

EMBEDDING SUBGRAPHS AND COLORING GRAPHS  
UNDER EXTREMAL DEGREE CONDITIONS

DISSERTATION

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## Publications

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Part I  
PRELIMINARIES

## 1. Notation

All graphs in this thesis are finite and undirected with no loops or multiple edges. Let  $V(G)$  denote the set of vertices of  $G$ . The edges of  $G$  are 2-element subsets of  $V(G)$ , and the set of all edges of  $G$  is  $E(G)$ . Two vertices  $u, v$  are adjacent if  $\{u, v\} \in E(G)$ .

For any set  $X$ , we let  $|X|$  denote the cardinality of  $X$ . Throughout this thesis,  $|V(G)|$  will be denoted by  $p$ , and we shall assume that  $p \geq 1$ .

The number of edges incident with a vertex  $v \in V(G)$  is called the degree of  $v$  in  $G$ , and is denoted  $\deg_G(v)$ . We define

$$\Delta(G) = \max_{v \in V(G)} \deg_G(v)$$

and

$$\delta(G) = \min_{v \in V(G)} \deg_G(v).$$

The complement of  $G$ , denoted  $G^c$ , is the graph on the same vertex set  $V(G)$ , in which  $\{u, v\} \in E(G^c)$  if and only if  $\{u, v\} \notin E(G)$ , where  $u, v \in V(G)$ . Clearly, for any graph  $G$ ,

$$\Delta(G^c) + \delta(G) + 1 = p.$$

For two graphs  $G$  and  $H$  with  $|V(H)| \leq |V(G)|$ , an embedding of  $H$  into  $G$  is an injection .

$$\pi: V(H) \longrightarrow V(G)$$

that maps edges of  $H$  into edges of  $G$ . If such an embedding exists, we say that  $H$  is a subgraph of  $G$ .

Note that when  $|V(H)| = |V(G)|$ ,  $H$  is a subgraph of  $G$  if and only if  $G^c$  is a subgraph of  $H^c$ .

Brackets will be used with two meanings, depending upon their context. For any rational number  $r$ ,  $[r]$  denotes the greatest integer less than or equal to  $r$ . For a subset  $X \subseteq V(G)$ , we denote by  $G[X]$  the subgraph of  $G$  induced by  $X$ : thus,  $V(G[X]) = X$  and if  $u, v \in X$ , then  $\{u, v\} \in E(G[X])$  if and only if  $\{u, v\} \in E(G)$ . We denote by  $G - X$  the graph  $G[V(G) - X]$ .

A complete graph on  $n$  vertices is a graph on  $n$  vertices in which any pair of distinct vertices are adjacent. Such a graph will be denoted by  $K_n$ . A complete bipartite graph on disjoint sets of  $n$  and  $m$  vertices is the graph on these vertices in which each vertex in the  $n$ -set is adjacent to every vertex in the  $m$ -set. Such a graph is denoted  $K_{n,m}$ .

A maximal complete subgraph induced by some vertices of a graph is called a clique. A maximal complete bipartite induced subgraph is called a biclique.

A set  $X$  of vertices is stable if  $G[X]$  is edgeless. The maximum cardinality of all stable sets  $X \subseteq V(G)$  is denoted  $\beta(G)$ , and is called the stability number of  $G$ . The maximum number of vertices in a clique of  $G$ , denoted  $\theta(G)$ , is called the clique number of  $G$ . Clearly,

$$\theta(G) = \beta(G^c); \quad \theta(G^c) = \beta(G).$$

A coloring of  $G$  is a partition of  $V(G)$  into stable subsets, where the partition is unordered and admits null sets. A set  $X \subseteq V(G)$  is monochromatic in a coloring of  $G$  if all vertices of  $X$  have the same color: i.e., they lie in the same set in the coloring partition. The chromatic number  $\chi(G)$  of  $G$  is the fewest possible number of sets in a coloring of  $G$ .

A path in  $G$  is a sequence of vertices  $v_0, v_1, \dots, v_n$  in  $V(G)$  for  $n \geq 1$  such that

$$(1.1) \quad v_i = v_j \text{ implies either } i = j \text{ or } \{i, j\} = \{0, n\};$$

$$(1.2) \quad \text{for } i = 1, 2, \dots, n, \quad v_i \text{ is adjacent in } G \text{ to } v_{i-1}.$$

The vertices  $v_0$  and  $v_n$  are said to be joined by the path. If  $v_0 = v_n$ , we say that the path is closed; otherwise, the path is open. A graph is connected if any two vertices are joined by a path. A component of  $G$  is a maximal connected subgraph of  $G$ . A vertex of a connected graph is a cutvertex if its removal disconnects the graph. A polygon is a subgraph determined by a set of vertices and edges joining consecutive vertices in a closed path.

The girth is the number of edges of the polygon. A polygon with odd girth is an odd polygon. An arc is a subgraph determined by the set of vertices and edges joining consecutive vertices in an open path. An odd arc is an arc with an odd number of edges.

A tree is a connected graph having no polygons.

A  $\theta$ -graph is a graph consisting of three distinct arcs, joining the same two vertices and having no other common vertices.

To simplify notation, we shall denote the singleton set  $\{x\}$  by  $x$ .

Given a set  $X$  and a subset  $\{x_1, \dots, x_n\}$ , let  $(x_1 \ x_2 \ \dots \ x_n)$  denote the cyclic permutation that sends  $x_i$  to  $x_{i+1}$ ,  $1 \leq i \leq n$ , that sends  $x_n$  to  $x_1$ , and that fixes all other elements of  $X$ . Given a permutation  $\alpha: X \rightarrow X$  and a function  $\pi: Y \rightarrow X$ , for sets  $X$  and  $Y$ , we denote by  $\alpha\pi$  the composition of  $\alpha$  and  $\pi$  which maps  $y \in Y$  to  $\alpha(\pi(y)) \in X$ .

Given a set  $X$  and a finite sequence  $x_1, x_2, \dots, x_n$  of members of  $X$ , such that  $x_i = x_j$ ,  $i < j$  imply  $x_i = x_{i+1} = \dots = x_j$ , let  $(x_1 \ x_2 \ \dots \ x_n)'$  denote the cyclic permutation obtained by deleting from  $x_1, x_2, \dots, x_n$  the terms which have previously appeared in the sequence.

## 2. Introduction

Two problems are considered in this dissertation. They concern somewhat separate topics, but both depend upon degree constraints, and there are several points of overlap. First, we consider the problem of estimating the chromatic number  $\chi(G)$ , knowing  $\Delta(G)$  and  $\theta(G)$ . Then, we consider the problem of giving sufficient conditions, in terms of  $\Delta(H)$  and  $\Delta(G^c)$ , for a graph  $H$  on  $p$  vertices to be a subgraph of a graph  $G$ , also on  $p$  vertices.

The basic result in the literature on the coloring problem is Brooks' Theorem [5]:

Theorem 2.1 Let  $G$  be a graph with maximum degree  $\Delta(G)$ . We have

$$(2.1) \quad \chi(G) \leq \Delta(G) + 1.$$

If  $\Delta(G) = 2$ , then equality holds in (2.1) if and only if  $G$  contains an odd polygon. If  $\Delta(G) \neq 2$ , then equality holds if and only if  $G$  contains a clique  $K_{\Delta(G)+1}$ .

Note that if  $\Delta(G) = 2$ , an odd polygon of  $G$  is necessarily a connected component of  $G$ . Also, a clique  $K_{\Delta(G)+1}$  is necessarily a component of  $G$ . Such components, which force equality in (2.1), are called  $B_{\Delta(G)}$ -components.

Since each component of a graph can be colored independently, we can assume without loss of generality, that  $G$  is connected.

We give a proof of Brooks' Theorem by induction on  $\Delta(G)$ , and in so doing, we obtain new information. For instance, we show that if  $G$  is not a  $B_{\Delta(G)}$ -component, then there is a coloring of  $G$  in  $\Delta(G)$  colors in which some monochromatic set contains  $\beta(G)$  vertices. Also, we characterize those connected graphs  $G$  for which there is a coloring of  $G$  in  $\Delta(G)$  colors such that some monochromatic set consists solely of vertices of degree  $\Delta(G)$ .

In section 4 we consider the problem of partitioning the vertices of a graph into sets  $X_1, X_2, \dots, X_n$  such that the numbers  $\Delta(G[X_i])$ ,  $i = 1, 2, \dots, n$  satisfy various constraints. One result will be used for a problem on subgraphs. Another result is a new proof of a partition theorem of Lovász [11].

We combine, in section 5, this partition theorem of Lovász with Brooks' Theorem to give an estimate of  $\chi(G)$  in terms of  $\Delta(G)$  and  $\theta(G)$ . The result improves (2.1) when  $\theta(G) < \frac{1}{2} \Delta(G)$ .

In section 6 we consider further the interrelationship between  $\chi(G)$ ,  $\Delta(G)$  and  $\theta(G)$ .

In [6], we considered the problem of giving a sufficient condition, based upon  $\Delta(H)$  and  $\Delta(G^c)$ , for

$H$  to be a subgraph of  $G$ . We continue here to obtain sharper results.

Our first result, which has recently been independently obtained by Sauer and Spencer [14], is that if  $G$  and  $H$  are graphs on  $p$  vertices satisfying

$$2\Delta(G^c)\Delta(H) \leq p-1,$$

then  $H$  is a subgraph of  $G$ . This is best possible only when  $\Delta(G^c)=1$  or  $\Delta(H)=1$ . We continue, in section 7, by discussing a conjectured improvement of this result that would be best possible if true, and we consider various special cases treated in the literature.

In section 8, we give a slightly sharper result when  $\Delta(H)=2$  whose proof is not long.

In section 10, we show that if  $\Delta(H)=2$  and if

$$\Delta(G^c) \leq \frac{1}{3}p - \max(9, \frac{3}{2}p^{1/3}),$$

then  $H$  is a subgraph of  $G$ . The coefficient  $\frac{1}{3}$  is best possible. However, the proof is quite long. In the special case where every component of  $H$  is either  $K_3$ ,  $K_2$ , or  $K_1$ , we obtain an even sharper result in section 9. We show that if  $\Delta(G^c) \leq \frac{p-1}{3}$  and if such a graph  $H$  is not a subgraph of  $G$ , then  $G$  lies in one of two classes which do not have  $H$  as a subgraph. We characterize these classes.

Part II  
CHROMATIC NUMBER

### 3. Brooks' graph-coloring theorem and the stability number

In this section, we shall consider a connected graph  $H$ , with at least one edge. To simplify notation, we denote  $\Delta(H)$  by  $h$ .

A maximum stable subset of the set of vertices of degree  $h$  will be called a superstable set.

A  $B_h$ -component of  $H$  was defined in section 2. The equivalence of (3.4) and (3.6) of Theorem 3.2 below is Brooks' Theorem (Theorem 2.1).

Albertson, Bollobás, and Tucker [1] showed first that with two exceptions  $H_1$  and  $H_2$ , defined below, every graph  $H$  with  $\Delta(H) = h$  and with no subgraph  $K_h$  has stability number

$$\beta(H) > |V(H)|/h,$$

and they conjectured that such graphs  $H$  have an  $h$ -coloring in which some monochromatic set has more than  $|V(H)|/h$  vertices. Second, they proved this conjecture for graphs that are not regular of degree  $h$ . Theorem 3.2, combined with the first result of Albertson, Bollobás, and Tucker shows that this conjecture is true, even for regular graphs.

The two exceptional graphs,  $H_1$  and  $H_2$ , may be defined as follows: let  $V(H_1)$  be the integers modulo 8, and let  $\{v, w\} \in E(H_1)$  if and only if

$$v - w \equiv 1, 2, 6, \text{ or } 7 \pmod{8}.$$

Let  $V(H_2)$  be the integers modulo 10, and let  $\{v, w\} \in E(H_2)$  if and only if

$$v - w \equiv 1, 4, 5, 6, \text{ or } 9 \pmod{10}.$$

A Brooks tree is any graph  $H$  with  $\Delta(H) = h$  that arises from a tree  $T$  satisfying  $\Delta(T) \leq h$  by the replacement of each vertex of  $T$  with

- (a) an odd polygon if  $h = 3$ ;
- (b) a clique  $K_h$  if  $h \neq 3$ ,

such that if  $x$  and  $y$  are adjacent vertices of  $T$ , then the polygons or cliques substituted for  $x$  and  $y$  are joined by an edge whose removal disconnects  $H$ . Thus,  $K_2$  is the only Brooks tree with  $h = 1$ ; odd arcs with at least 3 edges are the only Brooks trees with  $h = 2$ ; and if  $h \geq 3$ , then a Brooks tree is not a tree in the usual sense of the word.

Theorem 3.1 Let  $H$  be a connected graph with  $\Delta(H) = h \geq 1$ . The following are equivalent:

- (3.1)  $H$  is a  $B_h$ -component, or a Brooks tree;
- (3.2) There is no superstable set  $S$  such that  $H - S$  can be colored in  $h - 1$  colors;
- (3.3) There is no stable set  $S$  of vertices of degree  $h$  such that  $H - S$  can be colored in  $h - 1$  colors.

We also have

Theorem 3.2 Let  $H$  be a connected graph with  $\Delta(H) = h \geq 1$ . The following are equivalent:

(3.4)  $H$  is a  $B_h$ -component;

(3.5) There is no maximum stable set  $S$ , such that  $H - S$  can be colored in  $h - 1$  colors;

(3.6) There is no  $h$ -coloring of  $H$ .

Proof of Theorem 3.2 from Theorem 3.1: For  $\Delta(H) \leq 2$ , the theorem is easily verified. Assume therefore, that  $\Delta(H) \geq 3$ .

We show that if (3.1), (3.2), and (3.3) are equivalent for  $\Delta(H) = h$ , then (3.4), (3.5), and (3.6) are also equivalent for  $\Delta(H) = h$ . Since (3.4) implies (3.6) and (3.6) implies (3.5), it suffices to prove that (3.5) implies (3.4) if (3.1), (3.2), and (3.3) are equivalent.

Adjoin to  $H$  a set  $V$  of  $\sum (h - \deg_H(v))$  vertices disjoint from  $V(H)$ , where the sum runs over all  $v \in V(H)$ . We join each vertex  $v$  of  $H$  to exactly  $h - \deg_H(v)$  vertices of  $V$ , such that no vertex of  $V$  is joined to more than one vertex of  $H$ . Denote the resulting graph  $H'$ . Then,

$$(3.7) \quad H'[V(H)] = H;$$

$$(3.8) \quad \text{Any } v \in V(H) \text{ has degree } h \text{ in } H';$$

$$(3.9) \quad \text{Any } v \in V \text{ has degree } 1 \text{ in } H'.$$

By (3.7) and (3.8), a superstable set  $S$  in  $H'$  is a maximum stable set in  $H$ . Hence, (3.5) for  $H$  implies (3.2) for  $H'$ , whence by (3.1), either  $H'$  is a  $B_h$ -component, or it is a Brooks tree. Since Brooks trees have vertices of degree  $h-1$ , conditions (3.8), (3.9), and  $h \geq 3$  imply that  $H'$  is not a Brooks tree. Thus,  $H'$  is a  $B_h$ -component, and therefore, has no vertices of degree 1, whence  $H = H'$ . This proves (3.4), and thus the equivalence of (3.4), (3.5), and (3.6). Hence, Theorem 3.2 follows from Theorem 3.1.

Proof of Theorem 3.1: Again, we may suppose that  $h \geq 3$ . Since (3.1) implies (3.3) and (3.3) implies (3.2), it suffices to show that (3.2) implies (3.1).

Suppose inductively that the theorem is true for all graphs  $G$  with  $\Delta(G) < h$ . Then Theorem 3.2 is true for such graphs  $G$ . Let  $H$  be a graph with  $\Delta(H) = h$  such that  $H$  does not satisfy (3.1), and such that for any superstable set  $S$ ,  $H - S$  has no  $(h-1)$ -coloring. For a given superstable set  $S$ , Theorem 3.2 and

$$\Delta(H - S) \leq h - 1$$

imply that either  $H - S$  can be colored in  $h - 1$  colors, or  $H - S$  has a  $B_{h-1}$ -component. We have already precluded the first possibility. Hence,  $H - S$  has a  $B_{h-1}$ -component. Without loss of generality, we shall choose  $S$  to be a superstable set that minimizes the number of  $B_{h-1}$ -components in  $H - S$ .

Suppose that a vertex  $s \in V(H)$  is in no  $B_{h-1}$ -component in  $H - S$ , regardless of the choice of a superstable set  $S$  that minimizes the number of  $B_{h-1}$ -components in  $H - S$ . Since  $H$  is connected, such a vertex  $s$  exists that is adjacent to a vertex  $v$  lying in a  $B_{h-1}$ -component  $C$  of  $H - S$ , for some such  $S$ . Since the only vertex not in  $C$  that is adjacent to  $v$  lies in  $S$ , we must have  $s \in S$ . Then  $S + v - s$  is a superstable set, and either  $H - (S + v - s)$  has one fewer  $B_{h-1}$ -component than  $H - S$ , contrary to the choice of  $S$ , or  $s$  lies in a  $B_{h-1}$ -component of  $H - (S + v - s)$ , contrary to the choice of  $s$ . Hence, by contradiction, all vertices of  $H$  lie in  $B_{h-1}$ -components of  $H - S$ , for suitable  $S$ .

Let  $P$  be a polygon in  $H$  with the property that there is no superstable set  $S$  such that a  $B_{h-1}$ -component of  $H - S$  contains  $P$ . If  $h = 3$ , any polygon of

even girth will do; otherwise, any polygon not contained in a clique suffices. We will show that if  $H$  is not a  $B_h$ -component or a Brooks tree, then such a  $P$  must exist.

If  $P$  does not exist, then

(3.10) If  $P'$  is a polygon in  $H$  and if  $h = 3$ , then

~~any~~  $P'$  has odd girth

and

(3.11) If  $P'$  is a polygon in  $H$  and  $h \geq 4$ , then

~~any~~  $P'$  is contained in a clique.

Suppose, by way of contradiction, that there are distinct overlapping subgraphs  $C_1$  and  $C_2$  of  $H$ , where  $C_1$  is a  $B_{h-1}$ -component of  $H - S_1$ , for some superstable set  $S_1$ . If  $h \geq 4$ , then  $C_1$  and  $C_2$  are cliques on  $h$  vertices each. Since  $C_1$  and  $C_2$  overlap,  $\Delta(H) = h$  forces

$$|V(C_1) \cup V(C_2)| \leq h+1.$$

Since  $C_1$  and  $C_2$  are distinct, we have equality, and hence  $H[V(C_1) \cup V(C_2)]$  is either isomorphic to  $K_{h+1}$  or to  $K_{h+1}$  minus an edge. In the first case,  $H$  is a  $B_h$ -component. In the second case, let  $P'$  be a polygon on 4 vertices in  $H[V(C_1) \cup V(C_2)]$  containing the 2 non-adjacent vertices. This violates (3.11). If  $h = 3$ , then  $C_1$  and  $C_2$  are overlapping odd polygons, and  $h < 4$

forces them to overlap in an edge. Then  $C_1 \cup C_2$  contains a  $\theta$ -graph, and hence an even polygon. Thus, (3.10) is violated. Hence, if  $P$  does not exist, then, since each vertex of  $H$  lies in a  $B_{h-1}$ -component of  $H-S$  for a suitable superstable set  $S$ ,  $V(H)$  can be partitioned into sets  $V_1, V_2, \dots, V_n$ , such that  $H[V_i]$  is a  $B_{h-1}$ -component of  $H-S$ , for suitable superstable  $S$ . All polygons of  $G$  are contained in these  $H[V_i]$ . Moreover,  $H$  is connected, and so it is easily seen in this case that if (3.10) and (3.11) hold, then  $H$  must be a Brooks tree or a  $B_h$ -component. This is contrary to assumption, and we may therefore conclude that  $P$  does exist. To prove the theorem, we will derive a contradiction from the existence of  $P$ .

Let  $C_0$  be a  $B_{h-1}$ -component of  $H-S_0$ , such that  $C_0$  intersects  $P$ , and such that  $S_0$  is superstable and chosen to minimize the number of  $B_{h-1}$ -components in  $H-S_0$ . Since the degree of any vertex of  $C_0$  in  $H-S_0$  is  $h-1$ , and since  $\Delta(H) = h$ , an edge of  $P$  lies in  $E(C_0)$ . Since  $P$  is not contained in  $C_0$ , which is an induced subgraph of  $H$ , an edge of  $P$  lies outside  $E(C_0)$ . Therefore, there is a vertex  $v$  of  $V(P) \cap V(C_0)$  having one incident edge in  $E(C_0)$  and the other incident edge

$\{v, s\}$  outside  $E(C_0)$ . Since  $C_0$  is a component of  $H - S_0$ , we have  $s \in S_0$ .

