloring each vertex in $\{x\} \cup \{v_z\}_{z \in Z}$ with 1 gives a proper $(k - 1)$-
coloring of $G$. This is exactly what happens in the above proof of the strong coloring result.
To make this work more generally, we need to find situations where each $G[O_z]$ has high
minimum degree. Also, intuitively, the $O_z$ intersecting each other should make things easier
since recoloring a vertex in the intersection of $O_{z_1}$ and $O_{z_2}$ works for both $z_1$ and $z_2$. In our
application we will allow some restricted intersections.
8.3 A general decomposition

Let \( D_1 \) be the collection of graphs without induced \( d_1 \)-choosable subgraphs. Plainly, \( D_1 \) is hereditary. For a graph \( G \) and \( t \in \mathbb{N} \), let \( C_t \) be the maximal cliques in \( G \) having at least \( t \) vertices. We prove the following decomposition result for graphs in \( D_1 \) which generalizes Reed’s decomposition in [67].

**Lemma 8.3.1.** Suppose \( G \in D_1 \) has \( \Delta(G) \geq 8 \) and contains no \( K_{\Delta(G)} \). If \( \frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1 \), then \( \bigcup C_i \) can be partitioned into sets \( D_1, \ldots, D_r \) such that for each \( i \in [r] \) at least one of the following holds:

- \( D_i = C_i \in C_t \),
- \( D_i = C_i \cup \{x_i\} \) where \( C_i \in C_t \) and \( |N(x_i) \cap C_i| \geq t - 1 \).

Moreover, each \( v \in V(G) - D_i \) has at most \( t - 2 \) neighbors in \( C_i \) for each \( i \in [r] \).

**Proof.** Suppose \( |C_i| \leq |C_j| \) and \( C_i \cap C_j \neq \emptyset \). Then \( |C_i \cup C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 4 \). It follows from Corollary 5.3.25 that \( |C_i \cup C_j| \leq 1 \).

Now suppose \( C_i \) intersects \( C_j \) and \( C_k \). By the above, \( |C_i \cap C_j| \geq \frac{\Delta + 3}{2} \) and similarly \( |C_i \cap C_k| \geq \frac{\Delta + 3}{2} \). Hence \( |C_i \cap C_j \cap C_k| \geq \Delta + 3 - (\Delta(G) - 1) = 4 \). Put \( J := C_i \cap C_j \cap C_k \) and \( U := C_i \cup C_j \cup C_k \). By maximality of \( C_i, C_j, C_k \), \( U \) cannot induce an almost complete graph. Thus, by Corollary 5.3.25, \( |U| \in \{4, 5\} \) and the graph induced on \( U - I \) is \( E_3 \). But then \( t \leq 6 \) and hence \( \Delta(G) \leq 7 \), a contradiction.

The existence of the required partition is immediate. \( \square \)

When \( D_i \in C_t \), we put \( K_i := C_i : D_i \) and when \( D_i = C_i \cup \{x_i\} \), we put \( K_i := N(x_i) \cap C_i \).

8.4 Borodin-Kostochka when every vertex is in a big clique

Let \( G \) be a graph. For \( v \in V(G) \), let \( \omega(v) \) be the size of a largest clique in \( G \) containing \( v \).

The proofs of the results in this section go more smoothly when we strengthen the induction in terms of the parameter \( \rho(G) := \max_{v \in V(G)} d(v) - \omega(v) \).

**Theorem 8.4.1.** For \( k \geq 9 \), every graph satisfying \( \Delta \leq k \), \( \omega < k \) and \( \rho \leq \frac{k}{3} - 2 \) is \( (k-1) \)-colorable.

**Proof.** Suppose the theorem fails for some \( k \geq 9 \) and choose a counterexample \( G \) minimizing \( |G| + ||G|| \). Put \( \Delta := \Delta(G) \). If \( \Delta < k \), then \( \Delta = k - 1 \) and by Brooks’ theorem \( G \) contains \( K_k \), a contradiction. Thus \( \chi(G) = k = \Delta \). Also, for any \( v \in V(G) \) we have \( \rho(G - v) \leq \rho(G) \), applying our minimality condition on \( G \) implies that \( G \) is vertex critical.

Therefore \( \delta(G) \geq \Delta - 1 \) and \( G \in D_1 \). For any \( v \in V(G) \), we have \( \Delta - 1 - \omega(v) \leq d(v) - \omega(v) \leq \frac{\Delta}{3} - 2 \) and hence \( \omega(v) \geq \frac{\Delta}{3} + 1 \). Applying Lemma 8.3.1 with \( t := \frac{\Delta}{3} + 1 \) we get a partition \( D_1, \ldots, D_r \) of \( \bigcup C_i = V(G) \). Note that for \( i \in [r] \), if \( K_i \neq D_i \) then all vertices in \( K_i \) are high by Lemma 5.3.48. Pick \( x \in K_1 \). Then \( x \) has \( |C_1| - 1 \leq \Delta - 2 \) neighbors in \( D_1 \) if \( K_i = D_i \) and \( |C_1| \leq \Delta - 1 \) if \( K_i \neq D_i \). Hence, by our note, \( x \) has a neighbor \( w \in V(G) - D_1 \).

We now claim that \( xw \) is a critical edge in \( G \). Suppose otherwise that \( \chi(G - xw) = \Delta \). Then by minimality of \( G \) we must have \( \rho(G - xw) > \rho(G) \). Hence there is some vertex \( v \in N(x) \cap N(w) \) so that every largest clique containing \( v \) contains \( xw \). But \( v \) is in some \( D_j \) and all largest cliques containing \( v \) are contained in \( D_j \) and hence do not contain \( xw \), a contradiction.

Let \( \pi \) be a \( (\Delta - 1) \)-coloring of \( G - xw \) chosen so that \( \pi(x) = 1 \) and so as to minimize \( |\pi^{-1}(1)| \). Consider \( \pi \) as a coloring of \( G - x \). One key property of \( \pi \) we will use is that since \( x \) got 1 in the coloring of \( G - xw \) and \( x \in K_1 \), no vertex of \( D_1 - x \) gets colored 1 by \( \pi \).
Now put $Z := \pi^{-1}(1)$ and for $z \in Z$, let $O_z$ be as defined in Section 8.2. By minimality of $|Z|$, each $z \in Z$ has at least one neighbor in every color class of $\pi$. Hence $z$ has two or more neighbors in at most $2 + d(z) - \Delta$ of $\pi$'s color classes. For each $z \in Z$ we have $i(z)$ such that $z \in D_{i(z)}$. For $z \in Z$ such that $i(z) \not\in i(Z - z)$, put $V_z := O_z \cap C_{i(z)}$.

We have $|V_z| \geq \omega(z) - 1 - (2 + d(z) - \Delta)$. Since $\omega(z) \geq d(z) - \frac{1}{3} \Delta + 2$, we have $|V_z| \geq \frac{2}{3} \Delta - 1$. Each $y \in V_z$ is adjacent to all of $C_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - |C_{i(z)}|$ neighbors outside $D_{i(z)}$. Since $\omega(y) \geq d(y) + 2 - \frac{1}{3} \Delta$, we conclude that $y$ has at most $d(y) + 1 - (d(y) + 2 - \frac{1}{3} \Delta) = \frac{1}{3} \Delta - 1$ neighbors outside $D_{i(z)}$.

Now let $Z'$ be the $z \in Z$ with $i(z) \in i(Z - z)$. Then $Z'$ can be partitioned into pairs $\{z, z'\}$ such that $i(z) = i(z')$. For such a pair, one of $z, z'$ is $x_{i(z)}$ and the other is in $C_{i(z)} - K_{i(z)}$. Put $V_z := O_z \cap O_{z'} \cap K_{i(z)}$ and don’t define $V_{z'}$. We have $|V_z| \geq \min \{\omega(z), \omega(z')\} - 1 + (2 + d(z) - \Delta) - (2 + d(z')) - \Delta \geq -\frac{1}{3} \Delta + 2 - 1 - 2(2 - \Delta) - \max \{d(z), d(z')\} = \frac{2}{3} \Delta - \max \{d(z), d(z')\} - 3 \geq \frac{2}{3} \Delta - 3$. Each $y \in V_z$ is adjacent to all of $D_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - |D_{i(z)}|$ neighbors outside $D_{i(z)}$. Since $|D_{i(z)}| = \omega(y) + 1 \geq d(y) + 3 - \frac{1}{3} \Delta$, we conclude that $y$ has at most $\frac{2}{3} \Delta - 2$ neighbors outside $D_{i(z)}$.

