A common generalization of Hall’s theorem and Vizing’s edge-coloring theorem

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LBD Data

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Hall’s theorem

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- when can we pick an SDR?

- if $k$ of the sets together have fewer than $k$ elements, we can’t
  - $A_1 = \{1, 2\}, A_2 = \{1, 2\}, A_3 = \{1, 2\}$

- Hall’s theorem: this is the only thing that can go wrong

$$\text{SDR exists} \iff \left| \bigcup_{i \in I} A_i \right| \geq |I| \text{ for all } I \subseteq \{1, \ldots, n\}$$
some card games

the simplest variation

- Dealer vs. Player

[cards depicted]
some card games
the simplest variation

- Dealer vs. Player
- the deck has just many copies of the high spade cards

[Images of spade cards: Ace, King, Queen, Jack, 10]
some card games
the simplest variation

- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up
some card games
the simplest variation

- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up
- Player wins if he can pick a Royal Flush, one card from each stack
some card games

example, a Player win
some card games

example, a Player win
some card games
example, a Dealer win
Player cannot win if there is a set of $k$ stacks that together have fewer than $k$ different cards
some card games

winning condition

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some card games

winning condition

- Player cannot win if there is a set of $k$ stacks that together have fewer than $k$ different cards
- Hall’s theorem says: **Player wins otherwise**
some card games
making things harder for Dealer

- this isn’t a fun game, far too easy for Dealer to win
some card games
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- this isn’t a fun game, far too easy for Dealer to win
- to make a better game, we allow Player to modify some of the stacks
some card games
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Player’s Move

Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).
some card games
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*Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).*

Dealer’s Move

*Dealer can (i) do nothing or (ii) swap A and B in one other stack.*
some card games
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**Player’s Move**

*Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).*

**Dealer’s Move**

*Dealer can (i) do nothing or (ii) swap A and B in one other stack.*

**Winning**

*Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.*
some card games

example, a Player win

Player picks a King from the deck and swaps it for a Queen in the first stack. The Dealer can swap a King and Queen in one of the other stacks. Player wins no matter what the Dealer does.
some card games
example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack

![Image of card deck with a King swapped for a Queen]
some card games
example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack
some card games

example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks
some card games
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some card games
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example, a Dealer win
some card games
what was the difference?

In the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks.

In the bottom game, Dealer cannot prevent Player from increasing the number of different cards in the first three stacks.
some card games
what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
some card games
what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks.
- in the bottom game, Dealer cannot prevent Player from increasing the number of different cards in the first three stacks.
if the same card appears on three stacks, Player can force the addition of a new card to these stacks
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it is not hard to show that this is essentially all Player can do
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this suggests a necessary condition

Degree

The degree of a card $C$ in a set of stacks $S$ is the number of times $C$ appears in $S$. We write $d_S(C)$ for this quantity.

Necessary Condition

If Player can win, then for every set of stacks $S$ we must have $\sum_{C \in \bigcup S} \lceil d_S(C)^2 \rceil \geq |S|$.
some card games

necessary condition

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks
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The *degree* of a card $C$ in a set of stacks $S$ is the number of times $C$ appears in $S$. We write $d_S(C)$ for this quantity.
some card games
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Necessary Condition
If Player can win, then for every set of stacks $S$ we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$
some card games

intuition

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$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$ 

- in Hall’s theorem, each $C$ is ‘worth’ 1 in

$$\sum_{C \in \bigcup S} 1 = \left| \bigcup S \right| \geq |S|$$
some card games

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If Player can win, then for every set of stacks $S$ we must have

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- in Hall’s theorem, each $C$ is ‘worth’ 1 in $\sum_{C \in \bigcup S} 1 = \left| \bigcup S \right| \geq |S|$.
- Player can turn $2t + 1$ of the same card into $t + 1$ different cards, so $C$ is ‘worth’ $\left\lceil \frac{d_S(C)}{2} \right\rceil$. 

some card games

Dealer’s strategy

- given a set of stacks $S$ with
  \[
  \sum_{C \in \bigcup S} \left\lfloor \frac{d_S(C)}{2} \right\rfloor < |S|
  \]
some card games

Dealer’s strategy

- given a set of stacks $S$ with $\sum_{C \in \bigcup S} \left\lfloor \frac{ds(C)}{2} \right\rfloor < |S|$.
- Dealer’s strategy: maintain this invariant.
some card games

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  - this is good enough since then $|\bigcup S| \leq \sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$ always
some card games

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  - if Player swaps $A$ in for $B$, increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$, then $d_S(A)$ and $d_S(B)$ both changed from even to odd
some card games
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  - so, Dealer can swap $A$ for $B$ somewhere else, decreasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$
some card games