Define a sequence  $v_1, s_1, v_2, s_2, \dots, v_m, s_m$  of vertices along  $P$  as follows: Let

$$v_1 = v; \quad s_1 = s;$$

$$S_1 = S_0 + v_1 - s_1.$$

For each  $i=1, 2, \dots, m-1$ , there is a superstable set

$$S_i = S_{i-1} + v_i - s_i$$

and a (unique)  $B_{h-1}$ -component  $C_i$  of  $H - S_i$  containing  $s_i$ . If for some  $i$ ,  $s_i$  is not in a  $B_{h-1}$ -component of  $H - S_i$ , then  $H - S_i$  has fewer  $B_{h-1}$ -components than  $H - S_0$ , contrary to our choice of  $S_0$ . The polygon  $P$  intersects  $C_i$  in a path starting at  $s_i$  and ending at a vertex of  $S_i$ , which we shall call  $v_{i+1}$ . Thus, we have determined a vertex  $s_{i+1} \in V(P) \cap S_i$  that is adjacent in  $P$  to  $v_{i+1}$  and is not in  $C_i$ . Since  $v_{i+1}$  is adjacent to  $h-1$  vertices in  $C_i$  also,  $\deg_H(v_{i+1}) = h$ . Thus, since  $S_i$  is superstable,

$$S_{i+1} = S_i + v_{i+1} - s_{i+1}$$

is also a superstable set. We terminate the sequence at the first vertex  $s_m$  ( $m \geq 1$ ) that is adjacent to a vertex of the original  $B_{h-1}$ -component  $C_0$  of  $H - S_0$ .

To see that  $s_m$  exists, note that  $P$  determines a closed

path, and the first vertex along that path after  $v$  and  $s$  that is adjacent to a vertex of the original  $B_{h-1}$ -component is necessarily in  $S_0$ , and hence in  $S_i$  for each  $i < m$ .

Of course, since  $s_m \in S_{m-1}$  is the first vertex in the sequence to be adjacent to a vertex of  $V(C_0)$ , the vertices of  $V(C_0 - v)$  have not been moved into the superstable set  $S_i$ , as  $i$  runs from 0 to  $m-1$ , and no vertices adjacent to vertices of  $C_0$  have been moved out of the superstable set. Thus, in the  $B_{h-1}$ -component of  $H - S_m$  containing  $s_m$  and  $C_0 - v$ , any vertices other than  $s_m$  or  $V(C_0 - v)$  would be adjacent to  $s_m$  only. But no vertex of a  $B_{h-1}$ -component is a cutvertex, and so  $s_m$  and  $V(C_0 - v)$  together induce a  $B_{h-1}$ -component of  $H - S_m$ . Therefore, we must have

$$N(s_m) - v_m = N(v) - s,$$

where  $N(v)$  denotes the set of vertices of  $H$  adjacent to  $v$ .

If  $C_0$  is a polygon of girth at least 5, then  $s_m$  is adjacent to two nonadjacent vertices  $x_1, x_2$  of degree  $h=3$  that comprise  $N(v) - s$ . Since  $s_m$  is the only vertex in  $S_0$  to which  $x_1$  and  $x_2$  are adjacent,  $S_0 \cup \{x_1, x_2\} - s_m$  is a bigger superstable set than  $S_0$ , contrary to the maximality of  $S_0$ .

If  $C_0$  is a clique  $K_h$ , then  $s_m$  is adjacent to every vertex of  $C_0 - v_1$ . If  $v_1$  and  $s_m$  are adjacent, then  $m=1$ , and  $V(C_0) + s_m$  induces a clique  $K_{h+1}$  in  $H$ . Since  $H$  is connected,  $K_{h+1}$  is necessarily all of  $H$ , a case excluded since (3.1) is false. Suppose, therefore, that  $s_m$  and  $v_1$  are not adjacent. Let  $x$  be a member of the equal sets  $V(C_m - s_m) = V(C_0 - v)$ . Then  $H - (S_0 + x - s_m)$  has fewer  $B_{h-1}$ -components than  $H - S_0$ , and  $S_0 + x - s_m$  is a superstable set. Since this contradicts the choice of  $H$ ,  $P$  does not exist. But, as we have seen, this contradicts the assumption that  $H$  is a  $B_h$ -component or a Brooks tree. This proves the theorem.

#### 4. Some partition theorems

We consider the problem of partitioning the vertex set of a graph so that the subgraphs induced by the subsets of vertices will satisfy various constraints on the degree of their vertices.

Given sets  $X, Y \subseteq V(G)$ , we denote by  $E(X, Y)$  the set of edges in  $E(G)$  with one end in  $X$  and the other end in  $Y$ . Let  $E^c(X, Y)$  denote the set of edges in  $E(G^c)$  with one end in  $X$  and the other end in  $Y$ .

Given a partition  $X_1 \cup X_2$  of  $V(G)$ , we simplify notation by writing  $G_i$  for the subgraph  $G[X_i]$  induced by  $X_i$ , where  $i=1, 2$ .

Lovász [11] proved a variation on the first theorem below, except that he maximized an expression different than  $f_1(X_1, X_2)$ .

Let  $h_1$  and  $h_2$  be integers, and let

$$f_1(X_1, X_2) = |E(X_1, X_2)| + h_1|X_1| + h_2|X_2|.$$

Theorem 4.1 Let  $G$  be a graph with maximum degree  $\Delta(G) \geq 1$ , and let  $h_1, h_2$  be nonnegative integers such that

$$\Delta(G) = h_1 + h_2 + 1.$$

If  $X_1 \cup X_2$  is a partition of  $V(G)$  that maximizes  $f_1$ , then for  $i=1,2$ ,  $X_i$  is nonempty, and

$$\Delta(G_i) \leq h_i.$$

Proof: Of  $X_1, X_2$ , at least one set, say  $X_1$ , is nonempty. Later, we show that  $X_2$  is also nonempty, whence the following argument applies also to  $X_2$ . Let  $x \in X_1$ . By hypothesis,

$$\begin{aligned} 0 &\leq f_1(X_1, X_2) - f_1(X_1 - x, X_2 + x) \\ &= |E(X_1, X_2)| + h_1 |X_1| + h_2 |X_2| - |E(X_1 - x, X_2 + x)| \\ &\quad - h_1 (|X_1| - 1) - h_2 (|X_2| + 1) \\ &= |E(x, X_2)| - |E(x, X_1)| + h_1 - h_2. \end{aligned}$$

We add  $2 \deg_{G_1}(x) = 2 |E(x, X_1)|$  to each side and get

$$\begin{aligned} 2 \deg_{G_1}(x) &\leq |E(x, X_2)| + |E(x, X_1)| + h_1 - h_2 \\ &= \deg_G(x) + h_1 - h_2 \\ &\leq (h_1 + h_2 + 1) + h_1 - h_2 \\ &= 2h_1 + 1. \end{aligned}$$

Dividing by 2 and observing that the left side is an integer, we get

$$\deg_{G_1}(x) \leq h_1.$$

Since  $x \in X_1$  is arbitrary, we have

$$\Delta(G_1) \leq h_1 < \Delta(G),$$

whence,  $X_1$  is not  $V(G)$ . Thus,  $X_2$  is also not empty, and the theorem follows.

Corollary 4.2 (Lovász [11]) Let  $G$  be a graph with  $\Delta(G) = h$ , and let  $h_1, h_2, \dots, h_n$  be nonnegative integers satisfying

$$h = h_1 + h_2 + \dots + h_n + n - 1.$$

Then there is a partition  $V(G) = X_1 \cup X_2 \cup \dots \cup X_n$  such that for  $i \leq n$ , if  $X_i$  is not empty, then

$$\Delta(G[X_i]) \leq h_i.$$

Proof: Let Theorem 4.1, where  $n=2$ , be a basis for induction. Assume inductively that this corollary is true for  $n-1$ , and write

$$h = h_1 + (h_2 + \dots + h_n + (n-1) - 1) + 1.$$

Theorem 4.1 asserts that there is a partition  $X_1 \cup (V(G) - X_1)$  such that

$$\Delta(G[X_1]) \leq h_1$$

$$\Delta(G - X_1) \leq h_2 + \dots + h_n + (n-1) - 1.$$

By the induction hypothesis, there is a partition  $X_2 \cup \dots \cup X_n$  of  $V(G) - X_1$  such that

$$\Delta(G[X_i]) \leq h_i,$$

for  $i=1, 2, \dots, n$ . This proves the corollary.

Conjecture: Let  $G$  be a graph on  $p$  vertices. If neither  $G$  nor  $G^c$  is edgeless, then there are partitions  $X_1 \cup X_2$  and  $Y_1 \cup Y_2$  of  $V(G)$  such that

$$\Delta(G[X_1]) + \Delta(G[X_2]) + \Delta(G^c[Y_1]) + \Delta(G^c[Y_2]) \leq p - 3.$$

If  $G$  is regular, then this conjecture follows easily from Theorem 4.1.

Suppose that the conjecture is true. It is easily verified that for any graph  $G$ ,

$$\chi(G) \leq \Delta(G) + 1.$$

Thus, the inequality of the conjecture implies

$$\chi(G[X_1]) + \chi(G[X_2]) + \chi(G^c[Y_1]) + \chi(G^c[Y_2]) \leq p + 1.$$

Therefore, for any graph  $G$ ,

$$\chi(G) + \chi(G^c) \leq p + 1.$$

Since this inequality is the theorem of Nordhaus and Gaddum [12], the conjecture, if true, would generalize their theorem.

A nontrivial partition  $X_1 \cup X_2$  of  $V(G)$  is a partition in which both  $X_1$  and  $X_2$  are nonempty.

For any partition  $X_1 \cup X_2$  of  $V(G)$  we write

$$G_i = G[X_i], \quad i = 1, 2,$$

and

$$p_i = |X_i|, \quad i = 1, 2,$$

and define, for  $c \in (0, 1]$ ,

$$f_2(X_1, X_2) = |E(X_1, X_2)| + \frac{1}{2}cp_1^2 + \frac{1}{2}cp_2^2.$$

Theorem 4.3 Let  $G$  be a graph with

$$\Delta(G) = c(p-1)$$

for  $c \in (0, 1]$  and  $p \geq 2$ . For any partition  $X_1 \cup X_2$  of  $V(G)$  such that

$$(4.1) \quad f_2 \text{ is maximized, and}$$

$$(4.2) \quad \frac{1}{2}c(p_1^2 + p_2^2) \text{ is minimized, subject to (4.1),}$$

it follows that

$$(4.3) \quad X_1 \cup X_2 \text{ is a nontrivial partition;}$$

and for  $i = 1, 2$ ,

$$(4.4) \quad \Delta(G_i) \leq c(p_i - 1).$$

Proof: Define the linear function

$$(4.5) \quad c(t) = c - t,$$

where  $t \geq 0$ . Thus,

$$(4.6) \quad \Delta(G) = c(p-1) = c(t)(p-1) + t(p-1).$$

For any partition  $X_1 \cup X_2$  of  $V(G)$  and any  $t \geq 0$ , define

$$F_t(X_1, X_2) = |E(X_1, X_2)| + \frac{1}{2}c(t)(p_1^2 + p_2^2).$$

Thus, for  $X_1$  and  $X_2$  fixed,  $F_t$  is a linear function of  $t$  with  $F$ -intercept  $f_2(X_1, X_2)$  and with slope  $-\frac{1}{2}(p_1^2 + p_2^2)$ . Moreover,  $F_0$  is equal to  $f_2$ .

Therefore, if  $X_1 \sim X_2$  satisfies (4.1), then for any other partition  $Y_1 \cup Y_2$  of  $V(G)$ ,

$$F_0(X_1, X_2) \geq F_0(Y_1, Y_2).$$

Also, (4.2) assures that if  $Y_1 \cup Y_2$  is another partition that maximizes  $f_2(X_1, X_2)$ , then

$$F_t(X_1, X_2) \geq F_t(Y_1, Y_2).$$

Thus, the only way that we could have

$$F_t(X_1, X_2) < F_t(Y_1, Y_2)$$

if (4.1) and (4.2) hold is if

$$F_0(X_1, X_2) > F_0(Y_1, Y_2)$$

and if the slope of  $F_t(X_1, X_2)$  is strictly less than that of  $F_t(Y_1, Y_2)$ , and  $t$  is sufficiently large. Thus, for  $t \geq 0$  sufficiently close to 0, if (4.1) and (4.2) hold, then  $X_1 \sim X_2$  also maximizes  $F_t$ . We shall consider  $t$  to be small enough so that  $X_1 \sim X_2$  also maximizes  $F_t$ .

Reversing the indices if necessary, we may suppose without loss of generality that  $X_1$  is nonempty. Let  $x \in X_1$ . We have

$$\begin{aligned}
(4.8) \quad 0 &\leq F_t(X_1, X_2) - F_t(X_1 - x, X_2 + x) \\
&= |E(X_1, X_2)| + \frac{1}{2}c(t)(p_1^2 + p_2^2) \\
&\quad - |E(X_1 - x, X_2 + x)| \\
&\quad - \frac{1}{2}c(t)((p_1 - 1)^2 + (p_2 + 1)^2) \\
&= |E(x, X_2)| - |E(x, X_1)| + c(t)p_1 \\
&\quad - c(t)p_2 - c(t).
\end{aligned}$$

We add  $2 \deg_{G_1}(x) = 2|E(x, X_1)|$  to each side and get

$$\begin{aligned}
2 \deg_{G_1}(x) &\leq \deg_G(x) + c(t)p_1 - c(t)p_2 - c(t) \\
&\leq (c(t) + t)(p_1 + p_2 - 1) + c(t)p_1 \\
&\quad - c(t)p_2 - c(t) \\
&= 2c(t)(p_1 - 1) + t(p - 1).
\end{aligned}$$

We divide by 2 and substitute for  $c(t)$  to get

$$\begin{aligned}
(4.9) \quad \deg_{G_1}(x) &\leq c(t)(p_1 - 1) + \frac{1}{2}t(p - 1) \\
&= c(p_1 - 1) - t(p_1 - 1) + \frac{1}{2}t(p - 1) \\
&= c(p_1 - 1) + \frac{1}{2}t(p - 2p_1 + 1).
\end{aligned}$$

If  $G_1 = G$ , then  $p_1 = p$ , whence by (4.9), if  $x$  is a vertex of maximum degree in  $G$ , then

$$\begin{aligned}
\deg_G(x) &= \deg_{G_1}(x) \\
&\leq c(p - 1) + \frac{1}{2}t(1 - p) \\
&< c(p - 1) \\
&= \deg_G(x),
\end{aligned}$$

a contradiction. Hence, (4.3) holds, and (4.9) applies to either set  $X_1$  or  $X_2$ . Since (4.9) holds for  $t = 0$ , (4.4) follows.

Let  $X_1 \cup X_2$  be a nontrivial partition that maximizes  $f_j(X_1, X_2)$ , with  $j=1$  in Theorem 4.1 or with  $j=2$  in Theorem 4.3. If Theorem 4.3 applies, assume also that (4.2) holds. If  $x_1 \in X_1$  and  $x_2 \in X_2$  have the property that

$$(4.10) \quad |E(X_1, X_2)| = |E(X_1 + x_2 - x_1, X_2 + x_1 - x_2)|,$$

then  $(X_1 + x_2 - x_1) \cup (X_2 + x_1 - x_2)$  is also a partition of  $V(G)$  such that the above conditions hold. Any pair  $x_1, x_2$  of vertices satisfying condition (4.10) are called interchangeable. If  $x_1 \in X_1$  and  $x_2 \in X_2$  are interchangeable vertices, then  $G[X_1 + x_2 - x_1]$  and  $G[X_2 + x_1 - x_2]$  satisfy the same conclusions in Theorems 4.1 and 4.3 that apply to  $G[X_1]$  and  $G[X_2]$ .

Theorem 4.4 If in Theorem 4.1 or 4.3  $x_1 \in X_1$  and  $x_2 \in X_2$  are two adjacent vertices such that

$$(4.11) \quad \deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \Delta(G) - 1,$$

then  $x_1$  and  $x_2$  are interchangeable, and we have

$$\deg_{G_1}(x_1) = \begin{cases} h_1 & \text{in Theorem 4.1;} \\ [c(p_1 - 1)] & \text{in Theorem 4.3,} \end{cases}$$

and

$$\deg_G(x_1) = \Delta(G).$$

If  $x_3$  is another vertex that is interchangeable with  $x_1$ , then  $x_2$  and  $x_3$  are adjacent in  $G$ .

Proof: Let  $x_1 \in X_1$  and  $x_2 \in X_2$  be adjacent vertices satisfying (4.11), where  $X_1 \sim X_2$  is a partition of  $V(G)$  that maximizes  $f_1(X_1, X_2)$  in Theorem 4.1 or maximizes  $f_2(X_1, X_2)$  and satisfies (4.2) in Theorem 4.3. We have

$$\begin{aligned}
 (4.12) \quad & |E(X_1 + x_2 - x_1, X_2 + x_1 - x_2)| = |E(X_1, X_2)| \\
 & \quad + \deg_{G_1}(x_1) + \deg_{G_2}(x_2) \\
 & \quad - |E(x_1, X_2 - x_2)| - |E(x_2, X_1 - x_1)| \\
 & = |E(X_1, X_2)| + 2 \deg_{G_1}(x_1) + 2 \deg_{G_2}(x_2) \\
 & \quad - |E(x_1, V(G) - x_2)| - |E(x_2, V(G) - x_1)| \\
 & = |E(X_1, X_2)| + 2(\Delta(G) - 1) - (\deg_G(x_1) - 1) \\
 & \quad - (\deg_G(x_2) - 1) \quad (\text{by (4.11)}) \\
 & \geq |E(X_1, X_2)|.
 \end{aligned}$$

By the maximality of  $f_j(X_1, X_2)$  in Theorems 4.1 and 4.3,  $|E(X_1, X_2)|$  cannot be less than  $|E(X_1 + x_2 - x_1, X_2 + x_1 - x_2)|$ . Hence, (4.12) holds with equality. Thus,  $x_1$  and  $x_2$  are interchangeable. Also, since (4.12) holds with equality,

$$\Delta(G) - 1 = \deg_G(x_i) - 1 \quad (i=1,2),$$

whence,

$$\deg_G(x_i) = \Delta(G).$$

Observe that if (4.11) holds, then  $\deg_{G_1}(x_1)$  and  $\deg_{G_2}(x_2)$  attain the upper bound specified by Theorem 4.1 or 4.3, whichever is applicable. For instance,

from (4.11) and from (4.4) of Theorem 4.3,

$$\begin{aligned}
 \Delta(G) - 1 &= \deg_{G_1}(x_1) + \deg_{G_2}(x_2) \\
 &\leq \Delta(G_1) + \Delta(G_2) \\
 &\leq c(p_1 - 1) + c(p_2 - 1) \\
 &= c(p - 1) - c \\
 &= \Delta(G) - c \\
 &< \Delta(G).
 \end{aligned}$$

Thus, since  $\Delta(G)$  is an integer,

$$(4.13) \quad \deg_{G_i}(x_i) = \Delta(G_i) = [c(p_i - 1)],$$

for  $i=1$  and  $2$ . In Theorem 4.1, we can more easily obtain

$$(4.14) \quad \deg_{G_i}(x_i) = h_i \quad (i=1,2).$$

If, contrary to the conclusion of Theorem 4.4,  $x_2$  is not adjacent to  $x_3$ , then in  $G[X_2 + x_1 - x_3]$ ,  $x_2$  is adjacent to  $x_1$  and to  $h_2$  or  $[c(p_2 - 1)]$ , respectively, other vertices in  $G[X_2 + x_1 - x_3]$ , depending upon whether we consider Theorem 4.1 or Theorem 4.3, respectively.

However, we have

$$\Delta(G[X_2 + x_1 - x_3]) \leq \begin{cases} h_2 & \text{in Theorem 4.1;} \\ [c(p_2 - 1)] & \text{in Theorem 4.3,} \end{cases}$$

since  $x_1$  and  $x_2$  are interchangeable, and so we have a contradiction. Thus,  $x_2$  must be adjacent to  $x_3$ .

We shall use the following result in section 9.

Theorem 4.5 Let  $G$  be a graph with  $p \geq 2$  and

$$(4.15) \quad \delta(G) = c(p-1)$$

for some  $c \in [0,1)$ . There is a nontrivial partition

$X_1 \cup X_2$  of  $V(G)$  which maximizes

$$(4.16) \quad f_3(X_1, X_2) = \frac{1}{2}(1-c)(p_1^2 + p_2^2) - |E(G_1^c)| - |E(G_2^c)|$$

and satisfies

$$(4.17) \quad \delta(G_i) \geq c(p_i - 1),$$

for  $i=1$  and  $2$ . Furthermore, suppose  $x_1 \in X_1$  and  $x_2 \in X_2$  are adjacent in  $G^c$  and satisfy

$$(4.18) \quad \deg_{G_1}(x_1) + \deg_{G_2}(x_2) = \delta(G).$$

Then  $x_1$  and  $x_2$  are interchangeable,

$$(4.19) \quad \deg_G(x_1) = \deg_G(x_2) = c(p-1),$$

and the set of vertices in  $X_{3-i}$  interchangeable with  $x_i$  are adjacent in  $G_{3-i}^c$  to  $x_{3-i}$ .