Let $H$ be the subgraph of $G$ induced on the union of the $V_z$. Put $S := N(x) \cap V(H)$. Since $|S| \leq d(x) + 1 - |D_1| \leq d(x) + 1 - \omega(x) \leq \frac{2}{3} \Delta - 1 < |V_z|$ for all $V_z$ since $\Delta \geq 7$. Hence we may apply Lemma 8.1.3 on $H$ with $t := \frac{1}{4} \Delta - 1$ to get an independent set $\{v_z : z \in Z\}$ disjoint from $S$ where $v_z \in V_z$. Recoloring each $z \in Z$ with $\pi(z)$ and coloring $x \cup \{v_z : z \in Z\}$ with 1 gives a $(\Delta - 1)$-coloring of $G$, a contradiction. \hfill \square

The following special case is a bit easier to digest.

Corollary 8.4.2. Every graph with $\chi \geq \Delta \geq 9$ such that every vertex is in a clique on $\frac{2}{3} \Delta + 2$ vertices contains $K_{\Delta}$.

8.5 Reducing to the irregular case

It is easy to see that if there are irregular counterexamples to the Borodin-Kostochka conjecture, then there are regular examples as well: take an irregular counterexample $G$ clone it add an edge between any vertex with degree less than $\Delta(G)$ and its clone; repeat until you have a regular graph (from [54]).

But what about the converse? If there are regular examples, must there be (connected) irregular examples? We’ll see that the answer is yes, but we need to decrease the maximum degree by one.

Theorem 8.5.1. Every graph satisfying $\chi \geq \Delta = k \geq 9$ either contains $K_k$ or contains an irregular critical subgraph satisfying $\chi = \Delta = k - 1$.

Proof. Suppose not and choose a counterexample $G$ minimizing $|G|$. Then $G$ is vertex critical. If every vertex in $G$ were contained in a $(k - 1)$-clique, then Corollary 8.4.2 would give a $K_k$ in $G$, impossible. Hence we may pick $v \in V(G)$ not in a $(k - 1)$-clique. If $v$ is high, choose a $(k - 1)$-coloring $\pi$ of $G - v$ so that the color class $T$ of $\pi$ where $v$ has two neighbors is as large as possible; if $v$ is low, let $\pi$ be a $(k - 1)$-coloring of $G - v$ where some color class $T$ of $\pi$ is as large as possible. By symmetry, we may assume that $\pi(T) = k - 1$.

Now we have a $(k - 1)$-coloring $\zeta$ of $H := G - T$ given by $\zeta(x) = \pi(x)$ for $x \neq v$ and $\zeta(v) = k - 1$. Since $\chi(H) = k - 1$, the maximality condition on $T$ together with Brooks’ theorem gives $\Delta(H) = k - 1$. Note that $d_H(v) = k - 2$. Let $H'$ be a $(k - 1)$-critical subgraph of $H$. Then $H'$ must contain $v$ and hence is not $K_{k-1}$. Since $d_{H'}(v) = k - 2$ and $\Delta(H') = k - 1$ (by Brooks’ theorem), $H'$ is an irregular critical subgraph of $G$ satisfying $\chi = \Delta = k - 1$, a contradiction. \hfill \square

Since the only known critical (or connected even) counterexample to Borodin-Kostochka for $\Delta = 8$ is regular (see Figure 7.4) we might hope that the following strengthened conjecture is true.
Conjecture 8.5.2. Every critical graph satisfying $\chi \geq \Delta = 8$ is regular.

8.6 Dense neighborhoods

Here we show that the Borodin-Kostochka conjecture holds for graphs where each neighborhood has “most” of its possible edges. First, we need to convert high average degree in a neighborhood into a large clique in the neighborhood. We need the following extension of a fundamental result of Mader [53] (see Diestel [22] for some history of this result). We will also need $d_1$-choosability results from [20] as well as some ideas for dealing with average degree in neighborhoods used in [19].

Lemma 8.6.1. For $k \geq 1$, every graph $G$ with $d(G) \geq 4k$ has a $(k + 1)$-connected induced subgraph $H$ such that $d(H) > d(G) - 2k$.

Lemma 8.6.2. If $B$ is a graph with $d(B) \geq \omega(B) + 2$, then $B$ has an induced subgraph $H$ such that $K_1 * H$ is $f$-choosable where $f(v) \geq d(v)$ for the $v$ in the $K_1$ and $f(x) \geq d(x) - 1$ for $x \in V(H)$.

Proof. Let $B$ be such a graph. Applying Lemma 8.6.1 with $k := 1$, we get a 2-connected subgraph $H$ of $B$ with $d(H) > d(B) - 2 \geq \omega(B)$. Since $H$ is 2-connected, if it is not $d_0$-choosable, then it is either an odd cycle or complete. The former is impossible since $d(H) \geq 3$, hence $H$ would be complete and we’d have the contradiction $\omega(H) > \omega(B)$. Hence $H$ is $d_0$-choosable.

Suppose $K_1 * H$ isn’t $f$-choosable and let $L$ be a minimal bad $f$-assignment on $K_1 * B$. By Lemma 5.3.16, no nonadjacent pair in $H$ have intersecting lists and hence we must have $\sum_{v \in V(H)} |L(v)| \leq |\text{Pot}(L)| \omega(H)$. Since for each $v \in V(H)$ we have $|L(v)| = d_H(v)$ and by the Small Pot Lemma we have $|\text{Pot}(L)| \leq |H|$, we must have $d(H) \leq \omega(H) \leq \omega(B) < d(H)$, a contradiction.

Lemma 8.6.3. If $B$ is a graph with $d(B) \geq \omega(B) + 3$, then $B$ has an induced subgraph $H$ such that $K_1 * H$ is $d_1$-choosable.

Proof. Let $B$ be such a graph. Applying Lemma 8.6.1 with $k := 1$, we get a 2-connected subgraph $H$ of $B$ with $d(H) > d(B) - 2 \geq \omega(B) + 1$. As in the proof of Lemma 8.6.2, we see that $H$ is $d_0$-choosable. Suppose $K_1 * H$ is not $d_1$-choosable and let $L$ be a minimal bad $d_1$-assignment on $K_1 * H$. Combining Lemma 5.3.15 with the same argument as in the proof of Lemma 8.6.2 shows that $|\text{Pot}(L)| \leq |H| - 1$.

Now, for $c \in \text{Pot}(L)$, we consider how big the color graphs $H_c$ can be. All of the information comes from Lemma 5.3.14. We have $\alpha(G_c) \leq 2$ for all $c \in \text{Pot}(L)$. First, suppose we have $c \in \text{Pot}(L)$ such that $|H_c| \geq \omega(H) + 3$. Then, using Lemma 5.3.14, we see that $|H_c| \leq \omega(H)$ for all $c' \in \text{Pot}(L) - c$ and hence $\sum_{\gamma \in \text{Pot}(L)} |H_{\gamma}| \leq |H| + (|\text{Pot}(L)| - 1) \omega(H) \leq |H| \omega(H) + |H| - 2\omega(H)$. Now suppose we have $c \in \text{Pot}(L)$ such that $|H_c| = \omega(H) + 2$. Then, using Lemma 5.3.14 again, we see that $|H_c| \leq \omega(H) + 1$ for all $c' \in \text{Pot}(L) - c$ and hence $\sum_{\gamma \in \text{Pot}(L)} |H_{\gamma}| \leq 1 + |\text{Pot}(L)| (\omega(H) + 1) \leq |H| \omega(H) + |H| - \omega(H)$.

Therefore we must have $2||H|| \leq |H| (\omega(H) + 1) - \omega(H)$ and hence $d(H) \leq \omega(H) + 1 < d(H)$, a contradiction.