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- Dealer has maintained
  $$\sum_{C \in \bigcup S} \left\lfloor \frac{d_S(C)}{2} \right\rfloor < |S|$$
Winning Condition

- this necessary condition is also sufficient
Winning Condition

Player can win if and only if for every set of stacks $S$ we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$
Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them.
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some card games

proof idea

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some card games

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some card games

proof idea

1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them
2. Player calls those stacks done and never plays with those card types again
if no such set of card types exists, then Hall’s theorem shows that there is at least one card appearing on none of the remaining stacks
if no such set of card types exists, then Hall’s theorem shows that there is at least one card appearing on none of the remaining stacks, but then some card appears at least thrice, so Player can increase the number of card types in the stacks.
some card games

proof idea

3. If no such set of card types exists, then Hall’s theorem shows that there is at least one card appearing on none of the remaining stacks.

4. But then some card appears at least thrice, so Player can increase the number of card types in the stacks.

5. Goto step 1.
if no such set of card types exists, then Hall’s theorem shows that there is at least one card appearing on none of the remaining stacks.

but then some card appears at least thrice, so Player can increase the number of card types in the stacks.

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some card games

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goto step 1
A generalization of Hall’s theorem
making it harder for Player

- allow Dealer to make more swaps in response to Player’s move

Winning Condition
Player can win in the $t$-game if and only if for every set of stacks $S$ we have
$$\sum_{C \in \bigcup S} \left\lceil d_S(C) t + 1 \right\rceil \geq |S|.$$
A generalization of Hall’s theorem
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- allow Dealer to make more swaps in response to Player’s move
- for each $t \geq 1$, the $t$-game allows Dealer to make up to $t$ swaps

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Hall’s theorem is the winning condition in the $(k-1)$-game when there are $k$ total stacks:

$$1 \leq d_S(C) \leq k,$$

so

$$\left\lceil d_S(C) t + 1 \right\rceil = 1$$

so, the sum equals

$$|\bigcup S|.$$
A generalization of Hall’s theorem
making it harder for Player

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**Winning Condition**

*Player can win in the \( t \)-game if and only if for every set of stacks \( S \) we have*

\[
\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.
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A generalization of Hall’s theorem
making it harder for Player

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A generalization of Hall’s theorem
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A generalization of Hall’s theorem
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- Hall’s theorem is the winning condition in the \((k - 1)\)-game when there are \( k \) total stacks:
  - \( 1 \leq d_S(C) \leq k \), so \( \left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1 \)
  - so, the sum equals \( |\bigcup S| \)
  - Player’s moves are useless
edge coloring

setup

- assign colors to the edges of a graph so that incident edges get different colors

Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.
edge coloring

setup

- assign colors to the edges of a graph so that incident edges get different colors
- how few colors can we use?

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edge coloring

proof of Vizing’s theorem

- proceed by induction on the number of vertices
edge coloring

proof of Vizing’s theorem

- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree
edge coloring
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edge coloring
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edge coloring
proof of Vizing’s theorem

- exchanging colors on a two-colored path is just a Player move followed by a Dealer move

\[ \sum_{C \in \bigcup S} d_S(C) \geq 2 |S| \]

so, we have the desired winning condition
edge coloring
proof of Vizing’s theorem

- exchanging colors on a two-colored path is just a Player move followed by a Dealer move
- we can make any of Player’s legal moves this way, so if the winning conditions are satisfied, Vizing’s theorem is true
edge coloring
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- exchanging colors on a two-colored path is just a Player move followed by a Dealer move
- we can make any of Player’s legal moves this way, so if the winning conditions are satisfied, Vizing’s theorem is true
- each stack has at least two colors, so counting the ‘cards’ in two ways we get for each set of stacks $S$,

$$\sum_{C \in \bigcup S} d_S(C) \geq 2|S|$$
edge coloring
proof of Vizing’s theorem

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- so, we have the desired winning condition

$$\sum_{C \in \bigcup S} \frac{d_S(C)}{2} \geq |S|$$
we introduced a simple card game

- Player can pick any card \( A \) from the deck and swap it for another card \( B \) in one stack (not containing \( A \)).
- Dealer can (i) do nothing or (ii) swap \( A \) and \( B \) in one other stack.
- Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.
- Player can win exactly when a Hall-like condition is satisfied.
- Vizing's edge-coloring theorem is an easy corollary taking it further.
- Most other classical edge-coloring results follow easily.
- Generalizes easily to multigraphs.
- A more general game unifies much of edge-coloring theory.
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summary