Proof: By (4.15),  $G^c$  satisfies

$$(4.20) \quad \Delta(G^c) = (1-c)(p-1)$$

for some  $c \in [0,1)$ . Note that a partition that maximizes  $f_3(X_1, X_2)$  also maximizes

$$f_3(X_1, X_2) + |E(G^c)| = |E^c(X_1, X_2)| + \frac{1}{2}(1-c)(p_1^2 + p_2^2),$$

which is  $f_2(X_1, X_2)$  with  $1-c$  in place of  $c$  and  $E^c$  in place of  $E$ . Hence, by Theorem 4.3, there is a nontrivial partition of  $X_1 \cup X_2$  of  $V(G)$  that maximizes  $f_3(X_1, X_2)$

such that

$$(4.21) \quad \Delta(G_1^c) \leq (1-c)(p_1-1),$$

by (4.4), whence (4.17) follows.

If  $x_1 \in X_1$  and  $x_2 \in X_2$  satisfy (4.18), then

$$\deg_{G_1^c}(x_1) + \deg_{G_2^c}(x_2) = \Delta(G^c) - 1,$$

whence (4.11) of Theorem 4.4 holds for  $G^c$ . The remaining conclusions of Theorem 4.5 follow directly from Theorem 4.4 applied to  $G^c$ .

## 5. A bound on the chromatic number of a graph.

In this section we combine Theorem 2.1 of Brooks [5] and Corollary 4.2 of Lovász [11] to give an upper bound on the chromatic number of a graph  $G$ , in terms of  $\Delta(G)$  and  $\theta(G)$ .

Theorem 5.1 If  $G$  is a graph with no complete subgraphs on  $r$  vertices, where  $r \geq 4$ , then

$$\chi(G) \leq \Delta(G) + 1 - [(\Delta(G) + 1)/r].$$

Proof: To simplify notation, let

$$n = [(\Delta(G) + 1)/r].$$

If  $n = 0$ , then Theorem 5.1 follows. Thus, we can assume that  $n > 0$ .

By Corollary 4.2, there is a partition of  $V(G)$  into  $n$  sets  $X_1, X_2, \dots, X_n$ , such that if  $X_i$  is nonempty, then

$$\Delta(G[X_i]) \leq r - 1 \quad \text{for } i = 1, 2, \dots, n-1,$$

and such that if  $X_n$  is nonempty then

$$\Delta(G[X_n]) \leq \Delta(G) - r(n-1).$$

Since  $G$  contains no complete subgraphs on  $r$  vertices, neither do the subgraphs  $G[X_i]$ , for all  $i \leq n$ . Hence, by these inequalities and Brooks' Theorem,

$$\chi(G[X_i]) \leq r-1 \quad \text{for } i=1,2,\dots,n-1,$$

and

$$\chi(G[X_n]) \leq \Delta(G) - r(n-1).$$

The latter inequality follows because by definition of  $n$ ,

$$\Delta(G) - r(n-1) \geq r-1,$$

whence Brooks' Theorem may be applied to  $G[X_n]$ . Hence,

$$\begin{aligned} \chi(G) &\leq \sum_{i=1}^n \chi(G[X_i]) \\ &\leq (n-1)(r-1) + \Delta(G) - r(n-1) \\ &= \Delta(G) + 1 - n, \end{aligned}$$

and the theorem is proved.

We know of no examples with  $\chi(G) < \Delta(G)$  for which Theorem 5.1 holds with equality.

It has recently come to our attention that O. V. Borodin and A. V. Kostochka have independently obtained Theorem 5.1. Their result appears in a preprint titled "On an Upper Bound of the Graph's Chromatic Number Depending on Graph's Degree and Density."

## 6. The chromatic number, clique number and maximum degree of a graph.

In this section we obtain results concerning the structure of a graph  $G$  having the parameters

$$\Delta(G) = h, \quad \theta(G) = h - r, \quad \chi(G) = h - r + 1,$$

where  $h$  and  $r$  are integers. Our main concern is with  $h \geq 6$  and  $r = 1$ . The case  $r = 0$  is Brooks' Theorem (Theorem 2.1), when  $h \geq 3$ .

Theorem 6.1 Let  $r$  and  $h$  be integers, where  $0 \leq r < h$ . Let  $G$  be an edge-minimal graph satisfying

$$(6.1) \quad \Delta(G) \leq h, \quad \theta(G) \leq h - r, \quad \chi(G) \geq h - r + 1.$$

For each  $e \in E(G)$  there is a maximal stable set  $S_e$  such that either  $e$  lies in all cliques  $K_{h-r}$  of  $G - S_e$ , or  $e$  lies in an edge-minimal subgraph  $H$  of  $G - S_e$  satisfying

$$(6.2) \quad \Delta(H) \leq h - 1, \quad \theta(H) \leq h - r - 1, \quad \chi(H) = h - r.$$

Proof: Assume  $G$  to be an edge-minimal graph with

$$(6.3) \quad \Delta(G) \leq h,$$

$$(6.4) \quad \theta(G) \leq h - r,$$

$$(6.5) \quad \chi(G) \geq h - r + 1.$$

The edge-minimality of  $G$  implies that for any  $e \in E(G)$ ,

$$(6.6) \quad \chi(G - e) = h - r.$$

Hence, (6.5) becomes

$$(6.7) \quad \chi(G) = h - r + 1.$$

By (6.6) and (6.7), for any maximal stable set  $S \subseteq V(G)$ ,

$$(6.8) \quad \chi(G - S) = h - r.$$

By (6.6), for any  $e \in E(G)$ , there is a maximal stable set  $S_e$  such that  $S_e$  is monochromatic in an  $(h - r)$ -coloring of  $G - e$ . Therefore,

$$(6.9) \quad \chi(G - e - S_e) = h - r - 1,$$

and by (6.3) and the maximality of  $S_e$ ,

$$(6.10) \quad \Delta(G - S_e) \leq h - 1,$$

and by (6.8),

$$(6.11) \quad \chi(G - S_e) = h - r.$$

Since (6.11) precludes  $\theta(G - S_e) > h - r$ , either

$$(6.12) \quad \theta(G - S_e) = h - r,$$

or

$$(6.13) \quad \theta(G - S_e) < h - r.$$

If (6.12) holds, then (6.9) implies that  $e$  lies in all cliques  $K_{h-r}$  of  $G - S_e$ . If (6.13) holds, then by (6.10), (6.13), and (6.11),  $H = G - S_e$  satisfies the relations (6.2). Also, since the removal of  $e$  from  $G - S_e$  reduces the chromatic number of  $G - S_e$ , by (6.9),  $e$  is in an edge-minimal subgraph  $H$  of  $G - S_e$  that satisfies (6.2).

Lemma 6.2 Suppose that  $G$  is a connected graph with

$$(6.14) \quad \Delta(G) = h, \quad \theta(G) = h - r, \quad \chi(G) = h - r + 1,$$

such that every edge lies in a clique  $K_{h-r}$ . If

$$(6.15) \quad h \geq 3r + 3,$$

then every two cliques on  $h - r$  vertices intersect in at least  $h - 2r - 1$  vertices.

Proof: Suppose first that two cliques of  $G$  intersect at a vertex  $v$ . We claim that these two cliques must intersect in at least  $h - 2r - 1$  vertices. Note that  $v$  is adjacent to  $h - r - 1$  vertices in each clique. If these two cliques overlap at  $v$  and at most  $h - 2r - 3$  other vertices, then  $v$  is adjacent to at least

$$2(h - r - 1) - (h - 2r - 3) = h + 1$$

vertices of  $G$ , contrary to (6.14). This proves the claim.

Suppose that  $C_1$  and  $C_0$  are cliques on  $h - r$  vertices each, which do not overlap. Since  $G$  is connected, there is a minimum length path  $v_0, v_1, \dots, v_n$  in  $G$ , where  $\{v_0, v_1\} \in E(C_1)$  and  $\{v_{n-1}, v_n\} \in E(C_0)$ , and since  $C_1$  and  $C_0$  do not overlap,  $n \geq 3$ . We shall find a shorter path with these properties, contrary to the minimality of  $n$ .

For  $i = 1, 2, 3$ , denote by  $C_i$  the clique on  $h - r$  vertices containing the edge  $\{v_{i-1}, v_i\}$ . By hypothesis, and since  $n \geq 3$ , such cliques exist. By the claim,  $C_1$

and  $C_2$  overlap in at least  $h - 2r - 1$  vertices, as do  $C_3$  and  $C_2$ . Since  $|V(C_2)| = h - r$ , the number of vertices common to  $C_1$ ,  $C_2$ , and  $C_3$  is at least

$$\begin{aligned} |V(C_1 \cap C_2)| + |V(C_3 \cap C_2)| - |V(C_2)| \\ \geq 2(h - 2r - 1) - (h - r) \\ = h - 3r - 2 \\ \geq 1, \end{aligned}$$

by (6.15). Let  $v$  denote a vertex at which  $C_1$  and  $C_3$  overlap. The path  $v_0, v, v_3, \dots, v_n$  violates the minimality of  $n$ . This proves the lemma.

We do not assume Brooks' Theorem in the following:

Theorem 6.3 If  $h \geq 3$ , then there is no graph  $G$  with

$$(6.16) \quad \Delta(G) = h, \quad \theta(G) = h, \quad \chi(G) = h + 1,$$

in which each edge of  $G$  lies in a clique  $K_h$ .

Proof: Suppose that such a graph exists. Let  $C$  be a clique  $K_h$ . By Lemma 6.2, with  $r = 0$ , each vertex of  $G - C$  lies in a clique  $K_h$  that intersects  $C$  in at least  $h - 1$  vertices. Hence, each vertex of  $G - C$  is adjacent to at least  $h - 1$  vertices of  $C$ . If  $|V(G - C)| \geq 2$ , then there are at least  $2(h - 1)$  edges with exactly one end in  $C$ .

However, since each vertex of  $C$  has degree at most  $h$ , and is adjacent to  $h - 1$  vertices in  $C$ , each vertex of  $C$  is incident with at most one edge having just one

end in  $H$ . Thus, there are at most  $h$  edges with just one end in  $C$ . This contradiction shows that  $|V(G - C)| \leq 1$ . But since  $\theta(G) = h$ , this forces  $\chi(G) = h$ , and hence  $G$  does not exist.

Theorem 6.4 If  $h \geq 6$ , then there is no graph with

$$(6.17) \quad \Delta(G) = h, \quad \theta(G) = h - 1, \quad \chi(G) = h$$

in which each edge of  $G$  lies in a clique  $K_{h-1}$ .

Proof: Let  $C$  be a clique  $K_{h-1}$  of  $G$  chosen to have at least as many vertices of degree less than  $h$  as any other clique.

By Lemma 6.2, with  $r=1$ , each vertex of  $G - C$  lies in a clique that intersects  $C$  in at least  $h-3$  vertices. Hence, each vertex of  $G - C$  is adjacent to at least  $h-3$  vertices of  $C$ . Therefore, there are at least  $(h-3)|V(G - C)|$  edges with exactly one end in  $C$ .

Case I: Suppose that each vertex of  $C$  has degree  $h$ . By the choice of  $C$ , it follows that each vertex of  $G$  has degree  $h$ . Hence, each vertex of  $C$  is adjacent to 2 vertices outside of  $C$ , and so there are  $2|V(C)| = 2(h-1)$  edges with exactly one end in  $C$ . Thus,

$$2(h-1) \geq (h-3)|V(G - C)|,$$

whence,

$$|V(G - C)| \leq 2 \frac{h-1}{h-3} \leq \frac{10}{3},$$

since  $h \geq 6$ . If  $|V(G - C)| = 3$ , then since each vertex

of  $V(G - C)$  is adjacent to at most two vertices of  $V(G - C)$ , each is adjacent to at least  $h - 2$  vertices of  $C$ . This gives at least  $(h - 2)|V(G - C)|$  edges with exactly one end in  $C$ . Thus,

$$2(h - 1) \geq (h - 2)|V(G - C)|,$$

whence,

$$|V(G - C)| \leq 2 \frac{h - 1}{h - 2} \leq \frac{5}{2}.$$

Case II: Suppose that at least two vertices of  $C$  have degree less than  $h$ . Hence, the number of edges with exactly one end in  $C$  is at most  $2(h - 2)$ . Thus,

$$2(h - 2) \geq (h - 3)|V(G - C)|,$$

whence

$$|V(G - C)| \leq 2 \frac{h - 2}{h - 3} \leq \frac{8}{3}.$$

Case III: Suppose that exactly one vertex of  $C$  has degree less than  $h$ . Hence, the number of edges with exactly one end in  $C$  is at most  $2(h - 1) - 1 = 2(h - \frac{3}{2})$ .

Thus,

$$2(h - \frac{3}{2}) \geq (h - 3)|V(G - C)|,$$

whence,

$$|V(G - C)| \leq \frac{2h - 3}{h - 3} \leq 3,$$

with equality only if  $h = 6$  and each vertex of  $G - C$  is adjacent to exactly  $h - 3 = 3$  vertices of  $C$ . In this case, if  $v_1 \in V(G - C)$  is adjacent to  $h - 3 = 3$  vertices of  $C$ , then  $v_1$  is in the same clique  $K_5$  with another

vertex  $v_2 \in V(G - C)$ . By the choice of  $C$ , one of  $v_1, v_2$  has degree  $h = 6$  in  $G$ , for otherwise, we would be in Case II. This vertex is adjacent to at most two other vertices of  $V(G - C)$ , and hence to four vertices of  $C$ . But this contradicts the earlier remark that each vertex of  $G - C$  is adjacent to exactly three vertices in  $C$ .

Therefore, in any case,

$$|V(G - C)| \leq 2.$$

If  $|V(G - C)| \leq 1$ , then  $|V(G)| \leq h$ , and so  $\Delta(G) \leq h - 1$  and  $\theta(G) = h - 1$  imply  $\chi(G) = h - 1 < h$ . Thus, we may assume that  $|V(G - C)| = 2$  and  $|V(G)| = h + 1$ . Let  $S$  be a maximum stable set in  $V(G)$ . If  $|S| \geq 3$ , then  $\chi(G) < h$ . Since  $\theta(G) = h - 1$ ,  $|S| \geq 2$ . Suppose, therefore, that  $|S| = 2$ . Write  $S = \{s_1, s_2\}$ . If  $G - S$  is not a clique  $K_{h-1}$ , then  $\chi(G - S) \leq h - 2$ , whence  $\chi(G) < h$ . On the other hand, suppose that  $G - S$  is a clique  $K_{h-1}$ . Since  $\theta(G) = h - 1$ ,  $s_1$  is not adjacent to some vertex  $v_1 \in V(G - S)$ , and  $s_2$  is not adjacent to some point  $v_2 \in V(G - S)$ . Since  $S$  is a maximum stable set,  $v_1 \neq v_2$ . Thus, since

$$\chi(G - S - \{v_1, v_2\}) = |V(G - S - \{v_1, v_2\})| = h - 3,$$

and since  $\{s_1, v_1\}$  and  $\{s_2, v_2\}$  are stable sets,

$\chi(G) < h$ . Therefore,  $G$  does not exist, and the proof of Theorem 6.4 is complete.

Both Theorem 6.3 and 6.4 are best possible in a certain sense. If  $h=2$ , then Theorem 6.3 fails for an odd polygon of at least five vertices. Suppose that  $h=5$  in Theorem 6.4. We construct a counterexample  $G$  as follows. Let  $V(G)$  be a set of  $4n+2$  vertices,  $n \geq 2$ , and let  $\pi$  map them onto the vertices of a polygon  $G'$  on  $2n+1$  vertices so that exactly two vertices of  $V(G)$  are mapped to each vertex of  $G'$ . We define the edges of  $G$  to be the pairs  $v_1, v_2$  such that either  $\pi(v_1) = \pi(v_2)$  or  $\pi(v_1)$  and  $\pi(v_2)$  are adjacent in  $G'$ .

Theorem 6.5 Let  $r=0$  or  $1$ . If for some  $h \geq 3r+3$  there is a graph  $G$  with

$$(6.18) \quad \Delta(G) \leq h, \quad \theta(G) \leq h-r, \quad \chi(G) = h-r+1,$$

then there is a subgraph  $H$  of  $G$ , outside of a maximal stable set  $S$ , which is edge-minimal with respect to

$$(6.19) \quad \Delta(H) \leq h-1, \quad \theta(H) \leq h-r-1, \quad \chi(H) = h-r.$$

Proof: Without loss of generality, we may assume that  $G$  is edge-minimal with respect to (6.18). By Theorem 6.1, with  $r=0$  or  $1$ , each edge  $e$  of  $G$  either lies in a clique  $K_{h-r}$  of  $G - S_e$ , for some maximal stable set  $S_e \subseteq V(G)$ , or there is a subgraph  $H$  of  $G - S_e$  satisfying (6.19). By Theorems 6.3 and 6.4, it is not possible that each edge  $e \in E(G)$  lies in a clique

$K_{h-r}$ , for no such graph exists. Thus, there is an edge  $e$  contained in a subgraph  $H$  of  $G$  satisfying (6.19).

Corollary 6.6 If Brooks' Theorem holds for all graphs  $H$  with  $\Delta(H) = 3$ , then Brooks' Theorem holds for all graphs.

Proof: Brooks' Theorem (Theorem 2.1) for  $\Delta(H) = 3$  is a basis for induction. By Brooks' Theorem for  $\Delta(H) = h-1$ , there is no graph satisfying (6.19). Thus, by Theorem 6.5 with  $r=0$ , there is no graph  $G$  satisfying (6.18), and so Brooks' Theorem holds for  $\Delta(G) = h$ .

Corollary 6.7 If there is an integer  $n \geq 6$  such that there is no graph  $H$  satisfying

$$(6.20) \quad \Delta(H) = n, \quad \theta(H) = n-1, \quad \chi(H) = n,$$

then for all  $h \geq n$ , there is no graph  $G$  satisfying

$$(6.21) \quad \Delta(G) = h, \quad \theta(G) = h-1, \quad \chi(G) = h.$$

Proof: We use the nonexistence of a graph  $H$  satisfying (6.20) as a basis for induction. Suppose there is no graph  $H$  satisfying

$$\Delta(H) = h-1, \quad \theta(H) = h-2, \quad \chi(H) = h-1$$

where  $h \geq 7$ . By Theorem 6.5, with  $r=1$ , there is no graph  $G$  satisfying (6.21).

Benedict and Chinn [2] note that for  $n \leq 7$  there are graphs  $H$  satisfying (6.20). Thus, the induction suggested by Corollary 6.7 would have to start at  $n \geq 8$ , if at all.

We show that there are infinitely many graphs  $G$  satisfying

$$(6.22) \quad \Delta(G) = 6, \quad \theta(G) = 5, \quad \chi(G) = 6.$$

We define such graphs recursively. Let  $G'$  be the graph obtained from  $K_7$  by the removal of three edges that form a triangle in  $K_7$ . Let  $G_0$  be either  $K_6$  or a graph that satisfies (6.22). Given  $G_i$ , let  $G_{i+1}$  be obtained from  $G_i$  and  $G'$  by removing from  $G_i$  a vertex (but not its incident edges) and joining these incident edges to the three vertices of degree four in  $G'$  (called vertices of attachment), so that at most two edges from  $G_i$  are assigned to each of the three vertices of degree four in  $G'$ . Suppose, by way of contradiction, that  $\chi(G_{i+1}) = 5$ . Since 4 colors are assigned to the 4 vertices of degree 6 in  $G'$ , a fifth color must be assigned to each of the three vertices of attachment of  $G'$ . Hence, in a 5-coloring of  $G_{i+1}$ , the 7 vertices of  $G'$  behave like a single vertex of the fifth color. Therefore,  $\chi(G_{i+1}) = \chi(G_i) = 6$ , a contradiction. Since  $G_0$

satisfies  $\chi(G_0) = 6$ , we have  $\chi(G_{i+1}) = 6$ , by induction. It is clear that the other relations of (6.22) also hold for  $G_{i+1}$ .

We give seven nonisomorphic examples of connected graphs  $G$  with

$$\Delta(G) = 7, \quad \theta(G) = 6, \quad \chi(G) = 7.$$

Define the graph  $G'$  to be a clique  $K_8$  minus 3 edges which form a triangle in  $K_8$ . Thus,  $G'$  has 3 vertices of degree 5 and 5 vertices of degree 7. For any non-empty subset  $S$  of the set of vertices of a clique  $K_7$ , construct  $G$  by removing each vertex of  $S$  (but not the incident edges) and replacing it with a copy of  $G'$  so that the six edges incident with a removed vertex are instead made to be incident in pairs with the 3 vertices of degree 5 in the copy of  $G'$ . This gives a graph  $G$  having the desired parameters. The number of vertices of  $G$  is thus  $7(|S| + 1)$ . Benedict and Chinn obtained the graph with  $|S| = 1$  as an example  $G$  having these parameters, and noted that the method of construction does not generalize to  $n \geq 8$ .