Theorem 8.6.4. Every graph $G$ with $\omega(G) < \Delta(G)$ such that $d(G_v) \geq \frac{2}{3} \Delta(G) - 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$-colorable.

Proof. Suppose note and let $G$ be a counterexample. Put $\Delta := \Delta(G)$. Let $H$ be a $\Delta$-vertex-critical induced subgraph of $G$. Then $\delta(H) \geq \Delta - 1$ and $H$ has no $d_1$-choosable induced subgraphs. By Theorem 8.4.2, we must have $v \in V(H)$ with $\omega(v) < \frac{2}{3} \Delta + 2$. Suppose $d(H_v) < d(G_v)$. Then $d_H(v) = \Delta - 1$ and $||H_v|| \geq ||G_v|| - (\Delta - 1)$; therefore, $d(H_v) > d(G_v) - 1 \geq \frac{2}{3} \Delta + 3$. Applying Lemma 8.6.2 gives $\omega(v) > d(H_v) - 1 \geq \frac{2}{3} \Delta + 2$, a contradiction.
Hence we must have \( d(H_v) = d(G_v) \geq 2 \Delta + 4 \). Applying Lemma 8.6.3 gives \( \omega(v) > d(H_v) - 2 \geq \frac{2}{3} \Delta + 2 \), a contradiction. \( \Box \)

8.7 Bounding the order and independence number

Lemma 8.7.1. Let \( G \) be a vertex critical graph with \( \chi(G) = \Delta(G) + 1 - k \). For every \( v \in V(G) \) there is \( H_v \subseteq G_v \) with:

1. \[ |H_v| \geq \Delta(G) - 2k; \text{ and} \]
2. \[ \delta(H_v) \geq |H_v| - (k + 1)(\alpha(G) - 1) - 1; \text{ and} \]
3. \[ |H_v| \geq |H_v| (|H_v| - (k + 2)) - (k + 1) (|G| + 2k - (\Delta(G) + 1)). \]

Proof. Put \( \Delta := \Delta(G) \). Pick \( v \in V(G) \) and let \( \pi \) be a \((\Delta - k)\)-coloring of \( G - v \). Let \( H_v \) be the subgraph of \( G_v \) induced on \( \{x \in N(v) \mid \pi(x) \notin \pi(N(v) - x)\} \). Plainly, \( |H_v| \geq \Delta - 2k \).

By the usual Kempe chain argument, any \( x, y \in V(H_v) \) must be in the same component of \( C_{x,y} := G[\pi^{-1}(\pi(x)) \cup \pi^{-1}(\pi(y))] \). Thus if \( xy \notin E(G) \), there must be a path of length at least 3 in \( C_{x,y} \) from \( x \) to \( y \) and hence some vertex of color \( \pi(x) \) other than \( x \) must have at least two neighbors of color \( \pi(y) \) and some vertex of color \( \pi(y) \) other than \( y \) must have at least two neighbors of color \( \pi(x) \). We say that such an intermediate vertex proxies for \( xy \). Each \( xy \) with \( y \in V(H_v) \) must have some proxy \( z_{xy} \in \pi^{-1}(\pi(x)) - x \) such that \( z_{xy} \) proxies for at most \( k + 1 \) total \( xw \) with \( w \in V(H_v) \), for otherwise we could recolor all of \( xy \)'s proxies, swap \( \pi(x) \) and \( \pi(y) \) in \( x \)'s component of \( C_{x,y} \) and then color \( v \) with \( \pi(x) \) to get a \((\Delta - k)\)-coloring of \( G \). We conclude that \( x \) has at most \((k + 1)|\pi^{-1}(\pi(x))| - 1\) non-neighbors in \( H_v \). This gives (2) immediately.

For (3), note that \(|\pi(i)| \geq 2\) for each \( i \in [\Delta - k] - \pi(V(H_v)) \) and hence \( \sum_{j \in \pi(V(H_v))} |\pi^{-1}(j)| \leq |G| - 1 - (\Delta - k - |H_v|) \). Since

\[
|H_v| \geq \sum_{j \in \pi(V(H_v))} \left(|H_v| - 1 - (k + 1)(|\pi^{-1}(j)| - 1)\right),
\]

we see that (3) follows. \( \Box \)

Theorem 8.7.2. Every graph satisfies \( \chi \leq \max \{ \omega, \Delta - 1, 4\alpha \} \).

Proof. Suppose not and choose a counterexample \( G \) minimizing \( |G| \). Since none of the terms on the right side increase when we remove a vertex, \( G \) is vertex critical. Since the Borodin-Kostochka conjecture holds for graphs with \( \alpha = 2 \) and \( \Delta \geq 9 \), we must have \( \alpha(G) \geq 3 \) and hence \( \Delta(G) \geq 13 \). By Lemma 8.4.2, there must be \( v \in V(G) \) with \( \omega(v) < \frac{2}{3} \Delta(G) + 2 \). Applying (2) of Lemma 8.7.1, we get \( H_v \subseteq G_v \) with \( |H_v| \geq \Delta(G) - 2 \) and \( \delta(H_v) \geq |H_v| - 2 \alpha(G) + 1 \). Since \( \Delta(G) \geq \chi(G) \geq 4 \alpha(G) + 1 \), we have \( \delta(H_v) \geq |H_v| - \Delta(G) - 1 + 1 \geq |H_v| - \frac{1}{2} \).

Applying Lemma 9.2.1 shows that either \( H_v = K_3 \vee E_4 \) or \( \omega(H_v) \geq |H_v| - 1 \). The former is impossible since \( \Delta(G) > 9 \). Therefore \( \omega(v) \geq \omega(H_v) + 1 \geq \Delta(G) - 2 \geq \frac{2}{3} \Delta(G) + 2 \) since \( \Delta(G) \geq 12 \), a contradiction. \( \Box \)

Theorem 8.7.3. Every graph satisfies \( \chi \leq \max \left\{ \omega, \Delta - 1, \left[\frac{15 + \sqrt{45n + 73}}{4}\right]\right\} \).

Proof. Suppose not and choose a counterexample \( G \) minimizing \( |G| \). Put \( \Delta := \Delta(G) \) and \( n := |G| \). Since none of the terms on the right side increase when we remove a vertex, \( G \) is vertex critical. By Lemma 8.4.2, there must be \( v \in V(G) \) with \( \omega(v) < \frac{2}{3} \Delta + 2 \). Applying (3) of Lemma 8.7.1, we get \( H_v \subseteq G_v \) with \( |H_v| \geq \Delta - 2 \) and \( |H_v| \geq |H_v| (|H_v| - 3) - 2 (n + 1 - \Delta) \). By Lemma 8.6.3, we must have \( d(H_v) < \frac{2}{3} \Delta + 4 \) and hence we have
\[
\frac{2}{3} \Delta + 4 > 2 (|H_v| - 3) - \frac{4 (n + 1 - \Delta)}{|H_v|} \\
\geq 2 (\Delta - 5) - \frac{4 (n + 1 - \Delta)}{\Delta - 2}.
\]

Simplifying a bit, we get \(6(n - 1) > (2\Delta - 15)(\Delta - 2)\). Since \(\Delta \geq \chi(G) \geq \frac{19+\sqrt{48n+73}}{4}\), we have \(6(n - 1) > (\frac{-11+\sqrt{48n+73}}{2})(\frac{11+\sqrt{48n+73}}{4}) = \frac{48n-48}{8} = 6(n - 1)\), a contradiction. \qed
LIST BORODIN-KOSTOCHKA FOR LARGE $\Delta$

9.1 The setup

The aim of this chapter is to prove the following result.

**Theorem 9.1.1.** There exists $\Delta_0$ such that every graph $G$ with $\chi_l(G) \geq \Delta(G) \geq \Delta_0$ contains a $K_{\Delta(G)}$.

Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Put $\Delta := \Delta(G)$ and let $L$ be a bad $(\Delta - 1)$-assignment on $G$. Then, by minimality of $|G|$, any proper induced subgraph of $G$ is $L$-colorable. In particular, every vertex has degree either $\Delta$ or $\Delta - 1$, we call these high and low vertices respectively.