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  - Player can pick any card $A$ from the deck and swap it for another card $B$ in one stack (not containing $A$)
  - Dealer can (i) do nothing or (ii) swap $A$ and $B$ in one other stack
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taking it further
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- generalizes easily to multigraphs
- a more general game unifies much of edge-coloring theory
the more general game

- Fixer vs. Breaker
the more general game

- Fixer vs. Breaker
- played on a multigraph $G$
the more general game

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- assign a list of colors $L(v)$ to each vertex
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the more general game

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**Fixer’s turn**

*Pick \( \alpha \) in the pot and \( v \in V(G) \) with \( \alpha \notin L(v) \) and set*

\[
L(v) := L(v) \cup \{\alpha\} - \beta \quad \text{for some} \quad \beta \in L(v).
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**Fixer’s turn**

Pick $\alpha$ in the pot and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} - \beta$ for some $\beta \in L(v)$.

**Breaker’s turn**

If Fixer modified $L(v)$ by inserting $\alpha$ and removing $\beta$, then Breaker can either do nothing or pick $w \in V(G - v)$ and modify its list by swapping $\alpha$ for $\beta$ or $\beta$ for $\alpha$. 
the more general game
necessary condition

**Definition**

For $C \subseteq \text{Pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of $H$ induced on the vertices $v$ with $L(v) \cap C \neq \emptyset$. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$
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Superabundance

We say that \((H, L)\) is **abundant** if \( \psi_L(H) \geq \|H\| \) and that \((H, L)\) is **superabundant** if for every \( H' \subseteq H \), the pair \((H', L)\) is abundant.

If Fixer can win, then \((G, L)\) is superabundant.
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We say that $(H, L)$ is **abundant** if $\psi_L(H) \geq \|H\|$ and that $(H, L)$ is **superabundant** if for every $H' \subseteq H$, the pair $(H', L)$ is abundant.

**Necessary Condition**

*If Fixer can win, then $(G, L)$ is superabundant.*
the more general game
adding a chronicle

- we can get more power for Fixer and still imply edge-coloring results by modifying the game slightly
the more general game
adding a chronicle

- we can get more power for Fixer and still imply edge-coloring results by modifying the game slightly
- we do this by adding a chronicle

basically, this ensures that Breaker's moves are consistent with being embedded some graph the chronicle $C$ is a multigraph with vertex set $V(G) \cup \{\infty\}$ that will be updated as the game progresses. Each edge of $C$ will be labeled with a doubleton of colors $\{\alpha, \beta\} \subseteq \text{Pot}(L)$.

At the start of the game $C$ is edgeless. Breaker's turn

If there is a $v_x \in E(C - \infty)$ labeled $\{\alpha, \beta\}$, then Breaker swaps $\alpha$ and $\beta$ at $x$. If instead $v_\infty \in E(C)$, Breaker does nothing. Otherwise, Breaker can do nothing, or pick $w \in V(G - v)$ with $|\{\alpha, \beta\} \cap L(w)| = 1$ such that no edge incident to $w$ in $C$ has label $\{\alpha, \beta\}$, and swap $\alpha$ and $\beta$ at $w$. 
the more general game
adding a chronicle

- we can get more power for Fixer and still imply edge-coloring results by modifying the game slightly
- we do this by adding a **chronicle**
- basically, this ensures that Breaker’s moves are consistent with being embedded *some* graph
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the more general game
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Chronicle update

Remove all edges of $C$ whose label intersects $\{\alpha, \beta\}$ in exactly one color. If Breaker swapped $\alpha$ and $\beta$ at $z$ and there is no $vz$ edge in $C$ labeled $\{\alpha, \beta\}$, then add one. Otherwise, if Breaker did nothing and there is no $v\infty$ edge in $C$ labeled $\{\alpha, \beta\}$, then add one.
the more general game
an equivalent game

Necessary Condition

*If Fixer can win the chronicled game, then \((G, L)\) is superabundant.*
the more general game
an equivalent game

Necessary Condition

If Fixer can win the chronicled game, then \((G, L)\) is superabundant.

- there is a simpler-looking game that is equivalent to the chronicled game
the more general game
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Necessary Condition

If Fixer can win the chronicled game, then \((G, L)\) is superabundant.

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Equivalent game

Fixer picks different colors \(\alpha, \beta \in \text{Pot}(L)\). Let \(S\) be the \(w \in V(G)\) with 
\[|\{\alpha, \beta\} \cap L(w)| = 1.\] Breaker picks a partition \(P_1, \ldots, P_k\) of \(S\) where 
\[|P_i| \leq 2\] for all \(i\). For each \(i\), Fixer either chooses to swap \(\alpha\) and \(\beta\) on all vertices in \(P_i\) or on no vertices in \(P_i\).