Part III  
SUBGRAPHS

## 7. Subgraphs of graphs, II

In [6] we gave a sufficient condition for  $H$  to be a subgraph of  $G$  by showing that for any positive integer  $d$  there is a constant  $c_d < d$  such that  $\Delta(H) \leq d$  and  $\delta(G) \geq c_d p$  imply that  $H$  is a subgraph of  $G$ . We obtained this from a result on bipartite graphs that is analogous to Theorem 7.1, and is in a certain sense best possible. In a footnote in [6] we announced having improved  $c_d$  to the value given by Theorem 7.1 below. Like Theorem 7.1, our result in [6] on bipartite graphs may be obtained using a generalization of the concept of alternating paths, which is used extensively in studying matchings. In the special case when  $\Delta(H) = 1$ , the proof of Theorem 7.1 reduces to an argument involving an alternating path of length 4.

N. Sauer and J. Spencer [14] have independently obtained Theorem 7.1. This was announced in [13]. Erdős and Stone [9] gave a sufficient condition of a different nature for  $H$  to be a subgraph of  $G$ . Bollobás and Eldridge [3] and Sauer and Spencer [14] have considered

the problem of giving sufficient conditions, based on the number of edges of  $H$  and  $G^c$ , for  $H$  to be a subgraph of  $G$ .

After proving the main result of section 7, we give some special cases, and indicate what would be best possible. Our main result of this section is

Theorem 7.1 If  $G$  and  $H$  are graphs on  $p$  vertices such that

$$(7.1) \quad 2\Delta(G^c)\Delta(H) \leq p-1$$

then  $H$  is a subgraph of  $G$ .

Proof: Throughout the proof, the letter  $w$  will be used to denote vertices of  $H$  (i.e.,  $w' \in V(H)$ ), and the letters  $x$  and  $v$  will be used for vertices of  $G$ . Given a graph  $G$ , suppose that  $H$  is an edge-minimal graph that is not a subgraph of  $G$ , but suppose that  $H$  and  $G$  satisfy (7.1). By edge-minimality, we can pick any edge, say  $e \in V(H)$ , fixed throughout the proof, so that  $H - e$  is a subgraph of  $G$ . Let

$$\pi: V(H) \longrightarrow V(G)$$

be an embedding of  $H - e$  into  $G$ . Let  $w, w'$  be the ends of  $e$  in  $H$ . We shall alter  $\pi$  by transposing  $\pi(w)$  with another vertex of  $G$  so that the resulting embedding of  $H - e$  also maps  $e$  to an edge of  $G$ . This is a contradiction.

Define

$$M(v) = \{v'' \in V(G) : \{\pi^{-1}(v), \pi^{-1}(v'')\} \in E(H - e)\}.$$

A successor of  $v$  is any vertex  $v_1 \in V(G)$  such that for each  $v'' \in M(v)$ ,  $v_1$  is either equal or adjacent in  $G$  to  $v''$ . Denote by  $S(v)$  the set of all successors of  $v$ . We say that  $v$  is a predecessor of  $v_1$  if  $v_1$  is a successor of  $v$ . Denote by  $P(v_1)$  the set of all predecessors of  $v_1$ .

Let  $v = \pi(w)$ . Note that if  $v_1 \in S(v) \cap P(v)$  and if  $v_1 \neq v$ , then  $(v, v_1)\pi$  embeds  $H - e$  into  $G$ .

We estimate  $|S(v)|$  and  $|P(v)|$  by deriving upper bounds for  $|V(G) - S(v)|$  and  $|V(G) - P(v)|$ . A vertex  $x \in V(G)$  is outside  $S(v)$  if  $x$  is adjacent in  $G^C$  to a vertex  $x'$  of  $M(v)$ . For any given  $x' \in M(v)$ , there are  $\Delta(G^C)$  choices of  $x$  adjacent to  $x'$  in  $G^C$ . Since  $\deg_{H-e}(w) \leq \Delta(H) - 1$ , we must have  $|M(v)| \leq \Delta(H) - 1$  choices of  $x'$ . Hence, at most  $\Delta(G^C)(\Delta(H) - 1)$  vertices  $x$  are not in  $S(v)$ . If  $x \notin P(v)$ , then there is an  $x' \in M(x)$  such that  $x'$  is adjacent in  $G^C$  to  $v$ . There are at most  $\Delta(G^C)$  choices of  $x'$  adjacent to  $v$  in  $G^C$ , and one of them is  $\pi(w')$ , since  $\pi$  does not embed  $e$  into  $G$ . Each  $x'$  lies in at most  $\Delta(H)$  sets  $M(x)$ , where  $x \in V(G)$ , with strict inequality when  $\pi^{-1}(x') = w'$ , whence at most  $\Delta(G^C)\Delta(H) - 1$  vertices  $x$  of  $V(G)$  are not in  $P(v)$ .

Therefore,

$$\begin{aligned}
 |P(v) \cap S(v)| &\geq |V(G)| - |V(G) - P(v)| - |V(G) - S(v)| \\
 &\geq p - (\Delta(G^c) \Delta(H) - 1) \\
 &\quad - \Delta(G^c)(\Delta(H) - 1) \\
 &= p - 2 \Delta(H) \Delta(G^c) + \Delta(G^c) + 1 \\
 &\geq 2 + \Delta(G^c),
 \end{aligned}$$

by (7.1). At most  $1 + \Delta(G^c)$  vertices are not adjacent in  $G$  to  $\pi(w')$ . Therefore, there is a  $v_1 \in P(v) \cap S(v)$  that is adjacent to  $\pi(w')$  in  $G$ . Thus,  $(v, v_1)\pi$  is an embedding of  $H$  into  $G$ . This proves the theorem.

Conjecture If  $G$  and  $H$  are graphs on  $p$  vertices satisfying

$$(\Delta(H) + 1)(\Delta(G^c) + 1) \leq p + 1,$$

then  $H$  is a subgraph of  $G$ .

We give examples to show that the conjecture, if true, would be best possible. Let  $H$  be a graph on  $p$  vertices, and let  $d$  be an integer such that

$$p \equiv -2 \pmod{d+1}.$$

Then  $H$  is said to be in class  $C_1(d)$  if  $\Delta(H) = d$  and if  $H$  has  $\frac{p+2}{d+1} - 1$  components isomorphic to  $K_{d+1}$ ;  $H$  is in class  $C_2(d)$  if  $d$  is odd, if  $H$  has one component isomorphic to  $K_{d,d}$ , and if all  $\frac{p+2}{d+1} - 2$  other components are isomorphic to  $K_{d+1}$ . Thus, for any odd  $d$ , there is a unique graph in  $C_2(d)$ , and for  $d$  even,  $C_2(d)$  is empty. We also denote these classes by  $C_1$  and  $C_2$  (Figures 1,2,3).

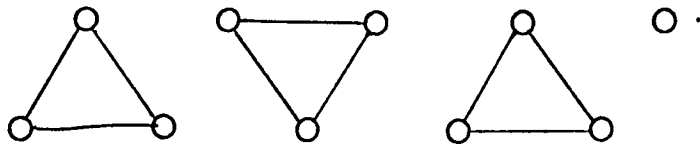


Figure 1: The graph in  $C_1(2)$  with  $p=10$ .

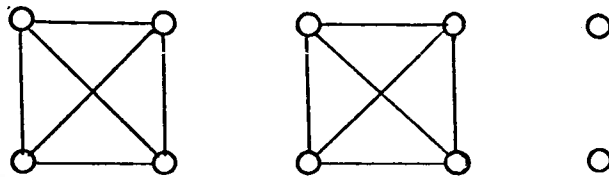


Figure 2: A graph in  $C_1(3)$  with  $p=10$ .

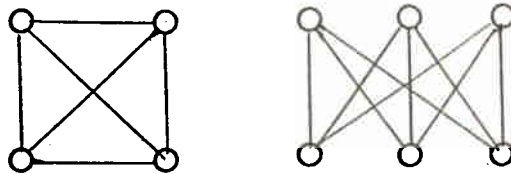


Figure 3: The graph in  $C_2(3)$  with  $p=10$ .

For any integers  $d, d'$  satisfying

$$(d+1)(d'+1) = p+2,$$

if  $H \in C_1(d) \cup C_2(d)$  and if  $G^c \in C_1(d') \cup C_2(d')$ , then it is easily verified that  $H$  is not a subgraph of  $G$ , unless  $H \in C_2(d)$  and  $G^c \in C_2(d')$ . Thus, the conjecture, if true, is best possible.

It is routine to show that if (7.1) of Theorem 7.1 is improved to

$$2\Delta(H) \Delta(G^c) \leq p,$$

then either  $H$  is a subgraph of  $G$ , or one of  $H$  or  $G^c$ , say  $H$ , is regular of degree 1, and the other graph lies in  $C_1(p/2)$  or  $C_2(p/2)$ . The proof, however, becomes much longer than the proof of Theorem 7.1.

We have recently been able to show that there is a function  $f(p)$ , on the order of  $p^{1/3}$ , such that if  $G$  and  $H$  are graphs on  $p$  vertices with  $\Delta(H) = 2$  and  $\Delta(G^c) \leq \frac{p}{3} - f(p)$ , then  $H$  is a subgraph of  $G$ . The coefficient  $\frac{1}{3}$  is best possible by above examples. A proof is in section 10. In the special case where  $H$  has  $\lfloor \frac{p}{3} \rfloor$  components isomorphic to  $K_3$  and  $G$  satisfies

$$\Delta(G^c) \leq \frac{p-1}{3},$$

either  $H$  is a subgraph of  $G$ , or equality holds and  $G^c \in C_1(\frac{p-1}{3}) \cup C_2(\frac{p-1}{3})$ . This characterizes the extremal graphs of a theorem of Corrádi and Hajnal [7]. We prove this in section 9.

There is a more general special case for which the conjecture has been proved. If

$$b = \frac{p+1}{\Delta(H)+1} - 1$$

is an integer, then this value of  $b$  in the following theorem of Hajnal and Szemerédi [10] gives the inequality of the conjecture. The case  $\Delta(H) = 2$  is the theorem of Corrádi and Hajnal.

Theorem 7.2 Let  $H$  be a graph with  $b$  components isomorphic to  $K_{\Delta(H)+1}$ , and with all other components isomorphic to  $K_{\Delta(H)}$ . If

$$\Delta(G^c) \leq \frac{p-b}{\Delta(H)} - 1,$$

then  $H$  is a subgraph of  $G$ .

Theorem 7.2 can be readily derived from the special case with  $b = \frac{p+1}{\Delta(H)+1} - 1$ . In this case  $H \in C_1$ . Hajnal and Szemerédi gave  $G^c \in C_1$  to show that their result is best possible. The example  $G^c \in C_2$  is new.

Given a graph  $H$ , the conjecture is not necessarily best possible for that particular graph. For instance, consider the following theorem of Bondy [4]:

Theorem 7.3 If  $H$  is a graph consisting of one polygonal component of girth  $g \leq p$  and of  $p-g$  components  $K_1$ , and if  $G$  satisfies  $\Delta(G^c) \leq \frac{1}{2}p - 1$  and also has  $p$  vertices, then either  $H$  is a subgraph of  $G$ , or  $g$  is odd,  $p$  is even and  $G$  is isomorphic to  $K_{\frac{1}{2}p, \frac{1}{2}p}$ .

Let  $n = \lfloor \frac{p}{2} \rfloor$ . If  $G^c$  contains a component isomorphic to  $K_{n+1}$  and  $g = p$ , or if  $G^c$  has a component isomorphic to  $K_{n,n}$  and  $g > p - n$ , then  $H$ , of Theorem 7.3, is not a subgraph of  $G$ . These examples are similar to the classes  $C_1$  and  $C_2$  which make Theorem 7.1 best possible for  $\Delta(H) = 1$ , and which make the conjecture best possible.

Next, we give a general construction of examples of graphs  $G$  such that a given graph  $H$  is not a subgraph of  $G$ . This construction generalizes class  $C_1$  given above.

Let  $r(H)$  denote the minimum number of vertices whose removal from  $H$  is necessary to lower the chromatic number  $\chi(H)$ . Clearly,

$$r(H) \leq \frac{p}{\chi(H)}.$$

Theorem 7.4 Let  $H$  be a graph on  $p$  vertices. There is a graph  $G$  on  $p$  vertices with

$$\Delta(G^c) = \left\lfloor \frac{p - r(H)}{\chi(H) - 1} \right\rfloor$$

such that  $H$  is not a subgraph of  $G$ .

Proof: Let  $n = \chi(H) - 1$ . Partition the  $p$  vertices of a set  $X$  into sets  $X_0, X_1, \dots, X_n$ , where

$$|X_0| = r(H) - 1,$$

where  $X_0$  is empty if  $r(H) = 1$ , where

$$|X_1| \leq |X_2| \leq \dots \leq |X_n|,$$

and  $|X_n| - |X_1| \leq 1$ . Let a graph  $G$  be defined on  $X$  so that  $G$  is the complete  $(n+1)$ -partite graph with  $(n+1)$ -partition  $X_0, X_1, \dots, X_n$ . Since

$$|X_0| = r(H) - 1 \leq \frac{p}{\chi(H)} - 1,$$

and since  $\frac{p}{\chi(H)}$  is the average of  $|X_i|$ ,  $i=0,1,\dots,n$ , we must have  $|X_n| > |X_0|$ . Thus,  $G^c[X_n]$  is a clique, and letting braces denote the least integer function, we have

$$\begin{aligned} \Delta(G^c) &= |X_n| - 1 \\ &= \{(p - |X_0|)/n\} - 1 \\ &= [(p - |X_0| + n - 1)/n] - 1 \\ &= [(p - |X_0| - 1)/n] \\ &= [(p - r(H))/(\chi(H) - 1)], \end{aligned}$$

which is the condition of the theorem.

Suppose that  $\pi$  embeds  $H$  into  $G$ . Let  $H'$  denote the subgraph of  $H$  induced by the preimage of  $V(G) - X_0$ .

Since  $H'$  contains  $|X_0| = r(H) - 1$  fewer vertices than  $H$ , by the definition of  $r$ ,

$$\chi(H') = \chi(H).$$

But for each  $i=1,2,\dots,n$ , no two vertices of  $X_i$  are adjacent in  $G$ , and so the embedding

$$\pi|_{H'}: V(H') \longrightarrow V(G) - X_0$$

is a  $(\chi(H) - 1)$ -coloring of  $H'$ , a contradiction. Hence,  $\pi$  does not exist, and the theorem is proved.

A  $B_h$ -component is defined in section 2.

**Corollary 7.5** Let  $H$  be a graph on  $p$  vertices of maximum degree  $\Delta(H)$  with  $b > 0$   $B_h$ -components. Then there is a graph  $G$  with  $\Delta(G^c) = [(p - b)/\Delta(H)]$ , such that  $H$  is a subgraph of  $G$ .

Proof: By Theorem 2.1, due to Brooks, we have  
 $\chi(H) = \Delta(H) + 1$ ,  $r(H) = b$  in Theorem 7.4.

Note that Theorem 7.4 and Corollary 7.5 contain, as special cases, some of the aforementioned examples showing that certain results are best possible.

Consider Bondy's Theorem. If  $p$  and  $g$  are odd, then  $b=1$  and Corollary 7.5 shows Theorem 7.3 to be best possible. If  $g$  is even and equal to  $p$ , then  $r(H) = \frac{1}{2}p$ , and Theorem 7.4 shows that  $\Delta(G^c) \leq \frac{1}{2}p$  is not sufficient to ensure that  $H$  is a subgraph of  $G$ .

In the case of Hajnal and Szemerédi's Theorem, Corollary 7.5 ensures that it is best possible for any value of  $\Delta(H)$  and any  $b \geq 1$ .

Let  $H$  be a graph satisfying the conditions of Corollary 7.5. We know of no graph on  $p$  vertices with

$$\Delta(G^c) \leq \frac{p-b}{\Delta(H)} - 1,$$

even when  $b=0$ , such that  $H$  is not a subgraph of  $G$ .

It would be interesting to know whether such graphs exist.

## 8. Subgraphs of maximum degree 2: a short proof.

We give in this section an improvement of Theorem 7.1 for the case  $\Delta(H) = 2$  that has a short proof, but is not best possible. In section 10, we prove a stronger result, which is best possible in a certain sense.

As in section 7, the graph  $H$  will be embedded in  $G$ , and the letters  $y$  and  $w$  will be used to denote the vertices of  $H$ , while  $x$  and  $v$  will be used for vertices of  $G$ . In this section, let

$$M(v) = \{v'' \in V(G) : \{\pi^{-1}(v), \pi^{-1}(v'')\} \in E(H)\},$$

where

$$\pi: V(H) \longrightarrow V(G)$$

is a fixed bijection.

In this section we used a slightly different definition of successors and predecessors. A vertex  $x' \in V(G)$  is a successor of  $x \in V(G)$  if  $x'$  is adjacent in  $G$  to every vertex of  $M(x)$ . (In section 7, we permitted  $x'$  to be a successor of  $x$  if  $x'$  were adjacent or equal to every vertex of  $M(x)$ .) The vertex  $x$  is a predecessor of  $x'$  if  $x'$  is a successor of  $x$ .

Theorem 8.1 Let  $G$  and  $H$  be graphs on  $p$  vertices, with  $\Delta(H) = 2$ . If

$$(8.1) \quad \Delta(G^c) \leq \frac{2p-11}{7},$$

then  $H$  is a subgraph of  $G$ .

Proof: Let  $H$  be an edge-minimal graph for which the theorem is false. Then there is a graph  $G$  satisfying (8.1) such that  $H$  is not a subgraph of  $G$ , but such that for any edge  $e \in E(H)$ ,  $H - e$  is a subgraph of  $G$ .

First, we show that if any vertex  $y_1 \in V(H)$  has degree 1 in  $H$ , then we are done. Let  $e = \{y_0, y_1\}$  be the edge incident with  $y_1$ , and let

$$\pi: V(H) \longrightarrow V(G)$$

be an embedding of  $H - e$  into  $G$ . Since

$$\Delta(G^c) < \frac{2p}{7},$$

$\pi(y_1)$  has at least  $\frac{3p}{7}$  predecessors in  $G$ , and (8.1) guarantees that among them lie successors of  $\pi(y_1)$  (i.e., vertices of  $G$  adjacent to  $\pi(y_0)$ ). Let  $x$  be such a vertex. Then  $(x \pi(y_1))\pi$  is an embedding of  $H$  into  $G$ .

Therefore, assume that each vertex of  $H$  has degree either 0 or 2. Thus, all components of  $H$  are either isolated vertices or polygons.

Let the edges of polygons of  $H$  be directed so that each vertex has one incoming edge and one outgoing edge.

Given a vertex  $y_0$  in  $H$ , we shall denote by  $y_1, y_2, y_3$  the next three successive vertices on the directed path in  $H$  starting at  $y_0$ . On a triangle,  $y_0 = y_3$ .

For the mapping

$$\pi: V(H) \longrightarrow V(G),$$

we shall simplify notation by writing  $\pi(y_i) = x_i$  for  $i = 0, 1, 2, 3$ .

By the minimality of  $H$ , we may assume that  $\pi$  embeds all but one edge, say  $e$ , of  $H$  into  $G$ . Denote the tail of  $e$  by  $w_0$ . Following the previous convention, the head of  $e$  is  $w_1$ , and the next two successive vertices after  $w_1$  are  $w_2$  and  $w_3$ . To simplify notation, we write  $\pi(w_i) = v_i$ ,  $i = 0, 1, 2, 3$ .

Since  $e$  is the only unembedded edge of  $H$ ,  $x_0, x_1, x_2, x_3$  and  $v_1, v_2, v_3$  are paths in  $G$ , and  $v_0$  and  $v_1$  are not adjacent in  $G$ .

Throughout the proof, we consider  $e, w_0, \pi$ , and  $v_0$  to be fixed. We shall choose  $y_0$  so that the paths  $y_0, y_1, y_2, y_3$  and  $w_0, w_1, w_2, w_3$  have no edge in common. Hence,  $w_3 \neq y_1, y_2$ , or  $y_3$ , and  $w_0 \neq y_1$  or  $y_2$  (the case  $w_0 = y_0$  is excluded by  $w_3 \neq y_3$ ). For any other choice of  $y_0 \in V(H)$ , the two paths have no common edge. Thus, we have  $p - 5$  choices for  $y_0$ . Each choice determines  $x_0, x_1, x_2$ , and  $x_3$  since  $\pi$  is fixed.