If $G$ had an induced $d_1$-choosable subgraph $H$, then we could $L$-color $G - H$ by minimality and then complete the $L$-coloring to all of $G$. So, $G$ has no $d_1$-choosable induced subgraphs.

The proof strategy is the same as Reed’s [67] for chromatic number, except some more care must be taken when lists have small intersection and $K_{\Delta - 1}$'s require special attention.

9.2 The decomposition

We use Lemma 8.3.1 and the notation from Section 8.3.

**Definition 9.2.1.** The cliques in $C_{\frac{4}{3}\Delta + 1}$ are called big.

Let $B$ be all vertices contained in a big clique; that is, $B := \bigcup C_{\frac{4}{3}\Delta + 1}$. For a vertex $v$, put $G_v := G[N(v)]$.

**Definition 9.2.2.** A vertex $v$ is called sparse if $\|G_v\| < \frac{2}{5}\Delta^2$.

**Lemma 9.2.1.** If $B$ is a graph with $\delta(B) \geq \frac{|B| + 1}{2}$ such that $K_1 \ast B$ is not $d_1$-choosable, then $\omega(B) \geq |B| - 1$ or $B = E_3 \ast K_4$.

**Proof.** Suppose the lemma is false and let $L$ be a minimal bad $d_1$-assignment on $B$. First note that if $B$ does not contain disjoint nonadjacent pairs $x_1, y_1$ and $x_2, y_2$, then $\omega(B) \geq |B| - 1$ or $B = E_3 \ast K_4$ by Corollary 5.3.25.

By Dirac’s theorem, $B$ is hamiltonian and in particular 2-connected. Since $B$ cannot be an odd cycle or complete, $B$ is $d_0$-choosable.

By the Small Pot Lemma, $|Pot(L)| \leq |B|$. Since $|L(x_1)| + |L(x_2)| \geq |B| + 1$, the lists intersect and thus Lemma 5.3.15 shows that $|Pot(L)| \leq |B| - 1$. But then $|L(x_i) \cap L(y_i)| \geq 2$ for each $i$ and Lemma 5.3.14 gives a contradiction.

Note that the neighborhoods we will be looking at are huge, so the $B = E_3 \ast K_4$ case will never happen here.
Lemma 9.2.2. Every vertex in $V(G) - B$ is sparse.

Proof. Suppose $x \in V(G) - B$. By applying Lemma 9.2.1 repeatedly, we get a sequence $y_1, \ldots, y_{\lfloor \frac{\Delta}{2} \rfloor} \in N(x)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence $x$ is sparse since

$$\|G_x\| \leq \left(\frac{\Delta}{2}\right) - \frac{1}{2} \sum_{i=1}^{\lfloor \frac{\Delta}{2} \rfloor} (\Delta - i) < \frac{2}{5} \Delta^2.$$

\[\square\]

Let $D_1, \ldots, D_r$ be the partition of $B$ guaranteed by Lemma 8.3.1 and put $S := V(G) - B$.

9.3 The random procedure

For each vertex $v$, pick $c \in L(v)$ at random to get a possibly improper coloring $\zeta$ of $G$ from $L$. Put $U := \{x \in V(G) \mid \zeta(x) = \zeta(y) \text{ for some } y \in N(x)\}$. Put $H := G - U$, $F := G[U]$ and let $\pi$ be $\zeta$ restricted to $V(H)$. We refer to $V(H)$ as the colored vertices and $V(F)$ as the uncolored vertices. Also, let $J$ be the resulting list assignment on $F$; that is, $J(x) := L(x) - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$ for $x \in V(F)$.

Definition 9.3.1. A vertex in $v \in V(G)$ is called safe if it is colored or $|J(v)| \geq d_F(v) + 1$.

Note that if every vertex is safe, then we can easily complete the $L$-coloring to all of $G$. Our goal will be to show that the random procedure will, with positive probability, produce a partial coloring where every sparse vertex is safe and the uncolored nonsparse vertices satisfy conditions that will allow the coloring to be completed. Now we make this precise. Consider the following events:

- $S_v$, for $v \in S$: the event that $v$ is not safe.
- $E_i$, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that $C_i$ does not contain two uncolored safe vertices.
- $Q_i$, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that $K_i$ does not contain two uncolored vertices.
- $F_i$, for $i \in [r]$ where $|C_i| = \Delta - 1$, every $x \in G - C_i$ has $|N(x) \cap C_i| \leq \sqrt{\Delta} \log(\Delta)$ and there are at most $\log^2(\Delta)$ vertices $x \in G - C_i$ with $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that $C_i$ does not contain two uncolored safe vertices.
- $P_i$, for $i \in [r]$ where $|C_i| = \Delta - 1$ and either some $x \in G - C_i$ has $|N(x) \cap C_i| > \sqrt{\Delta} \log(\Delta)$ or more than $\log^2(\Delta)$ vertices $x \in G - C_i$ have $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that every $x \in G - C_i$ has at most two “good clumps” in $K_i$.

It remains to define “good clumps”. To do so we need a lemma.

Lemma 9.3.1. Let $K$ be a $\Delta - 1$ clique in $G$ and $x \in G - K$ with $|N(x) \cap K| \geq 4$. Then every vertex in $|N(x) \cap K|$ is high and there is a partition $\{Z_1, \ldots, Z_m\}$ of $N(x) \cap K$ such that for each $i \in [m]$ we have $|Z_i| \leq 5$ and $L(u) = L(v)$ for all $u, v \in Z_i$. Moreover, $|L(v) - L(w)| \leq 1$ for all $v, w \in N(x) \cap K$. 
Proof. Put $A := N(x) \cap K$ and $Q := G[\{x\} \cup K]$. For any $L$-coloring $\gamma$ of $G - Q$, let $L_\gamma$ be the resulting list assignment on $Q$.

First, suppose there is an $L$-coloring $\gamma$ of $G - Q$ such that $L_\gamma(u) \neq L_\gamma(v)$ for some $u, v \in A$. Pick $y \in K - A$. If $L_\gamma(x) \cap L_\gamma(y) \neq \emptyset$, then coloring $x$ and $y$ the same leaves a list assignment on $K - y$ which is completable by Hall’s theorem. Hence we must have $L_\gamma(x) \cap L_\gamma(y) = \emptyset$. Thus $|L_\gamma(x) \cup L_\gamma(y)| \geq \Delta$. Put $Pot(T) := \bigcup_{v \in T} L_\gamma(v)$ for $T \subseteq A$. If there is $c \in (L_\gamma(x) \cup L_\gamma(y)) - Pot(A)$, then coloring $x$ and $y$ so that $c$ is used leaves a list assignment on $K - y$ which is completable by Hall’s theorem. In particular, we must have $|Pot(A)| \geq \Delta$. Now, if we color $x$ and $y$ arbitrarily we can complete the coloring unless there exists $T \subseteq A$ with $|T| = |A| - 1$ and $|Pot(T)| \leq \Delta - 2$. Thus we can pick a color in $L_\gamma(x) \cup L_\gamma(y)$ which is not in any of $T$’s lists giving a coloring that is again easily completable.

Therefore $L_\gamma(u) = L_\gamma(v)$ for all $u, v \in A$ for every $L$-coloring $\gamma$ of $G - Q$. In particular, no vertex of $A$ is low and $|L(v) - L(w)| \leq 1$ for all $v, w \in A$. Suppose there exists $Z \subseteq A$ with $|Z| \geq 6$ such that $L(u) = L(v)$ for all $u, v \in Z$. Then every $v \in Z$ has exactly one neighbor $z_v$ in $G - Q$. Put $N := \{z_v \mid v \in Z\}$. If $|N| = 1$, then $G$ contains $K_6 + E_3$ violating Lemma 5.3.25. If some $L$-coloring $\gamma$ of $G - Q$ assigned two vertices of $N$ different colors, then $L_\gamma$ would give different lists for two vertices of $A$, a contradiction. Hence $N$ is an independent set and adding an edge between two vertices of $N$ in $G - Q$ must create a $K_\Delta$ by minimality of $|G|$. By counting degrees this is plainly impossible for $|N| \geq 3$. For $|N| = 2$, both vertices have $\Delta - 2$ neighbors in $G - Q$ and one has at least 3 vertices in $Z$, impossible.

