Conjectures that should be true

August 24, 2016

1 Edges in list-critical graphs

A graph \(G\) is \(k\)-list-critical if \(G\) is not \((k - 1)\)-choosable, but every proper subgraph of \(G\) is \((k - 1)\)-choosable. Replace ‘\((k - 1)\)’ with ‘online \((k - 1)\)’ and ‘\(k\)’ with ‘online \(k\)’ in the previous sentence and read it.

**Conjecture 1.** Every incomplete \(k\)-list-critical graph has average degree at least

\[
k - 1 + \frac{k - 3}{(k - 1)^2}.
\]

**Background.** The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [11] proved that in a \(k\)-critical graph, the vertices of degree \(k - 1\) induce a disjoint union of Gallai trees. The same is true for \(k\)-list-critical graphs [1, 10]. This quickly implies a lower bound on the average degree of \(k\)-list-critical graphs of

\[
k - 1 + \frac{k - 3}{k^2 - 3}.
\]

In [21], R. improved this to

\[
k - 1 + \frac{k - 3}{k^2 - 2k + 2}
\]

using a lemma from Kierstead and R. [14] that generalizes a kernel technique of Kostochka and Yancey [15]. As noted at the end of [21], a small improvement to the argument would yield Conjecture [1]. This is now known to hold for \(k \geq 6\), only the \(k = 4\) and \(k = 5\) cases remain.

**Conjecture 2.** Every incomplete online \(k\)-list-critical graph \(G\) has

\[
2 \|G\| \geq (k - 1) |G| + k - 3.
\]

**Background.** Dirac [9] proved this for \(k\)-critical graphs. Kostochka and Stiebitz [16] proved it for \(k\)-list-critical graphs. Their proof does not seem to generalize. When \(|G|\) is large compared with \(k\), the conjecture holds by Gallai-type bounds on the average degree of online \(k\)-list-critical graphs [13, 2].

*(dis)proofs ⇒ landon.rabern@gmail.com*
1.1 The $\frac{5}{6}$ bound

**Conjecture 3.** Every vertex-transitive graph has $\chi \leq \max \{\omega, \left\lceil \frac{5\Delta + 3}{6} \right\rceil \}$.

**Background.** In [3], the following was proved

- the conjecture holds for the fractional chromatic number $\chi^*$,
- the conjecture holds with $\left\lceil \frac{5\Delta + 3}{6} \right\rceil$ replaced by $\varepsilon(\Delta + 1)$ for some $\varepsilon < 1$,
- the conjecture holds both if Reed’s $\omega, \Delta, \chi$ conjecture and the strong $2\Delta$-colorability conjecture hold for vertex-transitive graphs (only strong $\frac{5}{2}\Delta$-colorability is required).

Does the conjecture hold for Cayley graphs?

**Conjecture 4.** Every line graph (of a multigraph) has $\chi \leq \max \{\omega, \left\lceil \frac{5\Delta + 3}{6} \right\rceil \}$.

**Background.** In [18] this was proved with $\left\lceil \frac{5\Delta + 3}{6} \right\rceil$ replaced by $\frac{7\Delta + 10}{8}$. Conjecture 14 in [18] that implies this conjecture is now known to be false. This conjecture is true, recently proved with Dan Cranston.

2 Around Planar graphs

**Conjecture 5.** Every graph with no $K_5$-subdivision is 2-fold 9-colorable.

**Background.** In [1], Cranston and R. gave a short proof of this conjecture with $K_5$-minors excluded instead of $K_5$-subdivisions (which also follows from the Four Color Theorem). Hajós conjectured that every graph is $(k-1)$-colorable unless it contains a subdivision of $K_k$. This is known to be true for $k \leq 4$ and false for $k \geq 7$. The cases $k = 5$ and $k = 6$ remain unresolved.

3 Maximum degree, clique number and colorings

3.1 Around Borodin-Kostochka

**Conjecture 6.** Every graph with $\chi \geq \Delta \geq 8$ contains a $K_3 \lor H$ where $H$ is some graph on $\Delta - 3$ vertices.

**Background.** By results in [8], for $\Delta \geq 9$ the existence of $K_3 \lor H$ implies the existence of $K_{\Delta}$. So, this (seemingly weaker) conjecture for $\Delta \geq 9$ implies the Borodin-Kostochka conjecture. The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 6.

**Conjecture 7.** Every graph with $\chi \geq \Delta$ contains $K_{\Delta - 3}$.

**Background.** Results in [5] show that this holds with $K_{\Delta - 4}$ instead of $K_{\Delta - 3}$. Moreover, [6] proves the conjecture for all but $\Delta \in \{6,8,9,11,12\}$. Reed’s conjecture [22] that every graph satisfies $\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil$ implies this conjecture with $K_{\Delta - 2}$ instead of $K_{\Delta - 3}$.
**Conjecture 8.** Every graph with $\chi \geq \Delta$ either contains $K_{\Delta}$ or contains a $K_{\Delta-4}$ with all $\Delta$-vertices.

**Background.** Results in [5] show that this holds with $K_{\Delta-5}$ instead of $K_{\Delta-4}$. For $\Delta \leq 7$, the conjecture holds by [19] [17]. Also by [5], it holds when $\Delta = 3r + 1$ for $r \geq 3$. □

**Conjecture 9.** Every graph with $\Delta \geq 8$ and $\omega < \Delta$ is 2-fold $(2\Delta - 1)$-colorable.

**Background.** The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 9. □

**Conjecture 10.** Every graph with $\theta \geq 10$ and $\omega \leq \frac{\theta}{2}$ is $\lfloor \frac{\theta}{2} \rfloor$-choosable.

**Background.** Here $\theta$ is the Ore degree given by $\theta(G) := \max_{x,y \in E(G)} d(x) + d(y)$. This conjecture holds for ordinary coloring [12] [19] [17] [20]. In [14], the conjecture is proved for $\theta \geq 18$ for both list-coloring and online list-coloring. Further lowering of $\theta$ would follow from improved bounds on average degree of list-critical graphs [13]. □

**Conjecture 11.** Every claw-free graph with $\Delta \geq 9$ and $\omega < \Delta$ is $(\Delta - 1)$-choosable.

**Background.** In [7], this was proved for ordinary coloring. In [6], the conjecture was proved for $\Delta \geq 69$. Also, [6] proved that the full conjecture follows from the line-graph case. □

**Conjecture 12.** There is a polynomial time graph algorithm that finds either a $(\Delta - 1)$-coloring or a $K_{\Delta-3}$.

**Background.** In [5], the following was proved

- the conjecture holds with $K_{\Delta-3}$ replaced by $K_{\Delta-4}$,
- the conjecture holds for $\Delta \geq 25$ (the proof uses algorithmic versions of the local lemma),
- the conjecture holds when $\Delta = 3r + 1$.

□

**References**