For any choice of  $y_0$ , if  $(v_1 x_1)\pi$  or  $(v_1 x_1)(v_2 x_2)\pi$  is an embedding, then  $H$  is a subgraph of  $G$ , and we are done. Otherwise, if neither is an embedding, then  $E(G^c)$  includes some of the following six edges:

$$(8.2) \{v_0, x_1\}, \{v_1, x_0\}, \{v_1, x_2\}, \{v_2, x_1\}, \{v_2, x_3\}, \{v_3, x_2\}.$$

We shall estimate the number of values of  $y_0$  for which  $(v_2 x_2)\pi$  embeds  $H - e$  into  $G$ . Observe that for a given  $y_0$  (which determines  $x_0, x_1, x_2, x_3$ ),

(8.3) If exactly one of the six edges (8.2) is in  $E(G^c)$ , then it must be  $\{v_0, x_1\}$  or  $\{v_1, x_0\}$  (otherwise,  $(v_1 x_1)\pi$  or  $(v_1 x_1)(v_2 x_2)\pi$  embeds  $H$  into  $G$ ), and so  $(v_2 x_2)\pi$  embeds  $H - e$  into  $G$ .

Let  $n_1$  be the number of values of  $y_0$  such that (8.3) holds. This leaves  $p - 5 - n_1$  choices of  $y_0$  such that at least 2 of the 6 edges (8.2) lie in  $E(G^c)$ . We count occurrences of edges in  $E(G^c)$  among the 6 edges of (8.2) in two ways, as  $y_0$  runs over  $p - 5$  vertices in  $H$ . It is clear that their number is at least

$$1(n_1) + 2(p - 5 - n_1) = 2p - 10 - n_1.$$

Also, each of at most  $\Delta(G^c)$  edges incident with  $v_1$  is counted once among the 6 edges (8.2), if  $i = 0$  or  $3$ , and each is counted twice if  $i = 1$  or  $2$ . Hence, the number of edge-occurrences is at most  $6\Delta(G^c)$ .

Counting two ways, we get

$$\begin{aligned} 2p - 10 - n_1 &\leq \text{number of edge occurrences} \\ &\leq 6\Delta(G^c). \end{aligned}$$

Hence,

$$n_1 \geq 2p - 10 - 6\Delta(G^c),$$

and so by (8.1),

$$n_1 \geq \Delta(G^c) + 1.$$

Thus, there are at least  $\Delta(G^c) + 1$  values of  $y_0$ , and hence  $\Delta(G^c) + 1$  values of  $x_2$ , such that  $(v_2 x_2)\pi$  embeds  $H - e$  into  $G$ .

The number of vertices  $x$  such that both vertices of  $H(x)$  are adjacent in  $G$  to  $v_1$  (i.e., the number of predecessors of  $v_1$ ) is at least

$$p - 2\Delta(G^c) \geq \frac{3p+22}{7}.$$

At most  $(2p-11)/7$  of these are not adjacent to  $v_0$ , and so the number of predecessors of  $v_1$  that are adjacent to  $v_0$  in  $G$  is at least  $\frac{p+11}{7}$ . Let  $x$  be any one of these. Among the  $\Delta(G^c) + 1$  values of  $x_2$  such that  $(v_2 x_2)\pi$  embeds  $H - e$  into  $G$ , choose  $x_2$  to be adjacent to  $x$ . Then  $(v_1 x)(v_2 x_2)\pi$  embeds  $H$  into  $G$ . But this contradicts the assumption that  $H$  is not a subgraph of  $G$ . The proof is complete.

## 9. Subgraphs with triangular components

In section 7 we gave two classes of graphs, denoted  $C_1(d)$  and  $C_2(d)$ , such that if

$$(d+1)(d'+1) = p+2,$$

if  $H \in C_1(d') \cup C_2(d')$ , and if  $G^c \in C_1(d) \cup C_2(d)$ , then either  $H$  is not a subgraph of  $G$  or both  $H \in C_2(d')$  and  $G^c \in C_2(d)$ . We conjectured that if

$$(\Delta(G^c) + 1)(\Delta(H) + 1) \leq p+1,$$

then  $H$  is a subgraph of  $G$ . Thus, these two classes  $C_1$  and  $C_2$  make the conjecture best possible.

To simplify notation in this section, we shall say that  $G$  is of type 1 or type 2 if  $p = 3b+1$ ,  $b > 0$ , and either  $G^c \in C_1(b)$  or  $G^c \in C_2(b)$ , respectively. Thus, when  $G$  is of type 1, there is a set  $S$  of  $b-1$  vertices such that  $G-S$  is isomorphic to  $K_{b+1, b+1}$ . Also, when  $G$  is of type 2, there is a stable set  $S$  of  $b+1$  vertices such that  $G-S$  has 2 components, both isomorphic to  $K_b$ , and  $b$  is odd.

Suppose  $H \in C_1(2)$ . If  $G$  is of type 1 or type 2, then clearly  $H$  is not a subgraph of  $G$ . We shall show that if  $H \in C_1(2)$ , then graphs  $G$  of types 1 and 2 are

the only graphs with  $\delta(G) \geq \frac{p-1}{3}$  such that  $H$  is not a subgraph of  $G$ .

Theorem 9.1 Let  $G$  and  $H$  be graphs on  $p$  vertices, and suppose that every component of  $H$  is isomorphic to either  $K_1, K_2$ , or  $K_3$ . Let  $b = b(H)$  denote the number of triangular components of  $H$ , and suppose  $b \geq 0$ . If

$$\delta(G) \geq \left\lceil \frac{p+b}{2} \right\rceil,$$

and if  $H$  is not a subgraph of  $G$ , then either

(9.1) There is a set  $S$  of  $b-1$  vertices of  $G$  such that  $G-S$  is a complete bipartite graph; or

(9.2) There is a set  $S$  of  $b+1$  vertices,  $b$  odd,

~~such that  $G-S$  has two components, both isomorphic to  $K_b$ , and  $H$  has  $\frac{p-1}{3}$  triangles.~~

Theorem 9.2 Let  $G$  and  $H$  be graphs on  $p$  vertices and suppose that every component of  $H$  is a triangle  $K_3$ , except for one vertex  $K_1$  if  $p = 3b+1$ , or one edge  $K_2$  if  $p = 3b+3$ . If

$$\delta(G) \geq \frac{2}{3}(p-1),$$

then  $H$  is not a subgraph of  $G$  if and only if both

$$\delta(G) = \frac{2}{3}(p-1) = 2b$$

and  $G$  is of type 1 or type 2.

If  $H$  is the graph of Figure 1, then Figures 2 and 3, respectively, are the complements of corresponding graphs of types 1 and 2 such that  $H$  is not a subgraph.

Lemma 9.3 Let  $G$  be a graph with  $p = 3b + 1$  vertices, for some integer  $b$ , and with  $\delta(G) \geq 2b$ . If for some set  $S \subseteq V(G)$ , with  $|S| = b - 1$ ,  $G - S$  is bipartite, with bipartition  $V_1 \cup V_2$ , then the following conclusions hold:

Every vertex of  $S$  is adjacent to every vertex of  $G - S$ ;

$$|V_1| = |V_2|;$$

$G - S$  is a complete bipartite graph.

Thus,  $G$  is of type 1.

Proof: Without loss of generality, assume that

$|V_1| \geq |V_2|$ . We have

$$|V_1| \geq \frac{1}{2}(p - |S|) = \frac{1}{2}(3b + 1 - (b - 1)) = b + 1.$$

Let  $v_1 \in V_1$ . Since  $V_1 \cup V_2$  is a bipartition of  $G - S$ ,  $v_1$  is adjacent in  $G^c$  to every vertex of  $V_1 - v_1$ . But

$$\Delta(G^c) = p - \delta(G) - 1 \leq b,$$

and hence we must have  $|V_1| = b + 1$  and  $\delta(G) = 2b$ . Also, each  $v_1 \in V_1$  must be adjacent to every vertex of  $G - V_1$  (i.e., to every vertex of  $V_2$  and every vertex of  $S$ ).

The conclusions of the lemma follow directly.

Remarks: If  $G$  is of type 2, then  $p \equiv 4 \pmod{6}$ , and  $G$  is regular of degree  $2b = \frac{2}{3}(p - 1)$ . Note that the only graph that is both of type 1 and type 2 is the quadrilateral.

Lemma 9.4 Let  $G$  be a graph with  $p = 3b + 1$  vertices, for some integer  $b$ , and with  $\delta(G) \geq 2b$ . If for some set  $S \subseteq V(G)$ , with  $|S| = b + 1$ ,  $G - S$  has two components, then the following conditions hold:

Every vertex of  $S$  is adjacent to every vertex of  $G - S$ ;

$G - S$  has two components, both isomorphic to  $K_b$ .

If, furthermore,  $b$  pairwise disjoint triangles do not embed in  $G$ , then

$p \equiv 4 \pmod{6}$ ;

$S$  is a stable set;

$G$  is of type 1 only if  $G$  is a quadrilateral.

Thus,  $G$  is of type 2.

Proof: Let  $G$  and  $S$  satisfy the hypotheses. Since  $p = 3b + 1$  and  $\delta(G) \geq 2b$ , any vertex is adjacent in  $G^c$  to at most  $b$  vertices of  $G$ . Thus, since  $|V(G - S)| = 2b$  and since  $G - S$  has two components, any vertex in the smaller component is adjacent in  $G^c$  to at least  $\frac{1}{2}|V(G - S)| = b$  vertices in the larger component of  $G - S$ . But these statements force equality: both components have just  $b$  vertices. Also, the first two conclusions of the lemma follow immediately.

If  $S$  is not a stable set or if  $p \not\equiv 4 \pmod{6}$ , then either  $G[S]$  has an edge, or, since  $p = 3b + 1$ ,  $p \equiv 1 \pmod{6}$ . In either case, an embedding of  $b$  pairwise disjoint triangles is easily found. The rest is easy.

Proof of Theorem 9.1 from Theorem 9.2: Assume without loss of generality that the components of  $H$  consist of  $b$  triangles  $K_3$ ,  $\lfloor \frac{p-3b}{2} \rfloor$  edges  $K_2$ , and  $p-3b-2\lfloor \frac{p-3b}{2} \rfloor$  vertices  $K_1$ . By adding  $\lfloor \frac{p-3b}{2} \rfloor$  vertices to  $H$ , each adjacent to both ends of a  $K_2$ , we can construct a graph  $H'$  on  $p + \lfloor \frac{p-3b}{2} \rfloor$  vertices, where the components of  $H'$  consist of  $b + \lfloor \frac{p-3b}{2} \rfloor$  triangles  $K_3$  and  $p-3b-2\lfloor \frac{p-3b}{2} \rfloor$  ( $= 0$  or  $1$ ) vertices  $K_1$ . By adding a stable set of  $\lfloor \frac{p-3b}{2} \rfloor$  vertices to  $G$ , we construct a graph  $G'$  in which each added vertex is adjacent to every vertex of  $G$ . Thus,

$$|V(G')| = p + \lfloor \frac{p-3b}{2} \rfloor = \lfloor 3 \frac{p-b}{2} \rfloor,$$

and

$$\begin{aligned} \delta(G') &\geq \min(p, \delta(G) + \lfloor \frac{p-3b}{2} \rfloor) \\ &\geq \min(p, \lfloor \frac{p+b}{2} \rfloor + \lfloor \frac{p-3b}{2} \rfloor) \\ &= \min(p, 2 \lfloor \frac{p-b}{2} \rfloor) \\ &= 2 \lfloor \frac{p-b}{2} \rfloor \\ &= \frac{2}{3} \lfloor 3 \lfloor \frac{p-b}{2} \rfloor \rfloor \\ &\geq \frac{2}{3} (|V(G')| - 1). \end{aligned}$$

Thus, by Theorem 9.2, either  $H'$  is a subgraph of  $G'$ , or  $G'$  is a graph of type 1 or type 2. Suppose  $G'$  is a graph of type 2. If  $G' \neq G$ , then  $G'$  has a vertex of degree  $p$  and  $|V(G')| < \frac{3}{2}p$ . Hence,  $G'$  is not a graph

of type 2 unless  $G' = G$ . Then by Lemma 9.4,

$$|V(G')| \equiv 4 \pmod{6}.$$

In this case  $\lfloor \frac{p-3b}{2} \rfloor = 0$  vertices were added to  $G$  to get  $G'$ , whence  $p-3b = 1$ , and  $H$  has  $b = \frac{1}{3}(p-1)$  triangles, and we have the second case of Theorem 9.1.

Suppose  $G'$  is a graph of type 1. Then  $H'$  has

$$b' = b + \lfloor \frac{p-3b}{2} \rfloor = \lfloor \frac{p-b}{2} \rfloor$$

triangles. Moreover,  $|V(G')| = 3b' + 1$ , and there is a set  $S' \subseteq V(G')$ , with  $|S'| = b' - 1$ , whose removal leaves a complete bipartite graph  $G' - S' = K_{b'+1, b'+1}$ . We have

$$\delta(G') \geq 2 \lfloor \frac{p-b}{2} \rfloor = 2b'.$$

We claim that  $V(G') = V(G) \cup S'$ . To prove this, suppose that  $V(G) \cup S'$  does not contain a vertex  $v \in V(G') - V(G)$ . However,  $V(G') - V(G)$  has only  $\lfloor \frac{p-3b}{2} \rfloor$  vertices, and so some vertex  $w$  of  $G$  lies on the same side of the bipartition as  $v$ . But  $v$  is adjacent to all vertices of  $G$ , and in particular to  $w$ , and we have a contradiction, which proves the claim.

Let  $S = V(G) \cap S'$ . Then, by the claim,

$$\begin{aligned} |S| &= |S'| - (|V(G')| - |V(G)|) \\ &= (b + \lfloor \frac{p-3b}{2} \rfloor - 1) - \lfloor \frac{p-3b}{2} \rfloor \\ &= b - 1, \end{aligned}$$

and  $G - S$  is bipartite. This is a conclusion of 9.1.

The remaining possibility is that  $H'$  is a subgraph of  $G'$ . There is an embedding of  $H'$  into  $G'$  which extends an embedding of  $H$  into  $G$ . This proves Theorem 9.1.

Lemma 9.5 Let  $G$  be a graph, and  $X_1 \cup X_2$  be a partition of  $V(G)$  of the type described in Theorem 4.5, for which

$$(9.3) \quad \delta(G_1) + \delta(G_2) = \delta(G).$$

Suppose that sets  $Y_3 \subseteq X_1$ ,  $V_3 \subseteq X_2$  exist such that

$$(9.4) \quad G_2 - V_3 \text{ is a complete bipartite graph with nontrivial bipartition } V_1 \cup V_2;$$

$$(9.5) \quad G_1 - Y_3 \text{ is a complete bipartite graph with nontrivial bipartition } Y_1 \cup Y_2;$$

$$(9.6) \quad \text{If } v \in V_1 \cup V_2 \text{ then } \deg_{G_2}(v) = \delta(G_2);$$

$$(9.7) \quad \text{If } y \in Y_1 \cup Y_2 \text{ then } \deg_{G_1}(y) = \delta(G_1).$$

Then any vertex of  $Y_1 \cup Y_2$  is adjacent to every vertex in  $V_j$ , for some  $j \in \{1, 2\}$ . Suppose further that

$$(9.8) \quad \text{No vertex of } Y_1 \cup Y_2 \text{ is adjacent to vertices in both } V_1 \text{ and } V_2.$$

Then  $G - (Y_3 \cup V_3)$  is a complete bipartite graph.

Proof: By (9.3), (9.6), and (9.7), the latter part of Theorem 4.5 may be applied to the vertices of  $V_1 \cup V_2 \cup Y_1 \cup Y_2$ .

Suppose that the first conclusion of the lemma is false for some  $y \in Y_1 \cup Y_2$ . Thus,  $y$  is not adjacent in  $G$  to a vertex  $v_1$  of  $V_1$  and a vertex  $v_2$  of  $V_2$ . By Theorem 4.5,  $v_1$  and  $v_2$  are interchangeable with  $y$ , and are thus not adjacent in  $G$ . But, by (9.4),  $v_1$  is adjacent to  $v_2$ . This contradiction proves the first part of the lemma.

By (9.8), any vertex  $y \in Y_1 \cup Y_2$  is adjacent in  $G^c$  to every vertex of  $V_j$  for  $j \in 1, 2$ . By the first part of the lemma, which was just proved,  $y$  is adjacent to every vertex of  $V_{3-j}$ .

Thus, the vertices of  $Y_1 \cup Y_2$  fall into two classes: those, the set of which we denote  $Y_4$ , which are adjacent to vertices of  $V_1$  but not  $V_2$ ; and those the set of which we denote  $Y_5$ , which are adjacent to vertices of  $V_2$  but not  $V_1$ .

We claim that  $\{Y_4, Y_5\} = \{Y_1, Y_2\}$ . To see this, suppose that  $Y_4 \cap Y_1$  and  $Y_4 \cap Y_2$  are both nonempty. Then any vertex  $v_2 \in V_2$  is not adjacent to a vertex  $y_1 \in Y_4 \cap Y_1$ , nor to a vertex  $y_2 \in Y_4 \cap Y_2$ . By Theorem 4.5,  $y_1$  and  $y_2$  are interchangeable with  $v_2$  and are thus not adjacent. However, (9.5) implies that  $y_1$  and  $y_2$  are adjacent.

This contradiction shows that either  $Y_4 \cap Y_1$  or  $Y_4 \cap Y_2$  is empty. Similarly, either  $Y_5 \cap Y_1$  or  $Y_5 \cap Y_2$  is empty. Since  $V_1 \cap V_2$  and  $Y_1 \cap Y_2$  are nontrivial, and

$$Y_4 \cap Y_5 = Y_1 \cap Y_2,$$

the claim must follow.

In either case of this claim, there is a  $j \in \{1, 2\}$  such that  $(V_1 \cap Y_j) \cap (V_2 \cap Y_{3-j})$  is a bipartition of  $G - (Y_3 \cap V_3)$ , and this bipartite graph is complete. This proves Lemma 9.5.

We define for  $X_j' \subseteq V(G)$

$$G_j' = G[X_j'] \quad j = 1, 2,$$

and

$$p_j' = |X_j'| \quad j = 1, 2.$$

A vertex  $x$  of  $G$ ,  $G_j$  or  $G_j'$  is critical in  $G$ ,  $G_j$ ,  $G_j'$ , if

$$\deg_G(x) - 1 < \frac{1}{3}(p - 1),$$

$$\deg_{G_j}(x) - 1 < \frac{1}{3}(p_j - 1),$$

or

$$\deg_{G_j'}(x) - 1 < \frac{1}{3}(p_j' - 1),$$

respectively.

Lemma 9.6 Suppose Theorem 9.2 is valid for all graphs with less than  $p$  vertices. Suppose

$$p \equiv 1 \pmod{3}$$

and that  $X_1 \cup X_2$  is a partition of  $V(G)$  which satisfies the conditions of Theorem 4.5 with  $c = \frac{2}{3}$ . For  $\{z, z'\} \subseteq V(G)$ , write

$$X'_j = X_j - \{z, z'\} \quad \text{for } j = 1, 2,$$

and assume that

$$\begin{aligned} p'_j &\equiv 1 \pmod{3} && \text{for } j = 1, 2, \\ (9.9) \quad \delta(G'_j) &\geq \frac{2}{3}(p'_j - 1) && \text{for } j = 1, 2, \end{aligned}$$

and that  $p_j \equiv 0 \pmod{3}$  for  $j \in \{1, 2\}$  implies that  $z \in X_j$  and that there exist critical vertices  $x_3, x_4 \in X_{3-j}$  such that  $G[z, x_3, x_4]$  is a triangle. Then, if  $\frac{1}{3}(p'_j - 1)$  pairwise disjoint triangles cannot be embedded in  $G'_j$ , for  $j = 1$  and  $j = 2$ , both  $G'_1$  and  $G'_2$  are of type 1.

Proof: Since Theorem 9.2 holds for graphs on fewer than  $p$  vertices, since (9.9) holds, and since  $\frac{1}{3}(p'_j - 1)$  triangles cannot be embedded in  $G'_j$ ,  $j = 1, 2$ , it follows that  $G'_1$  is of type 1 or type 2, and  $G'_2$  is of type 1 or type 2. Thus,

$$\delta(G'_j) = \frac{2}{3}(p'_j - 1) \quad j = 1, 2,$$

whence

$$\begin{aligned} (9.10) \quad \delta(G'_1) + \delta(G'_2) &= \frac{2}{3}(p'_1 + p'_2 - 2) \\ &= \frac{2}{3}(p - 1) - 2. \end{aligned}$$

Moreover, by Theorem 4.5,

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1) - \frac{2}{3}.$$

The left side is an integer and  $p \equiv 1 \pmod{3}$ , whence

$$\delta(G_1) + \delta(G_2) \geq \frac{2}{3}(p-1).$$

So that (9.10) also holds, it follows that if  $z$  or  $z'$ , respectively, is in  $X_j$ , for  $j \in \{1, 2\}$ , then  $z$  or  $z'$  is adjacent in  $G$  to every critical vertex of  $G_j'$ . In fact

$$(9.11) \quad \delta(G_1) + \delta(G_2) = \frac{2}{3}(p-1),$$

and vertices critical in  $G_j'$  are critical in  $G_j$ , for  $j=1, 2$ . Also, since critical vertices of  $G_1$  and critical vertices of  $G_2$  are interchangeable if they are adjacent in  $G^c$ , critical vertices of  $G_1'$  and critical vertices of  $G_2'$  are also interchangeable if they are adjacent in  $G^c$ . By (4.19) of Theorem 4.5, such vertices are also critical in  $G$ .