Now taking maximal subsets of $A$ of vertices all having the same list gives the desired partition. 

The $Z_i$ in the partition in Lemma 9.3.1 are called clumps of $x$ in $K$. Note that there exists $Y$ such that for any $i \neq j$ we have $L(v) \cap L(z_i) = Y$ for $v \in Z_i$ and $w \in Z_j$. For $i \in [m]$ and $v \in Z_i$, we let $\alpha_i$ be the unique element of $L(v) - Y$. We call $\alpha_i$ the special color for $Z_i$.

Now let $i \in [r]$ where $|C_i| = \Delta - 1$ and some $x \in G - C_i$ has $|N(x) \cap C_i| \geq 4$. A clump $Z_j \subseteq N(x) \cap C_i$ is good if there is uncolored $z_i \in Z_i$ such that $\alpha_i$ is not used on any neighbor of $z_i$ and the unique $y$ in $N(z_i) - C_i - \{x\}$ is colored with a color that is either not in $L(z_i)$ or is used on $C_i$.

Now suppose we have a partial coloring $\pi$ where none of the bad events $S_u, E_i, Q_i, F_i$ and $P_i$ occur. We color the $D_i$ corresponding to $P_i$ events first. Suppose $x \in G - C_i$ has 3 good clumps $Z_1, Z_2, Z_3$ in $K_i$ with corresponding vertices $z_1, z_2, z_3$. Since $\alpha_1 \notin L(z_2). L(z_3)$, coloring $z_1$ with $\alpha_1$ leaves a list assignment we can complete greedily by coloring $z_2$ and $z_3$ last. However, we need to be careful to not break the other such $D_i$ in the process. So, we first color the respective $z_1$ in each such $D_i$. After all of those have been colored, we greedily color the rest of each $D_i$. It still needs to be checked that when we color $z_1$ with $\alpha_1$ we don’t lose the ability to do the same with $\alpha_1$ on some other $D_j$. To see this, note that $x$ has at least 3 neighbors in $C_i$ and thus is contain in no other $C_j$ with $|C_j| = \Delta - 1$. Moreover, $z_1$’s only possible other neighbor $y$ outside $C_i$ is already colored by assumption.

Now consider the $D_i$ that have two safe uncolored vertices in $C_i$. If $C_i \neq K_i$, then since $Q_i$ doesn’t happen $x_i$ has two uncolored neighbors, color it first. Now color $C_i$ greedily saving the two safe uncolored vertices in $C_i$ for last. Now we can finish the coloring on the sparse vertices greedily. Therefore if we can prevent all the bad events from happening we get our desired contradiction.

It is easy to see that any given event depends on less than $3\Delta^5$ others, so the result will follow by showing that $\Pr(S_u), \Pr(E_i), \Pr(Q_i), \Pr(F_i), \Pr(P_i) \leq \Delta^{-6}$. The following sections prove these bounds.
9.4 \( \Pr(S_v) \leq \Delta^{-6} \)

We know \( |G_v| < \frac{2}{3} \Delta^2 \). Put \( A := \{ x \in N(v) \mid |L(x) \cap L(v)| \geq \frac{2}{3} \Delta \} \) and \( B := N(v) - A \). Note that for \( x, y \in A \) we have \( |L(x) \cap L(y)| \geq \frac{1}{3} \Delta \) and for \( x \in B \) we have \( |L(x) - L(v)| \geq \frac{1}{3} \Delta \).

Let \( A_v \) be the random variable that counts the number of nonadjacent pairs \( x, y \in A \) such that, \( \zeta(x) = \zeta(y) \) and \( \zeta(z) \neq \zeta(x) \) for all \( z \in N(v) - \{ x, y \} \cup N(x) \cup N(y) \).

Let \( B_v \) be the random variable that counts the number of \( x \in B \) such that \( \zeta(x) \notin L(v) \) and \( \zeta(z) \neq \zeta(x) \) for all \( z \in N(v) - \{ x \} \cup N(x) \).

Put \( Z_v := A_v + B_v \). Then \( E(Z_v) = E(A_v) + E(B_v) \). We prove the bound \( E(Z_v) \geq \frac{\Delta}{1000} \) and then use Azuma’s inequality to prove that \( \Pr(|Z_v - E(Z_v)| > \frac{\Delta}{1000} - 2) \leq \Delta^{-6} \). The conclusion \( \Pr(S_v) \leq \Delta^{-6} \) is then immediate.

We know that \( G_v \) has at least \( (\Delta - 1) - \frac{1}{2} \Delta^2 \geq \frac{\Delta^2}{12} \) nonadjacent pairs. Let \( b \) be the number of nonadjacent pairs in \( G_v \) that intersect \( B \). Plainly, \( G[A] \) contains at least \( \frac{\Delta^2}{12} - b \) nonadjacent pairs and \( b \leq |B| \Delta \).

First let's consider \( E(A_v) \). Let \( x, y \in A \) be nonadjacent. Since \( |L(x) \cap L(y)| \geq \frac{1}{3} \Delta \), the probability that \( x \) and \( y \) get the same color and this color is not used on any of the rest of \( N(v) \cup N(x) \cup N(y) \) is at least \( (3\Delta)^{-1}(1 - (\Delta - 1)^{-1})3\Delta^{-3} \geq (3\Delta)^{-1}3^{-3} \). Thus \( E(A_v) \geq (\frac{\Delta^2}{12} - b)\Delta^{-1}3^{-4} \geq \frac{\Delta}{1000} - \frac{b}{812} \).

Now consider \( E(B_v) \). Let \( x \in B \). Since \( |L(x) - L(v)| \geq \frac{1}{3} \Delta \), the probability that \( x \) gets a color not in \( L(v) \) and this color is not used on any of the rest of \( N(v) \cup N(x) \) is at least \( \frac{1}{3}(1 - (\Delta - 1)^{-1})^{3\Delta - 2} \geq 3^{-4} \). Hence \( E(B_v) \geq \frac{|B|}{81} \geq \frac{b}{812} \). Therefore \( E(Z_v) \geq \frac{\Delta}{1000} \).

Now we need Azuma's inequality. The concentration analysis is identical to the coloring case in Reed’s proof. We reexpress it for completeness.

Lemma 9.4.1 (Azuma). Let \( X \) be a random variable determined by \( n \) trials \( T_1, \ldots, T_n \) such that for each \( i \) and any two possible sequences of outcomes \( t_1, \ldots, t_i \) and \( t_1, \ldots, t_{i-1}, t'_i \):

\[
|E(X \mid T_1 = t_1, \ldots, T_i = t_i) - E(X \mid T_1 = t_1, \ldots, T_i = t'_i)| \leq c_i,
\]

then \( \Pr(|X - E(X)| > t) \leq 2e^{-\frac{t^2}{2\sum c_i^2}} \).

Since we colored the vertices of \( G \) independently, we can apply Azuma using any ordering. Order \( V(G) \) as \( w_1, \ldots, w_n \) so that \( N(v) \) comes last and let \( w_s \) be the last vertex not in \( N(v) \). Changing \( \zeta(w_i) \) from \( \beta \) to \( \tau \) only affects the vertices using \( \beta \) or \( \tau \) and thus changes the conditional expected value by at most 2. For \( w_i \notin N(v) \), the probability that changing \( w_i \)'s color will affect \( Z_v \) is at most the probability that one of \( w_i \)'s two colors is also assigned to one of its neighbors in \( N(v) \). Say \( w_i \) has \( d_i \) neighbors in \( N(v) \). Then the most changing \( w_i \) can change \( E(Z_v) \) is \( c_i := 2 \frac{2d_i}{\Delta - 1} = \frac{4d_i}{\Delta - 1} \). Now \( \sum_{i=1}^s d_i \leq \Delta^2 \) and thus \( \sum_{i=1}^s c_i \leq 4\Delta + 4\frac{\Delta^2}{\Delta - 1} \leq 4\Delta + 5 \). As each \( c_i \leq 5 \), we have \( \sum_{i=1}^s c_i^2 \leq 21\Delta^2 \) and hence \( \sum_{i=1}^s c_i^2 \leq 25\Delta \).