Let  $y_1, y_2$  be any pair of adjacent critical vertices of  $G_1'$ . If  $G_1'$  is of type 2, then every vertex of  $G_1'$  is critical in  $G_1'$ , whence, any adjacent pair suffices.

If  $G_1'$  is of type 1, then  $p_1' \geq 4$ , and there is a set

$Y_3 \subseteq X_1'$  with

$$|Y_3| = \frac{1}{3}(p_1' - 1) - 1,$$

such that  $G_1' - Y_3$  is a complete bipartite graph with bipartition  $Y_1 \cup Y_2$ , where

$$|Y_1| = |Y_2| = \frac{1}{3}(p'_j - 1) + 1.$$

Since  $Y_1 \cup Y_2$  is the set of critical vertices in  $G'_1$ , if  $y_1 \in Y_1$  and  $y_2 \in Y_2$ , then  $y_1$  and  $y_2$  are adjacent critical vertices of  $G'_1$ .

Suppose by way of contradiction that  $G'_2$  is of type 2 and not of type 1. Then every vertex  $v$  of  $G_2$  is critical in  $G_2$ , and hence interchangeable with  $y_i$  ( $i=1,2$ ) if  $y_i$  is adjacent in  $G^c$  to  $v$ .

Since  $y_1$  is critical in  $G'_1$  and in  $G$ ,

$$\begin{aligned} |E(y_1, X'_2)| &= \deg_G(y_1) - \deg_{G'_1}(y_1) - |E(y_1, \{z, z'\})| \\ &= \frac{2}{3}(p-1) - \frac{2}{3}(p'_1-1) - |E(y_1, \{z, z'\})| \\ &= \frac{2}{3}(p'_2+2) - |E(y_1, \{z, z'\})|. \end{aligned}$$

Hence, the number of vertices of  $G'_2$  adjacent in  $G^c$  to  $y_1$  is at least

$$p'_2 - |E(y_1, X'_2)| \geq \frac{1}{3}(p'_2-1) + |E(y_1, \{z, z'\})| - 1,$$

and these vertices are interchangeable with  $y_1$  and thus form a stable set.

We have two cases: when  $\{z, z'\} \cap X_1$  is not empty, and when  $\{z, z'\} \subseteq X_2$ . In the first case, without loss of generality, suppose  $z \in X_1$ . In  $G_1$ ,  $z$  is adjacent to every critical vertex of  $G'_1$ , including  $y_1, y_2 \in X'_1$ . Hence,  $|E(y_1, \{z, z'\})| \geq 1$ . In the second case, by the hypotheses of the lemma,  $p_2 \equiv 0 \pmod{3}$  and  $z$  lies in a triangle  $G[z, x_3, x_4]$ , where  $x_3$  and  $x_4$  are adjacent

critical vertices of  $G_1$ . Pick  $y_1, y_2$  so that  $\{y_1, y_2\} = \{x_3, x_4\}$ , which is possible because  $G_1 = G_1'$  here. Then  $|E(y_1, \{z, z'\})| \geq 1$ . Therefore, in either case there are at least  $\frac{1}{3}(p_2' - 1)$  critical vertices of  $G_2'$  interchangeable with  $y_i$  ( $i = 1, 2$ ).

Since  $G_2'$  is of type 2,  $p_2' \equiv 4 \pmod{6}$ , and since  $G_2'$  is not of both type 1 and type 2, it follows that  $p_2' \geq 10$ . Hence, at least  $\frac{1}{3}(p_2' - 1) \geq 3$  critical vertices of  $G_2'$  are interchangeable with  $y_i$  ( $i = 1, 2$ ). By Theorem 4.5, this set of  $\frac{1}{3}(p_2' - 1)$  vertices is a stable set. But since  $G_2'$  is of type 2, there is only one maximal stable set  $S_2$  of more than 2 vertices, and  $S_2$  has  $\frac{1}{3}(p_2' - 1) + 1$  vertices. Therefore,  $y_1$  is interchangeable with all but at most one vertex of  $S_2$ . Since  $|S_2| \geq 3$ , there is a critical vertex  $v \in S_2$ , critical in  $G_2$ , interchangeable with both vertices  $y_1, y_2$  critical in  $G_1$ . By Theorem 4.5,  $y_1$  and  $y_2$  are not adjacent, contrary to the choice of  $y_1$  and  $y_2$ . Hence,  $G_2'$  is of type 1, and the lemma is proved.

We leave to the reader the proofs of the next two lemmas.

Lemma 9.7 Let  $G_0$  be a graph of type 1 on  $3b_0 + 1$  vertices. Let  $S_0$  be the set of  $b_0 - 1$  vertices whose removal leaves  $G - S_0 = K_{b_0+1, b_0+1}$ . Any embedding of  $b_0 - 1$  pairwise disjoint triangles into  $G_0$  uses all but four vertices  $v_1, v_2, v_3, v_4 \in V(G_0) - S_0$ , and these four vertices induce a quadrilateral in  $G_0$ . Furthermore,  $v_1, v_2, v_3, v_4$  may be chosen to be any four vertices of  $G_0 - S_0$  that induce a quadrilateral in  $G$ .

Lemma 9.8 Let  $G_0$  be a graph of type 2 on  $3b_0 + 1$  vertices. Let  $S_0$  be the stable set of  $b_0 + 1$  vertices such that  $G_0 - S_0$  consists of two components, each  $K_{b_0}$ . Any embedding of  $b_0 - 1$  pairwise disjoint triangles into  $G_0$  uses all but four vertices, two in  $S_0$ , and one in each  $K_{b_0}$ , and these four vertices induce a quadrilateral in  $G_0$ . Furthermore, for any four vertices of  $V(G_0)$  with two in  $S_0$  and one in each  $K_{b_0}$ , there is an embedding of  $b_0 - 1$  pairwise disjoint triangles into the remaining  $3b_0 - 3$  vertices of  $G_0$ .

To save work, we assume without proof the following result of Corradi and Hajnal [7]:

Theorem 9.9 Let  $G$  and  $H$  be graphs on  $p$  vertices such that every component of  $H$  is a triangle, except possibly for one component that is either  $K_1$  or  $K_2$ . If

$$\delta(G) \geq \frac{2p-1}{3},$$

then  $H$  is a subgraph of  $G$ .

Proof of Theorem 9.2: By Theorem 9.9, it suffices to consider graphs  $G$  for which

$$\delta(G) = \frac{2}{3}(p-1),$$

Equality implies that

$$p \equiv 1 \pmod{3}.$$

Thus, we can assume that  $H$  is a graph with  $b$  triangles and one isolated vertex, and that

$$p = 3b + 1,$$

$$\delta(G) = \frac{2}{3}(p-1) = 2b.$$

By Theorem 4.5 there are disjoint nonempty sets  $X_1, X_2$  such that  $V(G) = X_1 \cup X_2$  and the induced subgraphs  $G_i$ , for  $G_i = G[X_i]$ ,  $i=1,2$ , satisfy

$$(9.12) \quad \delta(G_i) \geq \frac{2}{3}(p_i - 1),$$

where  $p_i = |X_i|$ .

Assume inductively that Theorem 9.2 is true for graphs smaller than  $G$ , and suppose that  $H$  is not a

subgraph of  $G$ . Theorem 9.2 is true for  $p \leq 4$ , and so we have a basis for induction. We have two cases: either one of the sets  $X_i$  has cardinality a multiple of 3, or neither do. In one subcase (Subcase IIA), we show that if  $H$  is not a subgraph of  $G$ , then  $G$  is of type 2. In other subcases, we verify the hypotheses of Lemma 9.5, and hence there is a subset  $S = Y_3 \cup V_3$  of  $V(G)$ , with  $|S| = b - 1$ , such that  $G - S$  is a bipartite graph. Thus, by Lemma 9.3,  $G$  is of type 1. We consider each case below.

Case I: Suppose that

$$p_1 \equiv 0 \pmod{3}$$

and

$$p_2 \equiv 1 \pmod{3}.$$

Since  $\delta(G_1)$  is an integer and  $p_1 \equiv 0 \pmod{3}$ , (9.12) gives

$$\delta(G_1) \geq \frac{2}{3}(p_1 - 1) + \frac{2}{3} = \frac{2}{3}p_1,$$

and Theorem 9.9 implies that  $p_1/3$  triangles can be embedded in  $G_1$ . Write

$$b_1 = \frac{1}{3}p_1$$

and

$$(9.13) \quad b_2 = \frac{1}{3}(p_2 - 1),$$

and note that

$$b_1 + b_2 = b,$$

and that the  $b_1 \geq 1$  triangles embedded in  $G_1$  use each vertex of  $G_1$ . Since  $b$  triangles are assumed to not embed in  $G$ , it follows that  $b_2$  triangles do not embed in  $G_2$ . By the induction hypothesis, either  $G_2$  is of type 1, and there is a set  $V_3 \subseteq X_2$  with

$$(9.14) \quad |V_3| = b_2 - 1$$

such that

$$G_2 - V_3 = K_{b_2+1, b_2+1},$$

or  $G_2$  is of type 2 and there is a stable set

$$(9.15) \quad S_2 \subseteq X_2$$

such that  $G_2 - S_2$  has two components, each a clique on  $b_2$  vertices.

If  $G_2$  is of type 2, each vertex  $v \in X_2$  has degree

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1),$$

and so (9.6) holds. If this alternative applies, write

$$(9.16) \quad V_2 = S_2, \quad V_1 = G_2 - S_2.$$

If  $G_2$  is of type 1, let

$$(9.17) \quad V_1 \cup V_2 \text{ denote the bipartition of } G_2 - V_3.$$

Then

$$|V_1| = |V_2| = b_2 + 1,$$

and (9.12) and (9.13) give

$$\delta(G_2) \geq 2b_2,$$

which allows us to apply Lemma 9.3. Also, by Lemma 9.3,

each vertex of  $V_j$  ( $j=1,2$ ) is adjacent to every vertex of  $V_{3-j}$  and to every vertex of  $V_3$ , and if  $v \in V_1 \cup V_2$ ,

$$\deg_{G_2}(v) = 2b_2 = \frac{2}{3}(p_2 - 1)$$

whence (9.6) holds.

It follows in either alternative (either  $G_2$  of type 1 or type 2) that there must be at least

$$\begin{aligned} \deg_G(v) - \deg_{G_2}(v) &\geq \frac{2}{3}(p-1) - \frac{2}{3}(p_2-1) \\ &= \frac{2}{3}p_1 \end{aligned}$$

vertices in  $X_1$  adjacent to a given vertex  $v \in V_1 \cup V_2$ .

Denote by  $N(v_1, v_2)$  the vertices of  $X_1$  that are adjacent to both  $v_1 \in V_1$  and  $v_2 \in V_2$ . We have

$$(9.18) \quad |N(v_1, v_2)| \geq 2\left(\frac{2}{3}p_1\right) - p_1 = \frac{1}{3}p_1 = b_1.$$

Since  $G_2$  is of type 1 or type 2,  $b_2$  disjoint triangles do not embed in  $G_2$ . By Lemmas 9.7 and 9.8, there is an embedding of  $b_2 - 1$  pairwise disjoint triangles into  $G_2$  such that the four remaining vertices induce a quadrilateral in  $G_2$ , with two of its vertices in  $V_1$  and the other two in  $V_2$ . Let  $\{v_1, v_2\}$  and  $\{v'_1, v'_2\}$  be disjoint edges of this quadrilateral, where  $v_1, v'_1 \in V_1$ , and  $v_2, v'_2 \in V_2$ .

In the two subcases below, we establish that  $G_2$  is of type 1, and that the hypotheses of Lemma 9.5 apply to  $G_1$  and  $G_2$ . We have already established (9.4) and (9.6), and it remains to establish (9.3), (9.5), and (9.7). After the subcases, we prove (9.8).

Subcase IA: Suppose  $v_1, v_1' \in V_1$  are distinct and  $v_2, v_2' \in V_2$  are distinct. Suppose that  $N(v_1, v_2), N(v_1', v_2')$  possess a transversal  $\{y, y'\}$  in  $X_1$ ; i.e., distinct  $y, y' \in X_1$  such that

$$y \in N(v_1, v_2), \quad y' \in N(v_1', v_2')$$

Since

$$\delta(G_1) \geq \frac{2}{3} p_1,$$

we have

$$\begin{aligned} \delta(G_1 - \{y, y'\}) &\geq \frac{2}{3} p_1 - 2 \\ &= \frac{2}{3} (p_1 - |\{y, y'\}| - 1). \end{aligned}$$

Since  $b$  pairwise disjoint triangles do not embed in  $G$ , and since  $(b_2 - 1) + 2$  triangles can be embedded in  $G[X_2 \cup \{y, y'\}]$ , we cannot embed

$$b - (b_2 + 1) = b_1 - 1$$

triangles in  $G_1 - \{y, y'\}$ . By the induction hypotheses,  $G_1 - \{y, y'\}$  is a graph of type 1 or of type 2, and by Lemma 9.6 with  $\{y, y'\} = \{z, z'\}$ , and with  $\{v_1, v_2\} = \{x_3, x_4\}$ , both  $G_1 - \{y, y'\}$  and  $G_2$  are of type 1. Therefore, there is a set  $Y_3'$  of  $b_1 - 2$  vertices such that

$G_1 - Y_3$  is bipartite, where

$$(9.19) \quad Y_3 = Y_3' \cup \{y, y'\}.$$

Let  $Y_1 \sim Y_2$  be the bipartition of  $G_1 - Y_3$ . By definition,

$$|Y_1| = |Y_2| = b_1 = \frac{1}{3} p_1,$$

and by Lemma 9.3, each vertex  $y_j$  of  $Y_j$  ( $j=1,2$ ) is adjacent to every vertex of  $Y_{3-j} \cup Y_3'$  and has degree  $\frac{2}{3} p_1 - 2$  in  $G_1 - \{y, y'\}$ . Thus, (9.5) holds. Since

$$\deg_{G_1}(y_j) \geq \delta(G_1) = \frac{2}{3} p_1,$$

each vertex of  $Y_j$  is also adjacent to  $y$  and  $y'$ , and hence has degree  $\delta(G_1)$  in  $G_1$ , whence we have (9.7).

Therefore,

$$\begin{aligned} \delta(G_1) + \delta(G_2) &= \frac{2}{3} p_1 + \frac{2}{3} (p_2 - 1) \\ &= \frac{2}{3} (p - 1) \\ &= \delta(G), \end{aligned}$$

which is (9.3). We have thus proved (9.3) and (9.4) through (9.7) of Lemma 9.5. This completes Subcase IA.

Subcase IB: Suppose that there is no pair of disjoint edges  $\{v_1, v_2\}, \{v_1', v_2'\}$  in  $G_2[V_1 \sim V_2]$  such that  $N(v_1, v_2), N(v_1', v_2')$  possess a transversal.

Since  $p_1 > 0$ , (9.18) implies that  $b_1 \geq 1$  and that  $N(v_1, v_2)$  and  $N(v_1', v_2')$  are nonempty. Since  $N(v_1, v_2), N(v_1', v_2')$  possess no transversal, we have  $y \in X_1$  such that

$$N(v_1, v_2) = y = N(v_1', v_2').$$

Hence,  $x_1$  is not adjacent to itself in  $G_1$ , nor to

$$\frac{1}{3}(p-1) = \frac{1}{3}(p_2-1) + 1$$

vertices in  $X_2$ , which, by Theorem 4.5, must be a stable set. Since  $G_2$  is of type 2, there is only one stable set, namely  $S_2$ , by (9.15), of

$$b_2 + 1 = \frac{1}{3}(p_2 - 1) + 1$$

vertices, unless  $b_2 + 1 = 2$ . If  $b_2 = 1$ , then  $p_2 = 4$ , and  $G_2$  is a quadrilateral, which is also of type 1. If  $b_2 \geq 2$ , each vertex of  $X_1$  is interchangeable with any vertex of  $S_2$ , and since they are interchangeable, Theorem 4.5 implies that  $X_1$  is stable. This contradicts the fact that  $G_1$  is a triangle. Hence,  $G_2$  is of type 1.

Finally, we must show that (9.8) of Lemma 9.5 applies in either subcase. Let  $y$ ,  $N(v_1, v_2)$ , and  $N(v'_1, v'_2)$  be as in the subcases above. Suppose (9.8) is false.

There exists a vertex  $y'' \in Y_1 \cup Y_2$  that forms a triangle with vertices of  $V_1 \cup V_2$ . Then the first part of Lemma 9.4 implies that  $y''$  is adjacent to all vertices of  $V_j$  and to some of the vertices of  $V_{3-j}$ , for  $j=1$  or  $2$ . Choose

$$v = v_{3-j} \text{ or } v'_{3-j}$$

so that  $y''$  is adjacent to a vertex  $v_{3-j}$  of  $V_{3-j} - v$ .  
 Without loss of generality, suppose

$$v = v_{3-j}.$$

Then,  $v_{3-j}$ ,  $v_j$  and  $y$  form a triangle, and  $y''$  forms a triangle with  $v_j'$  and  $v_{3-j}''$ . Thus, there are two disjoint triangles, which together with the  $b_2 - 1$  triangles that can be embedded in  $G_2 - \{v_j, v_j', v_{3-j}, v_{3-j}'\}$ , and the  $b_1 - 1$  triangles that can be embedded in  $G_1 - \{y'', y\}$  constitute an embedding of

$$2 + b_2 - 1 + b_1 - 1 = b$$

triangles in  $G$ . We have contradicted the nonembeddability assumption, and hence (9.8) is true. Thus, all of the hypotheses of Lemma 9.5 hold. We conclude from Lemma 9.5 that  $G - (Y_3 \cup V_3)$  is a bipartite graph. By (9.14), (9.19), (9.20), we have

$$|Y_3 \cup V_3| = b - 1,$$

and so by Lemma 9.3,  $G$  is of type 1. This completes Case I.

Case II: Suppose that

$$p_1 \equiv p_2 \equiv 2 \pmod{3}.$$

Since  $\delta(G_i)$  is an integer, (9.12) implies

$$(9.22) \quad \delta(G_i) \geq \frac{2}{3}p_i - \frac{1}{3}$$

for  $i=1,2$ . Without loss of generality, assume

$$b_1 \leq b_2.$$

Write

$$b_1 = \frac{1}{3}p_1 - \frac{2}{3}, \quad b_2 = \frac{1}{3}p_2 - \frac{2}{3},$$

and note that  $b_1$  and  $b_2$  are integers such that

$$b_1 + b_2 + 1 = b.$$

If we form a graph  $G_i + z$ , adding to  $G_i$  ( $i=1,2$ ) a new vertex  $z$  adjacent to every vertex of  $G_i$ , then by (9.22),

$$\delta(G_i + z) = \frac{2}{3}|X_i + z|,$$

and by Theorem 9.9,  $b_i + 1$  pairwise disjoint triangles can be embedded in  $G_i + z$ . Therefore,  $b_i$  pairwise disjoint triangles and an edge disjoint from the  $b_i$  triangles, which we shall call the free edge, can be embedded in  $G_i$ . We shall attempt to use the vertices of the two free edges to form an extra triangle, disjoint from the  $b_1$  triangles in  $G_1$  and the  $b_2$  triangles in  $G_2$ , thus constituting  $b_1 + b_2 + 1 = b$  pairwise disjoint triangles in  $G$ . By assuming that  $b$  pairwise disjoint

triangles do not embed in  $G$ , we shall determine the structure of  $G$  in the attempt to find such an embedding.

We show in the two subcases below that either  $G$  is of type 2, or there is a vertex  $x_3 \in X_2$  such that the free edge in  $G_1$  together with  $x_3$  form a triangle in  $G$ . It may be necessary to alter the embedding of  $b_1$  triangles and the free edge into  $G_1$  in order to accomplish this.