Now using \( t := \frac{\Delta}{1000} - 2 \) in Azuma gives \( \Pr(Z_v < 2) < 2e^{-\frac{(\frac{\Delta}{1000} - 2)^2}{(25\Delta)^2}} \leq \Delta^{-6} \) for large enough \( \Delta \).

9.5 \( \Pr(E_i) \leq \Delta^{-6} \)

We first need a couple structural lemmas.

Lemma 9.5.1. Each \( v \in C_i \) has at most one neighbor outside of \( C_i \) with more than 4 neighbors in \( C_i \) and no such neighbor if \( v \) is low.

Proof. Suppose otherwise that we have \( v \in C_i \) with two neighbors \( w_1, w_2 \in V(G) - C_i \) each with 5 or more neighbors in \( C_i \). Put \( Q := G[\{ w_1, w_2 \} \cup C_i - v] \), then \( v \) is joined to \( Q \) and hence \( K_1 + Q \leq G \). We show that \( K_1 + Q \) must be \( d_1 \)-choosable.
CHAPTER 9. LIST BORODIN-KOSTOCHKA FOR LARGE $\Delta$

First, suppose there are different $z_1, z_2 \in C_i$ such that \{$w_1, z_1$\} and \{$w_2, z_2$\} are independent. Since $Q$ contains an induced diamond, it is $d_0$-choosable. Let $L$ be a minimal bad $d_1$-assignment on $K_1 * Q$. Then $|L(w_1)| + |L(z_1)| \geq 4 + |Q| - 3 = |Q| + 1$. By the Small Pot Lemma, $|\text{Pot}(L)| \leq |Q|$. Hence $L(w_1) \cap L(z_1) \neq \emptyset$ and Lemma 5.3.15 shows that $|\text{Pot}(L)| \leq |Q| - 1$, but then $|L(w_1) \cap L(z_1)| \geq 2$ and Lemma 5.3.14 gives a contradiction.

By maximality of $C_i$, neither $w_1$ nor $w_2$ can be adjacent to all of $C_i$; hence it must be the case that there is $y \in C_i$ such that $w_1$ and $w_2$ are joined to $C_i - y$. If $w_1$ and $w_2$ aren’t adjacent, then $G$ contains $K_6 + E_3$ contradicting Corollary 5.3.25. Hence $C_i$ intersects the larger clique \{$w_1, w_2$\} $\cup C_i - \{y\}$, this is impossible by the definition of $C_i$.

When $v$ is low, an argument similar to the above shows that there can be no $z_1$ in $C_i$ so that \{$w_1, z_1$\} is independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of $C_i$.

Lemma 9.5.2. For $C_i$ with $|C_i| \leq \Delta - 2$, there are at least $\frac{3}{2\Delta} \Delta$ disjoint $P_3$’s $xyz$ with $y \in C_i$ and $x, z \notin C_i$ such that each $x$ and $z$ have at most 4 neighbors in $C_i$.

Proof. Consider a maximal such set of $P_3$’s. Let $A$ be all the central vertices of these $P_3$’s and $X$ all the ends. Then each $v \in X$ has at most 3 neighbors in $C_i - A$ and by Lemma 9.5.1 and maximality, each $v \in C_i - A$ has at most 2 neighbors in $G - C_i - B$ and at most 1 if $v$ is low. Thus $6|A| = 3|X| \geq ||C_i - A, X|| \geq (\Delta - |C_i| - 1)|C_i - A| \geq |C_i| - |A|$. Hence $|A| \geq \frac{3}{2\Delta} \Delta$.

We need to force safe uncolored vertices in $C_i$. If the lists have small intersections this might not happen with high probability. We handle this case using minimality of $|C|$ instead.

Lemma 9.5.3. There exists $C_i' \subset C_i$ with $|C_i'| = |C_i| - 1$ such that for $x, y \in C_i'$ we have $|L(x) \cap L(y)| \geq \frac{3}{2\Delta} \Delta$.

Proof. Suppose not and consider a $L$-coloring of $G - C_i$. Let $L'$ be the resulting list assignment on $C_i$. Then $|L'(v)| \geq |C_i| - 2$ for all $v \in C_i$. By assumption, for each $v \in C_i$ we have $x, y \in C_i - \{v\}$ with $|L(x) \cap L(y)| < \frac{3}{4\Delta}$. But then $|L'(x) \cup L'(y)| \geq 2(\Delta - 1 - (\Delta + 1 - |C_i|)) - \frac{3}{4\Delta} \geq |C_i|$. Hence we can complete the $L$-coloring to $C_i$ by Hall’s theorem, a contradiction.

We will find the desired uncolored safe vertices in $C_i'$. By Lemma 9.5.2, there are at least $\frac{3}{10} \Delta$ paths $acb$ where $c \in C_i'$ and $a, b \notin C_i'$ such that $a$ and $b$ each have at most 4 neighbors in $C_i$. Let $T_i$ be the union of all the vertices in these paths. For some fixed path we want to bound the probability that $c$ is uncolored and safe and the colors used on $a$ and $b$ are used on none of the rest of $T_i$. To do so, we distinguish three cases.

Case 1. $|L(a) \cap L(c)| < \frac{3}{4\Delta}$ and $|L(b) \cap L(c)| < \frac{3}{4\Delta}$

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$ and $\gamma \in L(c) \cap L(z)$ where $\alpha, \beta, \gamma$ are all different, let $A_{\alpha, \beta, \gamma, z}$ be the event that all of the following hold:

1. $\alpha$ is assigned to $a$ and none of the rest of $T_i \cup N(a)$,
2. $\beta$ is assigned to $b$ and none of the rest of $T_i \cup N(b)$,
3. $\gamma$ is assigned to $c$ and $z$ and none of the rest of $T_i \cup N(c)$.

Then $\Pr(A_{\alpha, \beta, \gamma, z}) \geq (\Delta - 1)^{-1} (1 - 3(\Delta - 1)^{-1}) |T_i \cup N(a)| (\Delta - 1)^{-1} (1 - 3(\Delta - 1)^{-1}) |T_i \cup N(b)| (\Delta - 1)^{-2} (1 - 3(\Delta - 1)^{-1}) |T_i \cup N(c)| \geq (\Delta - 1)^{-3} 3^{-18}$. Plainly, the $A_{\alpha, \beta, \gamma, z}$ are disjoint for different sets of indices.

Since $|L(a) - L(c)| \geq \frac{A}{3}$, we have $\frac{A}{3}$ choices for $\alpha$. Similarly we then have $\frac{A}{3} - 1$ choices for $\beta$. For $z$ we have at least $\frac{A}{10} \Delta - \frac{1}{10} \Delta \geq \frac{A}{10}$ choices. Since $|L(z) \cap L(c)| \geq \frac{3}{4\Delta}$, we then have at least $\frac{3}{4\Delta} - 2$ choices for $\gamma$ for each $z$. In total we have at least $\Delta^3 3^{-18}$ choices and thus the probability that $A_{\alpha, \beta, \gamma, z}$ holds for some choice of indices is at least $3^{-22}$.

Case 2. $|L(a) \cap L(c)| < \frac{2}{3} \Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3} \Delta$

For $y \in C_i' - T_i - N(b)$, $z \in C_i' - T_i$, $\alpha \in L(a) - L(c)$, $\beta \in L(b) \cap L(y)$ and $\gamma \in L(c) \cap L(z)$ where $\alpha, \beta, \gamma$ are all different, let $A_{\alpha, \beta, \gamma, y, z}$ be the event that all of the following hold:
1. \( \alpha \) is assigned to \( a \) and none of the rest of \( T_i \cup N(a) \),
2. \( \beta \) is assigned to \( b \) and \( y \) and none of the rest of \( T_i \cup N(b) \cup N(y) \),
3. \( \gamma \) is assigned to \( c \) and \( z \) and none of the rest of \( T_i \cup N(c) \).

Then \( \Pr(A_{\alpha,\beta,\gamma,y,z}) \geq (\Delta - 1)^{-1}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(a)|}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(b) \cup N(y)|}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(c)|} \geq (\Delta - 1)^{-53-24} \).

Again the \( A_{\alpha,\beta,\gamma,y,z} \) are disjoint for different sets of indices. For \( y \) we have at least \( |C'_i| - |T_i \cap C'_i| - |N(b) \cap C'_i| \geq \frac{3}{4}\Delta - 1 - \frac{\Delta}{3} - 4 \geq \frac{\Delta}{4} \) choices. For each \( y \) we have at least \( \frac{\Delta}{4} \) choices for \( \beta \). The rest are similar to above and in total we have at least \( \Delta^33-6 \) choices and thus the probability that \( A_{\alpha,\beta,\gamma,y,z} \) holds for some choice of indices is at least \( 3^{-27} \).

**Case 3.** \( |L(a) \cap L(c)| \geq \frac{\Delta}{3} \) and \( |L(b) \cap L(c)| \geq \frac{\Delta}{3} \)

For \( x \in C'_i - T_i - N(a) \), \( y \in C'_i - T_i - N(b) \), \( z \in C'_i - T_i \), \( \alpha \in L(a) \cap L(c) \), \( \beta \in L(b) \cap L(y) \) and \( \gamma \in L(c) \cap L(z) \) where \( \alpha, \beta, \gamma \) are all different, let \( A_{\alpha,\beta,\gamma,x,y,z} \) be the event that all of the following hold:

1. \( \alpha \) is assigned to \( a \) and \( y \) and none of the rest of \( T_i \cup N(a) \cup N(x) \),
2. \( \beta \) is assigned to \( b \) and \( y \) and none of the rest of \( T_i \cup N(b) \cup N(y) \),
3. \( \gamma \) is assigned to \( c \) and \( z \) and none of the rest of \( T_i \cup N(c) \).

Then \( \Pr(A_{\alpha,\beta,\gamma,x,y,z}) \geq (\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(a) \cup N(x)|}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(b) \cup N(y)|}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{|T_i \cup N(c)|} \geq (\Delta - 1)^{-63-24} \).

Again the \( A_{\alpha,\beta,\gamma,x,y,z} \) are disjoint for different sets of indices. In total we get at least \( \Delta^33-8 \) choices and thus the probability that \( A_{\alpha,\beta,\gamma,x,y,z} \) holds for some choice of indices is at least \( 3^{-32} \).

Now we have at least \( \frac{\Delta}{3} \) such triples. So if \( M_i \) counts the number of uncolored safe vertices in \( C_i \), we have \( E(M_i) \geq 3^{-35}\Delta \). The concentration details are identical to Reed’s proof and we conclude \( \Pr(M_i < 2) < \Delta^{-6} \).

### 9.6 \( \Pr(Q_i) \leq \Delta^{-6} \)

If \( \zeta(x) = \zeta(y) \) for different \( x, y \in K_i \), then \( x \) and \( y \) will be uncolored and \( Q_i \) cannot hold. Thus it is enough to show that all vertices of \( K_i \) getting different colors is unlikely. Just like Lemma 9.5.3, we can find \( K'_i \subset K_i \) with \( |K'_i| = |K_i| - 1 \) such that for \( x, y \in K'_i \) we have \( |L(x) \cap L(y)| \geq \frac{\Delta}{3} \).

Let \( x, y \in K'_i \). The probability that \( x \) and \( y \) get the same color and this color is used on none of the rest of \( N(x) \cup N(y) \) is at least \( \frac{\Delta}{2\Delta}(1 - (\Delta - 1)^{-1})^{2\Delta-2} \geq \frac{\Delta^2}{2\Delta} \). Since there are at least \( \frac{1}{2}\Delta^2 \) such pairs, the expected number of pairs getting the same color is at least \( 3^{-4}\Delta \). An application of Azuma’s inequality very similar to the sparse case now proves \( \Pr(Q_i) \leq \Delta^{-6} \).

### 9.7 \( \Pr(F_i) \leq \Delta^{-6} \)

In this case we must have \( C_i = K_i \) since no vertex outside \( C_i \) has \( \frac{\Delta}{2} \) neighbors in \( C_i \). Since low vertices don’t make things harder, we will assume there are no low vertices in \( C'_i \). In particular, for a low vertex, we don’t need a triple as in the follow lemma, but just one good neighbor outside because we only need to save one color on a low vertex’s neighborhood to make it safe.

**Lemma 9.7.1.** There are at least \( \frac{1}{2}\sqrt{\Delta \log \Delta} \) disjoint \( P_3 \)'s \( xyz \) with \( y \in C_i \) and \( x, z \notin C_i \) such that \( x \) and \( z \) each have at most \( \frac{\sqrt{\Delta}}{\log(\Delta)} \) neighbors in \( C_i \).
Proof. Since there are at most \(\log^2(\Delta)\) vertices outside \(C_i\) which have more than \(\sqrt{\Delta}\) neighbors in \(C_i\) and all of these vertices have at most \(\sqrt{\Delta}\log(\Delta)\) neighbors in \(C_i\), removing all their neighbors from \(C_i\) we are left with a set \(A\) of of vertices all of whose neighbors outside \(C_i\) have at most \(\sqrt{\Delta}\) neighbors in \(C_i\). Now \(|A| \geq \Delta - 1 - \log^2(\Delta)\sqrt{\Delta}\log(\Delta) \geq \frac{3}{4}\).

Now pick \(P_3\)'s \(xyz\) with \(y \in A\) in turn removing the neighbors of \(x\) and \(z\) each time. We get at least \(\frac{|A|}{2 \log(\Delta)} \geq \frac{\sqrt{\Delta}}{4} \log \Delta\) disjoint \(P_3\)'s.

Now the proof of the expected value is the same as the proof of \(\Pr(E_i) \leq \Delta^{-6}\), except that we have fewer \(P_3\)'s to multiply by at the end. So, if \(M_i\) counts the number of uncolored safe vertices in \(C_i\), we have \(\E(M_i) \geq 3^{-35}(\frac{1}{2}) \sqrt{\Delta}\log \Delta \geq 3^{-37}\Delta\log \Delta\).

Now, the application of Azuma is the same as in the \(\Pr(E_i) \leq \Delta^{-6}\) case, except we use \(t := 3^{-37}\Delta\log \Delta - 2\), which gives gives \(\Pr(M_i < 2) < 2e^{-(\frac{1}{3} \sqrt{\Delta}\log \Delta - 2)^2} \leq \Delta^{-6}\) for large enough \(\Delta\).

\(\square\)

9.8 \(\Pr(P_i) \leq \Delta^{-6}\)

Case 1. Some \(x \in G - C_i\) has \(|N(x) \cap C_i| > \sqrt{\Delta}\log(\Delta)\).

If \(C_i \neq K_4\), then take \(x\) to be \(x_i\). Let \(Z_1, \ldots, Z_m\) be the clumps of \(x\) in \(K_4\) and for each \(j \in [m]\), let \(\alpha_j\) be the color the \(Z_j\) clump has that the others do not. By Lemma 9.5.3, we may as well assume that \(|L(x) \cap L(y)| \geq \frac{4}{3} \Delta\) for all \(x, y \in K_4\) (the cost is one vertex which changes nothing).

Pick \(z_j \in Z_j\) arbitrarily. By Lemma 9.5.1, any neighbor of \(z_j\) in \(G - C_i - x\) (of which there is at most one) has at most \(4\) neighbors in \(C_i\). Thus, by symmetry, for each \(j \in [\frac{n}{4}]\) we can pick \(y_j \in G - C_i - x\) such that \(y_jz_j \in E(G)\) and the \(y_j\) are all different. Put \(A := N(x) \cap K_4\). Then \(\frac{n}{4} \geq \left(\frac{4}{3}\right) \left(\frac{1}{2}\right) |A| \geq \frac{\sqrt{\Delta}}{2 \log(\Delta)}\).

Now, for fixed \(j \in [\frac{n}{4}]\), we bound the probability that \(z_j\) is uncolored, \(\alpha_j\) is not used on any neighbor of \(z_j\) and \(y_j\) is colored with a color that is either not in \(L(z_j)\) or is used on \(C_i\). Let \(T_i\) be the union of all the \(y_j\)'s and \(z_j\)'s. We distinguish two cases.