Let  $x_1, x_2$  be the ends of the free edge in  $G_1$ . Without loss of generality, choose the free edge from among all possible free edges so that

$$\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$$

is minimized. If  $x_1$  and  $x_2$  are adjacent in  $G$  to a vertex  $x_3 \in X_2$ , then  $x_1, x_2, x_3$  is the desired triangle. Otherwise,  $x_1$  and  $x_2$  are adjacent to no common vertex in  $X_2$ . Then

$$\begin{aligned} (9.23) \quad \deg_{G_1}(x_1) + \deg_{G_1}(x_2) &\geq 2\delta(G) - p_2 \\ &\geq \frac{4}{3}(p-1) - p_2 \\ &= p_1 + \frac{1}{3}p - \frac{4}{3}. \end{aligned}$$

Also, without loss of generality, assume that

$$\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2).$$

These inequalities imply

$$\begin{aligned} (9.24) \quad 2 \deg_{G_1}(x_1) &\geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \\ &\geq p_1 + \frac{1}{3}p - \frac{4}{3}. \end{aligned}$$

We define

$$\pi: V(H_1) \rightarrow V(G_1)$$

to be an embedding of  $b_1$  triangles  $K_3$  and one edge-component  $K_2$ , constituting  $H_1$ , into  $G_1$  such that the edge-component  $K_2$  is mapped to the free edge  $x_1, x_2$  that minimizes  $\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$ . We shall alter  $\pi$  if necessary, and then either we shall extend  $\pi$  to an embedding of  $H$  into  $G$ , where  $H$  consists of  $b$  triangular components and one isolated vertex, or we shall show (Subcase IIA) that  $G$  is of type 2 or (following the subcases) that  $G$  is of type 1.

Define

$$M(x) = \{x' \in X_1: \pi^{-1}(x) \text{ and } \pi^{-1}(x') \text{ are adjacent in } H_1\}.$$

For  $i = 1, 2$ , and  $x \in V(G)$ , define

$$N_i(x) = \{x' \in X_i: x \text{ and } x' \text{ are adjacent in } G\}.$$

We say that  $x \in X_1$  is a successor of  $x_1 \in X_1$  if each vertex of  $M(x_1)$  is adjacent in  $G_1$  to  $x$ . Denote the set of successors of  $x_1$  by  $S(x_1)$ . We say that  $x_1 \in X_1$  is a predecessor of  $x \in X_1$  if  $x$  is a successor of  $x_1$ . Denote the set of predecessors of  $x$  by  $P(x)$ .

Recall from section 1 that if  $x_1, x_4 \in X_1$  are equal, then  $(x_1 \ x_4)'$  is the identity permutation on  $X_1$ , but if  $x_1, x_4$  are distinct, then  $(x_1 \ x_4)' = (x_1 \ x_4)$ .

Subcase IIA: Suppose that

$$\deg_{G_1}(x_2) \leq \frac{1}{3}(p-1).$$

First, we eliminate the possibility of strict inequality.

If the inequality above is strict, then

$$\begin{aligned} |E(x_2, X_2)| &= \deg_G(x_2) - \deg_{G_1}(x_2) \\ &> \frac{2}{3}(p-1) - \frac{1}{3}(p-1) \\ &= \frac{1}{3}(p-1). \end{aligned}$$

Since  $x_1$  is not adjacent to at most  $\frac{1}{3}(p-1)$  vertices of  $G$  other than itself,  $x_1$  is adjacent to one of the more than  $\frac{1}{3}(p-1)$  vertices  $x_3$  of  $X_2$  incident with an edge of  $E(x_2, X_2)$ . Hence,  $G[x_1, x_2, x_3]$  is a triangle on the free edge in  $G_1$  and a vertex of  $G_2$ .

Henceforth in this subcase, we shall suppose

$$\deg_{G_1}(x_2) = \frac{1}{3}(p-1).$$

By (9.23),

$$\deg_{G_1}(x_1) + \frac{1}{3}(p-1) \geq p_1 + \frac{1}{3}p - \frac{4}{3}.$$

Hence,

$$\deg_{G_1}(x_1) \geq p_1 - 1,$$

and so  $x_1$  must be adjacent to each vertex of  $G_1$ .

Therefore,  $P(x_1) = G_1 - x_2$ . Since  $S(x_1) = N_1(x_2)$ , we conclude that for any  $x_4 \in N_1(x_2)$ ,  $(x_1 x_4)^\pi$  is an embedding of the  $b_1$  triangles and free edge into  $G_1$ . Note that

the embedding  $(x_1 x_4)'\pi$  makes  $\{x_4, x_2\}$  the free edge.

By the minimality of  $\deg_{G_1}(x_1) + \deg_{G_1}(x_2)$ ,

$$\deg_{G_1}(x_4) + \deg_{G_1}(x_2) \geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2),$$

whence,

$$\deg_{G_1}(x_4) = p_1 - 1.$$

Since  $x_4$  may be any of the  $\frac{1}{3}(p-1)$  vertices of  $N_1(x_2)$ , we know that the vertices of  $X_1 - N_1(x_2)$  must be adjacent to each vertex of  $N_1(x_2)$ , a set of  $\frac{1}{3}(p-1)$  vertices adjacent to all of  $G_1$ . Hence,

$$\delta(G_1) \geq \frac{1}{3}(p-1) = \deg_{G_1}(x_2).$$

Define the sets

$$T_1 = N_1(x_2),$$

$$T_2 = N_2(x_2),$$

$$S_1 = X_1 - T_1,$$

$$S_2 = X_2 - T_2.$$

We have already shown that  $G[T_1]$  is a complete graph, and each vertex of  $S_1$  is adjacent to every vertex of  $T_1$ . If there is an  $x_4 \in T_1 = S(x_1)$  and a vertex  $x_3 \in X_2$  such that  $G[x_2, x_3, x_4]$  is a triangle in  $G$ , then we have accomplished the goal of this subcase, since  $(x_1 x_4)'\pi$  is an embedding of  $b_1$  triangles and a disjoint edge mapped to  $\{x_2, x_4\}$ , which is the edge forming the triangle with  $x_3$ . Otherwise, no  $x_4 \in T_1$  forms a triangle with  $x_2$  and any vertex in  $X_2$ . Hence, no  $x_4 \in T_1$  is adjacent to vertices

of  $T_2$ . Now,

$$|T_2| = \deg_G(x_2) - \deg_{G_1}(x_2) \geq \frac{1}{3}(p-1),$$

and hence, any  $x_4 \in T_1$ , having degree at least  $\frac{2}{3}(p-1)$

in  $G$ , must be adjacent to every vertex of  $S_1 \cup T_1 \cup S_2 - x_4$ .

A similar argument shows that any vertex of  $T_2$ , not being adjacent to any vertex of  $T_1$ , a set of  $\frac{1}{3}(p-1)$

vertices, is adjacent to any vertex of  $S_1 \cup T_2 \cup S_2$

except itself. Note that this implies that  $G[T_2]$  is,

like  $G[T_1]$ , a complete graph on  $\frac{1}{3}(p-1)$  vertices.

Also, note that any vertex of  $S_1 \cup S_2$  is adjacent to every vertex of  $T_1 \cup T_2$  in  $G$ .

Hence,  $S_1 \cup S_2$  is a set of

$$\begin{aligned} |V(G) - (T_1 \cup T_2)| &= p - \frac{2}{3}(p-1) \\ &= \frac{1}{3}(p-1) + 1 \\ &= b + 1 \end{aligned}$$

vertices whose removal from  $G$  leaves two components  $G[T_i]$ ,  $i=1,2$ , each a complete graph on  $\frac{1}{3}(p-1) = b$  vertices.

By Lemma 9.4, either  $b$  pairwise disjoint triangles embed in  $G$ , or  $G$  is of type 2. The first possibility is contrary to hypothesis. The other possibility is a desired conclusion of Theorem 9.1. Hence, we can assume that there is a free edge in  $G_1$ , which together with some  $x_3 \in X_2$ , forms a triangle in  $G$ .

Subcase IIB: Suppose that

$$(9.25) \quad \deg_{G_1}(x_2) > \frac{1}{3}(p-1).$$

Let  $x_3$  be a vertex of  $X_2$  that is adjacent in  $G$  to  $x_2$ .

Since  $p_1 \leq p_2$ ,

$$\begin{aligned} \deg_G(x_2) &\geq \frac{2}{3}(p-1) \\ &\geq \frac{2}{3}(2p_1-1) \\ &= p_1 + \frac{1}{3}p_1 - \frac{2}{3} \\ &> p_1 - 1, \end{aligned}$$

and so  $x_3$  exists. The successors  $S(x_1)$  of  $x_1$  in  $G_1$  are the vertices of  $G_1$  adjacent to  $x_2$ . We see that

$S_1(x_1) = N_1(x_2)$ . We have

$$\begin{aligned} |S(x_1) \cap N_1(x_3)| &\geq \deg_{G_1}(x_2) + \deg_G(x_3) \\ &\quad - (p_2 - 1) - |S(x_1) \cup N_1(x_3)| \\ &\geq \deg_{G_1}(x_2) + \frac{2}{3}(p-1) - (p_2 - 1) - p_1 \\ &= \deg_{G_1}(x_2) - \frac{1}{3}(p-1) \\ &> 0, \end{aligned}$$

by (9.25). Hence, there is a vertex  $x_4 \in X_1$  that forms a triangle with  $x_2$  and  $x_3$  and is a successor of  $x_1$ .

If  $x_1 \in S(x_4)$ , then the embedding  $(x_1 x_4)\pi$  maps the free edge in  $G_1$  to  $\{x_2, x_4\}$ , which forms with  $x_3 \in X_2$  a triangle in  $G$  as desired. Otherwise,

$$(9.26) \quad x_1 \notin S(x_4).$$

We shall find a vertex  $x_5 \in X_1$  with  $x_5 \in S(x_4) \cap P(x_1)$ , whence  $(x_1, x_4, x_5)$  is the desired embedding of  $b_1$  triangles and one edge into  $G_1$ .

In the image of the triangle embedded into  $G_1$  having vertex  $x_4$  are two other vertices, which we call  $x_6, x_7$ . The successors of  $x_4$  are those vertices in  $G_1$  adjacent to both  $x_6$  and  $x_7$ . Hence,  $x_1, x_6, x_7 \notin S(x_4)$ , and

$$(9.27) \quad |S(x_4)| \geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1.$$

The predecessors  $P(x_1)$  of  $x_1$  in  $G_1$  are those vertices  $v \in X_1$  such that  $x_1$  is adjacent to all vertices of  $M(v)$ . Now,  $x_1$  is adjacent in  $G_1^c$  to  $p_1 - \deg_{G_1}(x_1) - 1$  vertices  $v' \in X_1$ . Any such  $v'$  lies in exactly two sets  $M(v)$ ,  $v \in X_1$ . Thus,  $x_1 \notin S(v)$  for at most

$$2p_1 - 2 \deg_{G_1}(x_1) - 2$$

vertices  $v$  of  $X_1 - M(x_1) = G - x_2$ . Since the remaining vertices of  $G_1 - x_2$  are in  $P(x_1)$ , we have  $x_2 \notin P(x_1)$ , and

$$\begin{aligned} (9.28) \quad |P(x_1)| &\geq |X_1 - x_2| - (2p_1 - 2 \deg_{G_1}(x_1) - 2) \\ &= 2 \deg_{G_1}(x_1) - p_1 + 1 \\ &\geq \frac{1}{3}(p - 1), \end{aligned}$$

by (9.24).

Suppose first that  $x_4$  is not adjacent to  $x_1$ . Then

$$x_1, x_6, x_7 \notin P(x_1),$$

and we combine (9.27), (9.28), (9.22), and  $2p_1 \leq p$  to get

$$x_1, x_2, x_6, x_7 \notin S(x_4) \cap P(x_1);$$

$$x_6, x_7 \notin S(x_4) \cup P(x_1),$$

and

$$\begin{aligned}
 (9.29) \quad |S(x_4) \cap P(x_1)| &\geq |S(x_4)| + |P(x_1)| - |X_1 - \{x_6, x_7\}| \\
 &\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 \\
 &\quad + \frac{p}{3} - \frac{1}{3} - p_1 + 2 \\
 &\geq 2\delta(G_1) - 2p_1 + \frac{p}{3} + \frac{5}{3} \\
 &\geq 2(\frac{2}{3}p_1 - \frac{1}{3}) - 2p_1 + \frac{p}{3} + \frac{5}{3} \\
 &= \frac{p}{3} - \frac{2}{3}p_1 + 1 \\
 &\geq 1.
 \end{aligned}$$

Suppose, otherwise, that  $x_4$  is adjacent to  $x_1$ . Then  $G[x_1, x_2, x_4]$  is a triangle, and  $\{x_6, x_7\}$  is a free edge. Thus, by choice of  $\{x_1, x_2\}$  and (9.23),

$$\begin{aligned}
 (9.30) \quad \deg_{G_1}(x_6) + \deg_{G_1}(x_7) &\geq \deg_{G_1}(x_1) + \deg_{G_1}(x_2) \\
 &\geq p_1 + \frac{p}{3} - \frac{4}{3}.
 \end{aligned}$$

We combine (9.27), (9.28), (9.30) and

$$p_1 + p_2 = p$$

to obtain

$$\begin{aligned}
 (9.31) \quad |S(x_4) \cap P(x_1)| &\geq |S(x_4)| + |P(x_1)| - p_1 \\
 &\geq \deg_{G_1}(x_6) + \deg_{G_1}(x_7) - p_1 \\
 &\quad + \frac{p}{3} - \frac{1}{3} - p_1 \\
 &\geq p_1 + \frac{p}{3} - \frac{4}{3} - 2p_1 + \frac{p}{3} - \frac{1}{3} \\
 &= \frac{2p}{3} - \frac{2}{3}p_1 - \frac{1}{3}p_1 - \frac{5}{3} \\
 &\geq \frac{2}{3}p_2 - \frac{1}{3}p_1 - \frac{5}{3} \\
 &= \frac{1}{3}(p_2 - p_1) + (\frac{1}{3}p_2 - \frac{5}{3}).
 \end{aligned}$$

Note that both of the terms in the last line of (9.31) are nonnegative if  $p_2 \geq 5$ , and if  $p_2 > 5$ , then the last line is positive. If  $p_2 \leq 5$ , then  $p_1 \leq p_2$  and  $p_1 \equiv 2 \pmod{3}$  imply one of the following three cases:

$$p_2 = p_1 = 5;$$

$$p_2 = 5, \quad p_1 = 2;$$

or

$$p_2 = p_1 = 2.$$

If  $p_2 = p_1 = 5$ , then (9.23) gives

$$\deg_{G_1}(x_1) + \deg_{G_1}(x_2) \geq 7,$$

whence  $\deg_{G_1}(x_1) \geq \deg_{G_1}(x_2)$  implies that  $x_1$  is adjacent to every vertex of  $G_1$  except itself, whence  $x_4 \in P(x_1)$ , in violation of (9.26). If  $p_2 = 5$ ,  $p_1 = 2$ , then the last line of (9.31) is 1, which is as desired. If  $p_1 = p_2 = 2$ , then  $p = 4$  and  $\delta(G) \geq \frac{2}{3}(p-1)$  imply  $G$  is  $K_4$ ,  $K_4 - e$  ( $e$  an edge), or a quadrilateral, all of which satisfy the theorem. Hence, under our hypotheses, the last line of (9.31) and the last line of (9.29) may be assumed to be positive.

Therefore, whether or not  $x_4$  and  $x_1$  are adjacent, there is a vertex  $x_5 \neq x_1$  or  $x_2$ , such that

$$x_5 \in S(x_4) \cap P(x_1),$$

and so we have a closed alternating chain in  $G_1$  represented

by the permutation

$$\alpha = (x_1 \ x_4 \ x_5).$$

Hence,  $\alpha\pi$  is an embedding of the  $b_1$  triangles and one edge into  $G_1$ . The free edge is determined by  $\alpha\pi$  to be  $\{x_2, x_4\}$ , since  $x_1$  is permuted to  $x_4$  and since  $x_2 \neq x_5$  guarantees that  $x_2$  is fixed. Thus, the free edge is part of a triangle  $G[x_2, x_3, x_4]$ , as desired. This concludes Subcase IIB.

To complete Case II and the proof of the theorem, we verify that all the hypotheses, and hence the final conclusion, of Lemma 9.5 apply to  $G_1$  and  $G_2$ , and then we show that  $G$  is of type 1.

Since we have assumed that

$$b = b_1 + 1 + b_2$$

triangles do not embed in  $G$ , and since  $b_1 + 1$  triangles embed in  $G_1 + x_3 = G[X_1 + x_3]$ , we know that we cannot embed  $b_2$  triangles in  $G_2 - x_3$ . Now,

$$|V(G_2) - x_3| = 3b_2 + 1,$$

and by (9.22),

$$\begin{aligned} \delta(G_2 - x_3) &\geq \delta(G_2) - 1 \\ &\geq \frac{2}{3}p_2 - \frac{1}{3} - 1 \\ &= \frac{2}{3}(|X_2 - x_3| - 1) \\ &= 2b_2. \end{aligned}$$

Since

$$\delta(G) = 2b = 2b_1 + 2 + 2b_2,$$

and since

$$\deg_{G_2}(v_j) = 2b_2 + 1 \quad (j=1,2),$$

each  $v_j$  ( $j=1,2$ ) is adjacent to at least  $2b_1 + 1$  vertices of  $G_1$ . Hence, there are at least

$$\begin{aligned} |E(v_1, X_1)| + |E(v_2, X_1)| - p_1 \\ &\geq 2(2b_1 + 1) - (3b_1 + 2) \\ &= b_1 \\ &\geq 1 \end{aligned}$$

choices  $y_3 \in X_1$  such that  $G[v_1, v_2, y_3]$  is a triangle.

Therefore, as we already remarked, we may apply Lemma 9.6 and conclude that both  $G_1 - y_3$  and  $G_2 - x_3$  are of type 1.

Next, we establish the hypotheses of Lemma 9.5.

Since  $G_1 - y_3$  and  $G_2 - v_3$  are of type 1, where

$$v_3 = x_3,$$

and since they have  $3b_1 + 1$ ,  $3b_2 + 1$  vertices, respectively,

there are sets  $Y'_3 \subseteq X_1 - y_3$  and  $V'_3 \subseteq X_2 - v_3$  with

$$|Y'_3| = b_1 - 1,$$

$$|V'_3| = b_2 - 1,$$

such that  $G_1 - y_3 - Y'_3$  and  $G_2 - v_3 - V'_3$  are complete bipartite graphs  $Y_1 \cup Y_2$  and  $V_1 \cup V_2$ , respectively. Define

$$Y_3 = Y'_3 + y_3,$$

$$V_3 = V'_3 + v_3.$$

Therefore, by the induction hypothesis,  $G_2 - x_3$  is of type 1 or type 2. Hence,

$$\delta(G_2 - x_3) = \frac{2}{3}(p_2 - 2) = \frac{2}{3}p_2 - \frac{4}{3},$$

whence, by (9.22),  $x_3$  is adjacent to every vertex of  $G_2 - x_3$  having degree  $\frac{2}{3}p_2 - \frac{4}{3} = \delta(G_2 - x_3)$  in  $G_2 - x_3$ .

By Lemmas 9.7 and 9.8, we know that  $b_2 - 1$  triangles embed in  $G_2 - x_3$ , and that such an embedding uses all but 4 vertices of  $G_2 - x_3$ . Moreover, these 4 vertices all have degree  $\delta(G_2 - x_3)$  in  $G_2 - x_3$ , and they induce a quadrilateral. Now,  $x_3$  is adjacent to all four of these vertices, and hence forms a triangle with 2 of them.

Let  $v_1$  and  $v_2$  denote the other 2 vertices on this quadrilateral. Note that  $v_1$  and  $v_2$  are adjacent. We shall show that there are  $b_1$  choices of a vertex  $y_3 \in X_1$  such that  $G[v_1, v_2, y_3]$  is a triangle. If  $b_1$  disjoint triangles can be embedded in  $G_1 - y_3$ , then, counting the triangle containing  $x_3$ , the triangle  $G[v_1, v_2, y_3]$ , and the  $b_2 - 1$  triangles of  $G_2 - x_3$ , we have  $b$  pairwise disjoint triangles in  $G$ , contrary to assumption. Hence,  $b_1$  triangles do not embed in  $G_1 - y_3$ . For this to happen,

$$b_1 \geq 1.$$

Thus, by (9.22), we may apply Lemma 9.6, with

$$\{z, z'\} = \{x_3, y_3\},$$

and conclude that both  $G_1 - y_3$  and  $G_2 - x_3$  are of type 1.

Thus, (9.4) and (9.5) of Lemma 9.5 hold, and also

$$(9.32) \quad |Y_3 \cup V_3| = b_1 + b_2 = b - 1.$$

Since  $G_1 - y_3$  is of type 1, if  $y \in Y_1 \cup Y_2$ , then

$$\deg_{G_1 - y_3}(y) = 2b_1 = \frac{2}{3}(p_1 - 2).$$

Now,  $\delta(G_1) \geq \frac{2}{3}p_1 - \frac{1}{3}$ , and hence  $y$  is adjacent to  $y_3 \in X_1$ .