Subcase 1a. \(|L(y_j) \cap L(z_j)| < \frac{3}{4} \Delta\)

For \(\beta \in L(y_j) - L(z_j), w \in C_i - T_i\) and \(\gamma \in L(z_j) \cap L(w)\) where \(\beta, \gamma \neq \alpha_j\) and \(\beta \neq \gamma\), let \(F_{\beta, \gamma, w}\) be the event that all of the following hold:

1. \(\beta\) is assigned to \(y_j\) and none of the rest of \(T_i \cup N(y_j)\),
2. \(\gamma\) is assigned to \(z_j\) and \(w\) and none of the rest of \(T_i \cup N(z_j)\),
3. \(\alpha_j\) is assigned to no neighbor of \(z_j\).

The probability of (3) is at least \(\left(\frac{\Delta - 2}{\Delta - 1}\right)^{\Delta} = (1 - (1 - \Delta)^{-1})^{\Delta} \geq \frac{1}{3}\). Hence \(\Pr(F_{\beta, \gamma, w}) \geq \frac{1}{3} (1 - 3(\Delta - 1)^{-1})^{T_i \cup N(y_j)}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})^{T_i \cup N(z_j)} \geq (\Delta - 1)^{-3} 3^{-13} \Delta^{-13}\).

Now we have at least \(\frac{\Delta}{3}\) choices for \(\beta, \frac{\Delta}{3}\) choices for \(w\) and \(\frac{\Delta}{3}\) choices for \(\gamma\) for each \(w\). Thus the probability that \(F_{\beta, \gamma, w}\) holds for some choice of indices is at least \(3^{-15}\).

Subcase 1b. \(|L(y_j) \cap L(z_j)| \geq \frac{3}{4} \Delta\)

For \(y \in C_i - T_i\) - \(N(y_j), \beta \in L(y_j) \cap L(y), w \in C_i - T_i\) and \(\gamma \in L(z_j) \cap L(w)\) where \(\beta, \gamma \neq \alpha_j\) and \(\beta \neq \gamma\), let \(F_{\beta, \gamma, y, w}\) be the event that all of the following hold:

1. \(\beta\) is assigned to \(y_j\) and \(y\) and none of the rest of \(T_i \cup N(y_j) \cup N(y)\),
2. \(\gamma\) is assigned to \(z_j\) and \(w\) and none of the rest of \(T_i \cup N(z_j)\),
3. \(\alpha_j\) is assigned to no neighbor of \(z_j\).
We have \( \Pr(F_{\beta, \gamma, y, w}) \geq \frac{1}{3}(\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})|T_i \cup N(y_i) \cup N(y)| (\Delta - 1)^{-2}(1 - 3(\Delta - 1)^{-1})|T_i \cup N(z_j)| \geq (\Delta - 1)^{-4}3^{-16}. \)

Now for \( y \) we have at least \( |K_i| - |C_i \cap T_i| - |N(y_j) \cap C_i| \geq \frac{\Delta}{2} \) choices and for each \( y \) we have at least \( \frac{2}{3} \Delta \) choices for \( \beta \). Thus the probability that \( F_{\beta, \gamma, y, w} \) holds for some choice of indices is at least \( 3^{-18} \).

Therefore the expected number of good clumps is at least \( 3^{-18}(\frac{1}{3\Delta})\sqrt{\Delta \log(\Delta)} \geq 3^{-21}\sqrt{\Delta \log(\Delta)}. \) Changing any color will affect the conditional expectations by at most 2 and a similar computation for Azuma shows that \( \Pr(F_i) \leq \Delta^{-6}. \) The key here is that \( (\sqrt{\Delta \log(\Delta)})^2 \) grows faster than \( \Delta. \)

**Case 2. More than \( \log^2(\Delta) \) vertices \( x \in G - C_i \) have \( |N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}. \)**

We must have \( C_i = K_i \). Let \( x_1, \ldots, x_k \) be \( k := \lceil \log^2(\Delta) \rceil \) different vertices in \( G - C_i \) which have \( |N(x_j) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)} \) for each \( j \in [k] \).

The computation for the expected number of good clumps for each \( x_j \) is the same as Case 1 and so we expect at least \( 3^{-21}\frac{\sqrt{\Delta}}{\log(\Delta)} \) good clumps for each \( x_j \). Thus in total we expect \( 3^{-21}\sqrt{\Delta \log(\Delta)} \) good clumps over the \( \log^2(\Delta) \) sets. Let \( X \) count this total number of good clumps. We show that \( \Pr(X < 3\log^2(\Delta)) \leq \Delta^{-6} \) and hence at least one \( x_j \) has at least 3 good clumps with high enough probability.

If we applied Azuma with the information we have now we’d be in trouble because many of the \( x_j \)’s could use the same special color and hence changing a vertex to that color would change the conditional expectation by a lot. We need one further structural lemma that guarantees at most 4 of the \( x_j \)’s use any given special color.

**Lemma 9.8.1.** Let \( K \) be a \( \Delta - 1 \) clique in \( G \) and \( x_1, x_2, x_3, x_4, x_5 \in G - K \) with \( |N(x_j) \cap K| \geq 5 \) such that the \( N(x_j) \cap K \) are pairwise disjoint. Then no color is special for all the \( x_j \).

**Proof.** Suppose otherwise that some color \( \alpha \) is special for all the \( x_j \). Put \( A_j := N(x_j) \cap K \). Just like in the proof of Lemma 9.3.1, any \( L \)-coloring of \( G - (K \cup \{x_1, \ldots, x_5\}) \) must not leave \( \alpha \) available on any of the vertices in \( A_j \) for any \( j \in [5] \). Pick \( z_j \in A_j \) for each \( j \) and let \( y_j \) be the neighbor of \( z_j \) in \( G - (K \cup \{x_1, \ldots, x_5\}) \). Put \( N := \{y_1, \ldots, y_6\} \). By Lemma 9.5.1, \( |N| \geq 2 \). Now just like in Lemma 9.3.1, by using minimality of \( |G| \) we see that adding any edge between vertices in \( N \) must create a \( K_\Delta \) and then counting degrees gives a contradiction. \qed

Now when we change a color we change the conditional expectation by at most 8. A similar computation to before bounds \( \sum_j c_j^2 \leq 500\Delta \). Applying Azuma with \( t = 3^{-21}\sqrt{\Delta \log(\Delta)} - 3\log^2(\Delta) \) gives \( \Pr(X < 3\log^2(\Delta)) < 2e^{-\frac{(3^{-21}\sqrt{\Delta \log(\Delta)} - 3\log^2(\Delta))^2}{500\Delta}} \leq \Delta^{-6} \) for large \( \Delta \).
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<td>$x$ and $y$ are adjacent</td>
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<td>$G[S]$</td>
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<td>$A * B$</td>
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<td>$\underbrace{G + G + \cdots + G}_{k \text{ times}}$</td>
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<td>$G^k$</td>
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<tr>
<td>$H \subset G$</td>
<td>$H$ is a proper induced subgraph of $G$</td>
</tr>
<tr>
<td>$H \prec G$</td>
<td>$H$ is a child of $G$</td>
</tr>
<tr>
<td>$f: S \mapsto T$</td>
<td>an injective function from $S$ to $T$</td>
</tr>
<tr>
<td>$f: S \rightarrow T$</td>
<td>a surjective function from $S$ to $T$</td>
</tr>
<tr>
<td>$X := Y$</td>
<td>$X$ is defined as $Y$</td>
</tr>
<tr>
<td>$K_k$</td>
<td>the complete graph on $k$ vertices</td>
</tr>
<tr>
<td>$E_k$</td>
<td>the edgeless graph on $k$ vertices (that is, $\overline{K_k}$)</td>
</tr>
<tr>
<td>$P_k$</td>
<td>the path on $k$ vertices</td>
</tr>
<tr>
<td>$C_k$</td>
<td>the cycle on $k$ vertices</td>
</tr>
<tr>
<td>$K_{a,b}$</td>
<td>the complete bipartite graph with parts of size $a$ and $b$ (that is, $E_a \ast E_b$)</td>
</tr>
<tr>
<td>$[n]$</td>
<td>${1, 2, \ldots, n}$</td>
</tr>
</tbody>
</table>
Bibliography


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