Therefore, for any  $y \in Y_1 \cup Y_2$ ,

$$\deg_{G_1}(y) = \frac{2}{3}(p_1 - 2) + 1 = \delta(G_1),$$

and (9.7) of Lemma 9.5 is established. Similarly,

since  $G_2 - v_3$  is of type 1, (9.6) may be established,

and also for any  $v \in V_1 \cup V_2$ ,

$$\deg_{G_2}(v) = \delta(G_2).$$

By (9.22),

$$\begin{aligned} \delta(G_1) + \delta(G_2) &\geq \frac{2}{3}p_1 - \frac{1}{3} + \frac{2}{3}p_2 - \frac{1}{3} \\ &= \frac{2}{3}(p - 1) \\ &= \delta(G), \end{aligned}$$

and (9.3) is established. Thus, having proved (9.3)

through (9.7) of Lemma 9.5, we conclude from Lemma 9.5

that any vertex  $y \in Y_1 \cup Y_2$  is adjacent to every vertex

in  $V_j$  for some  $j \in \{1, 2\}$ .

Suppose by way of contradiction that some  $y \in Y_1 \cup Y_2$  is adjacent in  $G$  to vertices  $v_1 \in V_1$  and  $v_2 \in V_2$  (i.e., suppose that (9.8) is false). Thus,  $G[y, v_1, v_2]$  is a triangle. By Lemma 9.7, for any vertices  $v_3 \in V_1 - v_1$  and  $v_4 \in V_2 - v_2$ , there is an embedding of  $b_2 - 1$  triangles into  $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$ , since  $G_2 - v_3$  is of type 1. Note that  $G[v_3, v_4, v_5]$  is also a triangle. We conclude from Lemma 9.7 that for any vertices  $y_1 \in Y_1$ ,  $y_2 \in Y_2$ , there is an embedding of  $b_1 - 1$  pairwise disjoint triangles into  $G_1 - \{y, y_1, y_2, y_3\}$ , since  $G_1 - y_3$  is of type 1. Including the  $b_2 - 1$  triangles of  $G_2 - \{v_1, v_2, v_3, v_4, v_5\}$  and the 3 triangles  $G[y_1, y_2, y_3]$ ,  $G[y, v_1, v_2]$ , and  $G[v_3, v_4, v_5]$ , we have

$$(b_1 - 1) + (b_2 - 1) + 3 = b$$

pairwise disjoint triangles embedded in  $G$ , contrary to assumption. Hence, (9.8) holds, and by Lemma 9.5,  $G - (Y_3 \cup V_3)$  is a complete bipartite graph. By (9.32) and Lemma 9.3,  $G$  is of type 1. This completes the proof of Theorem 9.2.

## 10. Subgraphs of graphs, III

We give one of our main results in this section.

Theorem 10.1 Let  $G$  and  $H$  be graphs on  $p$  vertices.

If  $\Delta(H) \leq 2$  and if

$$(10.1) \quad \Delta(G^c) \leq \frac{p}{3} - \max(k, \frac{3}{2} p^{1/3}),$$

where  $k = 9$ , then  $H$  is a subgraph of  $G$ .

Proof: The entire chapter is devoted to the proof of this Theorem. First, we introduce notation.

Recall that for a bijection

$$\pi: V(H) \longrightarrow V(G),$$

if  $v \in V(G)$ , then

$$M(v) = \{x: \pi^{-1}(x) \text{ and } \pi^{-1}(v) \text{ are adjacent in } H\}.$$

We shall use the notation

$$M(v_1, v_2, \dots, v_n) = \bigcup_{i=1}^n M(v_i),$$

and

$$M(A) = \bigcup_{v \in A} M(v),$$

where the latter union is over the vertices  $v \in A$ , where

$A \subseteq V(G)$ .

Successors and predecessors are defined as in section 7, except that we do not permit vertices of  $M(v)$  to be successors or predecessors of  $v$ . Thus,  $x \in V(G)$  is a successor of  $v \in V(G)$  if  $x$  is adjacent in  $G$  to each

vertex of  $M(v)$ . The set of successors of  $v$  is denoted  $S(v)$ . Also,  $v$  is a predecessor of  $x$  whenever  $x$  is a successor of  $v$ . The set of predecessors of  $x$  is denoted  $P(x)$ .

Suppose that  $H$  is an edge-minimal graph for which the theorem is false. If  $y \in V(H)$  is a vertex of degree 1, then let  $e \in E(H)$  be the incident edge. Otherwise, all vertices of  $H$  have degree either 0 or 2. If this is the case, let  $y \in V(H)$  be a vertex of degree 2, and let  $e \in E(H)$  be an edge incident with  $y$ . By the edge-minimality of  $H$ , there is, in either case, an embedding

$$\pi: V(H) \longrightarrow V(G)$$

of  $H - e$  into  $G$ . Let  $\pi(y)$  be denoted by  $x$ . The bijection  $\pi$  and the vertex  $x$  are considered fixed throughout the proof. At a relatively early stage in the proof (prop. 10.5), we shall dispose of the case in which  $H$  has a vertex of degree 1.

Henceforth, all vertices denoted in this proof by a letter are vertices of  $G$ .

We define an alternating chain from  $x_0$  to  $v$  to be any finite sequence of at least 2 distinct vertices  $x_0, x_1, \dots, x_m$  of  $V(G)$ , with  $x_m = v$ , such that

$$(10.2) \quad x_i \in S(x_{i-1}) \quad \text{for } i = 1, 2, \dots, m;$$

$$(10.3) \quad \text{If } x_i = x_j \text{ and } i < j, \text{ then either } x_i = x_{i+1} = \dots = x_j, \text{ or both } x_0 = x_1 = \dots = x_i \text{ and } x_j = \dots = x_m = x_0;$$

$$\text{and } (10.4) \quad x_i \notin M(x_j) \quad \text{for } 0 \leq i < j \leq m.$$

Note that (10.2) is equivalent to  $x_{i-1} \in P(x_i)$ , and (10.4) is equivalent to  $x_j \notin M(x_i)$ . If  $x_0 = v$  in an alternating chain from  $x_0$  to  $v$ , we say that the chain is closed.

The proof of Theorem 10.1 will rest upon the observation that for a closed alternating chain  $x_0, x_1, \dots, x_m$ , with  $x = x_0 = x_m$ ,  $(x_0 x_1 \dots x_{m-1})\pi$  is an embedding of  $H$  into  $G$ .

Define, for each integer  $t \geq 1$ , the set  $D_t(x)$  to be the set of all vertices  $z$  such that for any  $t-1$  vertices  $w_1, w_2, \dots, w_{t-1} \in V(G) - M(x, z)$ , there is an alternating chain from  $x$  to  $z$  containing no vertex of  $M(w_1, \dots, w_{t-1})$ . Define  $D_0(x) = V(G)$ . Thus, we have

$$(10.5) \quad D_0(x) \supseteq D_1(x) \supseteq D_2(x) \supseteq \dots \supseteq D_t(x) \supseteq \dots \supseteq S(x).$$

We first prove 15 propositions. Then we break the proof into six cases, and use the propositions and two key lemmas about alternating chains to show that in each case there is a permutation  $\alpha: V(G) \rightarrow V(G)$  such that  $\alpha$  embeds  $H$  into  $G$ .

Recall that in section 1 we defined, for  $x_1, x_2, \dots, x_n \in X$ , where  $x_i = x_j$  and  $i < j$  imply  $x_i = x_{i+1} = \dots = x_j$ , the symbol  $(x_1 x_2 \dots x_n)'$  to be the permutation obtained by suppressing multiple successive occurrences of the same member of  $X$  in  $x_1, x_2, \dots, x_n$ . Thus, if  $v_0, v_1, v_2 \in V(G)$  are distinct and if  $v_2 = v_3$ , then

$$(v_0 v_1 v_2 v_3)' = (v_0 v_1 v_2).$$

Prop. 10.2 The number  $|P(v)|$  of predecessors of a vertex  $v \in V(G)$  is at least

$$\frac{p}{3} + \max(2k, 3p^{1/3}) - 2.$$

Proof: A vertex  $v'$  is not a predecessor of  $v$  if there is a vertex  $u \in V(G)$ , either equal or adjacent in  $G^c$  to  $v$ , such that  $v' \in M(u)$ . Since  $|M(u)| \leq 2$ , (10.1) implies that there are at most

$$\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$$

non-predecessors of  $v$ . Prop. 10.2 follows.

Prop. 10.3 The number  $|S(v)|$  of successors of a vertex  $v$  is at least

$$\frac{p}{3} + \max(2k, 3p^{1/3}) - 2.$$

Proof: The non-successors of  $v$  are the vertices which are either equal or adjacent in  $G^c$  to a vertex of  $M(v)$ . For each  $u \in M(v)$ , the number of vertices either equal or not adjacent to  $u$  is at most

$\frac{p}{3} - \max(k, \frac{3}{2}p^{1/3}) + 1$ . Since  $|M(v)| \leq 2$ , there are at most  $\frac{2p}{3} - \max(2k, 3p^{1/3}) + 2$  non-successors of  $v$ .

Prop. 10.3 follows.

Prop. 10.4 If  $z \in D_t(x)$  and  $t \geq 1$ , then  $z \notin P(x) + x$ .

Proof: If Prop. 10.4 is false, then there is a closed alternating chain  $x = x_0, x_1, \dots, z, x$ , and so  $(x_0 \ x_1 \ \dots \ z)'\pi$  embeds  $H$  into  $G$ . This is the conclusion of Theorem 10.1, which we have assumed to be false.

Prop. 10.5 Every vertex of  $H$  has degree either 0 or 2. In particular,  $|M(x)| = 2$ , and if  $|M(v)| = 0$  for some  $v \in V(G)$ , then  $v \in P(x)$ .

Proof: If  $|M(x)| = 1$ , then let  $M(x) = x'$ . The successors of  $x$  are the vertices adjacent in  $G$  to  $x'$ . Thus, by (10.1),  $|S(x)| > \frac{2p}{3}$ , and since, by Prop. 10.2, there are more than  $\frac{p}{3}$  predecessors of  $x$ , there is a vertex  $x_1 \in S(x) \cap P(x)$ . Then  $(x x_1)\pi$  embeds  $H$  into  $G$ , contrary to the assumption that  $H$  is a graph for which the theorem is false. Hence,  $|M(x)|$  is not 1, and by the original choice of  $x$ , every vertex of  $H$  has degree 0 or 2. The final statement of the proposition follows because, by the definition of successors, if  $|M(v)| = 0$ , then  $S(v) = V(G)$ .

Definitions If  $z^* \in D_1(x) - D_t(x)$ , for some  $t \geq 2$ , define  $\mathcal{C}(z^*)$  to be the set of all alternating chains from  $x$  to  $z^*$ . Since  $z^* \notin D_t(x)$ , then by definition of  $D_t(x)$ , there is a set  $A(z^*) = \{w_1, \dots, w_s\} \subseteq V(G) - M(x, z^*)$ , with minimum possible integer  $s \leq t-1$ , such that every chain in  $\mathcal{C}(z^*)$  has a vertex in  $M(A(z^*))$ . Of course,  $A(z^*)$  is not necessarily uniquely determined. However, we shall consider the set  $A(z^*)$  to be fixed, for each  $z^* \in D_1(x) - D_t(x)$ . We have

$$|A(z^*)| \leq t - 1,$$

and we have

$$|M(A(z^*))| \leq 2t - 2.$$

Prop. 10.6 Let  $t \geq 2$  be an integer. If

$$(10.6) \quad z \in D_t(x)$$

and if

$$(10.7) \quad z^* \in S(z) - D_t(x),$$

then one of the following three statements holds:

$$(10.8) \quad z \notin M(A(z^*)) \cap D_{t+1}(x) \text{ and } z^* \in D_{t-1}(x);$$

$$(10.9) \quad z \in M(A(z^*));$$

$$(10.10) \quad z^* \in M(x).$$

If (10.10) is false, then also,

$$(10.11) \quad z^* \notin P(x) + x.$$

Proof: Let  $z, z^* \in V(G)$  satisfy (10.6) and (10.7).

First, we claim that either (10.10) holds, or  $\mathcal{C}(z^*)$  is not empty. By (10.6) and  $t \geq 2$ , there is a chain  $C$  from  $x$  to  $z$  avoiding  $M(z^*)$ . If  $C$  passes through  $z^*$ , then  $\mathcal{C}(z^*)$  is not empty. Otherwise, we extend the chain  $C$  by adding  $z^*$  at the end and we denote the resulting sequence by  $C^*$ . Since  $z^* \in S(z)$ ,  $C^*$  is an alternating chain, provided some vertex in  $M(z^*)$  does not already occur in  $C^*$ . By our choice of  $C$ , unless (10.10) holds, this condition is satisfied. This justifies the claim.

Henceforth, we assume that (10.10) is false and thus that  $C(z^*)$  is not empty. Therefore,  $A(z^*)$  exists, and  $z^* \in D_1(x)$ , whence (10.11) follows, by Prop. 10.4.

By (10.6) and (10.7),

$$(10.12) \quad z \in P(z^*) \cap D_t(x).$$

Suppose by way of contradiction that (10.9) is false and

$$(10.13) \quad z \in D_{t+1}(x).$$

Then either there is an alternating chain  $C$  from  $x$  to  $z$  that misses  $M(A(z^*) + z^*)$ , or by definition of  $D_{t+1}(x)$ , the sets  $A(z^*) + z^*$  and  $M(x, z)$  intersect.

We quickly dispose of the latter possibility.

From (10.12),  $z \in P(z^*)$ , and hence,  $z^* \notin M(z)$ . Since (10.10) is assumed to be false,  $z^* \notin M(x)$ . Since (10.9) is assumed to be false,  $M(z)$  cannot intersect  $A(z^*)$ . Finally, the definition of  $A(z^*)$  assures us that  $M(x)$  and  $A(z^*)$  do not overlap.

Thus, we can assume that there is an alternating chain  $C$  from  $x$  to  $z$  that misses  $M(A(z^*) + z^*)$ . If  $z^*$  does not occur in  $C$ , let  $C^*$  denote the sequence obtained by appending  $z^*$  to the end of the sequence  $C$ . Since  $C$  misses  $M(z^*)$ ,  $C^* \in C(z^*)$ , unless  $z^*$  occurs in  $C$ . But if  $z^*$  occurs in  $C$ , let  $C^*$  instead denote the subsequence of  $C$  terminating at  $z^*$ . Since  $C$  misses  $M(A(z^*))$  and since, by definition of  $A(z^*)$ ,  $z^*$  also misses  $M(A(z^*))$ ,

so does  $C^* \in \mathcal{C}(z^*)$ , contrary to the definition of  $A(z^*)$ . This contradiction shows that (10.13) is false, and thus the first part of (10.8) holds.

Next, supposing that (10.9) and (10.10) are false, we shall prove the last part of (10.8). We proceed by induction on  $t$ .

As a basis for induction, we note that the fact that  $\mathcal{C}(z^*)$  is nonempty implies that  $z^* \in D_1(x)$ , by definition of  $D_1(x)$ . Therefore, the last part of (10.8) holds when  $t = 2$ .

Suppose that Prop. 10.6 is true for integers less than  $t$ , where  $t \geq 3$ . Suppose, contrary to (10.8), that

$$(10.14) \quad z^* \notin D_{t-1}(x).$$

Thus, (10.5) and (10.6) imply

$$(10.15) \quad z \in D_{t-1}(x),$$

and (10.14) and (10.7) imply

$$(10.16) \quad z^* \in S(x) - D_{t-1}(x).$$

Note that (10.15) and (10.16) are simply (10.6) and (10.7) with  $t-1$  in place of  $t$ . Hence, by the induction hypothesis, since (10.9) and (10.10) are false, we must have  $z \notin D_t(x)$ . But this contradicts (10.6) itself. Therefore, (10.8) is proved. This proves Prop. 10.6.

We define a succession to be an ordered pair  $(u,v)$  of vertices such that  $v \in S(u)$ .

Prop. 10.7 There is an integer  $t \geq 2$  such that the number of successions  $(u,v)$  with  $u \in D_t(x)$ ,  $v \notin D_t(x)$  is at most  $p^{4/3} + \frac{2}{3}p$ .

Proof: We have the following three upper bounds on different types of successions, for  $t \geq 2$ .

(10.17) The number of successions  $(u,v)$ , with  $u \in D_t(x)$ ,  $v \in M(x)$  is at most  $2|D_t(x)|$ ;

(10.18) The number of successions  $(u,v)$  with  $v \notin D_{t-1}(x) \cup M(x)$  and  $u \in D_t(x) \cap P(v)$ , with  $|D_t(x) \cap P(v)| \leq 2t - 2$ , is at most  $(p - |P(x)| - 1 - |D_{t-1}(x)|)(2t - 2)$ ;

(10.19) The number of successions  $(u,v)$ , with  $u \in D_t(x)$ ,  $v \notin D_t(x)$  and  $|D_t(x) \cap P(v)| > 2t - 2$  is at most  $(|D_{t-1}(x) - D_t(x)|)(2t - 2 + |D_t(x) - D_{t+1}(x)|)$ .

Statement (10.17) holds because by Prop. 10.5,

$$|M(x)| = 2.$$

We obtain (10.18) by using (10.11) of Prop. 10.6, which asserts that  $v \notin P(x) + x$ . Next, we justify the bound of (10.19).

Suppose that for a given vertex  $v \notin D_t(x) \cup M(x)$ , there are more than  $2t-2$  successions of the form  $(u,v)$ , where  $u \in D_t(x)$ . Thus, we have excluded successions counted in (10.17) and (10.18). By Prop. 10.6 with  $u=z$  and  $v=z^*$ , there is a set  $M(A(v))$  of at most  $2t-2$  vertices in  $V(G)$  such that (10.8), (10.9), or (10.10) of Prop. 10.6 holds. We cannot have (10.10), since  $v \in M(x)$  has been excluded. Note that if for each  $u$ , condition (10.9) holds, then there are at most  $2t-2$  values of  $u$ , another case already excluded. Hence, there is a value of  $u$ , say  $u=u_0$ , such that  $u_0 \notin M(A(v))$ . Then we have (10.8), whence  $u_0 \in D_t(x) - D_{t+1}(x)$ , and  $v \in D_{t-1}(x)$ . Therefore, in general, since

$$u \in M(A(v)) \cup (D_t(x) - D_{t+1}(x))$$

and

$$v \in D_{t-1}(x) - D_t(x)$$

for all successions  $(u,v)$  not counted in (10.17) or (10.18), the bound of (10.19) holds.

We write

$$(10.20) \quad a_t = |D_t(x) - D_{t+1}(x)|.$$

The total number of successions  $(u,v)$  with  $u \in D_t(x)$ ,  $v \notin D_t(x)$  is, by (10.17), (10.18), and (10.19), at most

$$\begin{aligned}
 (10.21) \quad & a_{t-1}(2t-2+a_t) + (p-|P(x)|-1-|D_{t-1}(x)|)(2t-2) \\
 & \quad + 2|D_t(x)| \\
 & = a_{t-1}a_t + (p-|P(x)|-1-|D_t(x)|)(2t-2) \\
 & \quad + 2|D_t(x)|.
 \end{aligned}$$

Let

$$(10.22) \quad b = p^{4/3}$$

and let

$$(10.23) \quad c = 2p/3.$$

Suppose, by way of contradiction, that for all  $t$  satisfying  $2 \leq t \leq b/c$ ,

$$(10.24) \quad a_t a_{t-1} \geq b - ct.$$

Since

$$0 \leq (\sqrt{a_t} - \sqrt{a_{t-1}})^2 = a_t - 2\sqrt{a_t}\sqrt{a_{t-1}} + a_{t-1},$$

we have from this and (10.24),

$$(10.25) \quad \sqrt{(b-ct)} \leq \sqrt{(a_t a_{t-1})} \leq \frac{1}{2}(a_t + a_{t-1}).$$

Summing (10.25) from  $t=2$  to  $n=[b/c]$ , we get

$$\begin{aligned}
 (10.26) \quad \sum_{t=2}^n \sqrt{(b-ct)} & \leq -\frac{1}{2}a_1 - \frac{1}{2}a_n + \sum_{t=2}^n a_t \\
 & < \sum_{t=2}^n a_t.
 \end{aligned}$$

By the Fundamental Theorem of Calculus, and (10.26),

$$\begin{aligned}
 (10.27) \quad \frac{2b^{3/2}}{3c} & = \int_0^{b/c} \sqrt{(b-cx)} \, dx \\
 & = \int_0^2 \sqrt{(b-cx)} \, dx + \int_2^{b/c} \sqrt{(b-cx)} \, dx \\
 & \leq \int_0^2 \sqrt{b} \, dx + \sum_{t=2}^n \sqrt{(b-ct)} \\
 & < 2\sqrt{b} + \sum_{t=2}^n a_t.
 \end{aligned}$$

We combine (10.22), (10.23), (10.27), Prop. 10.4, and Prop. 10.2 to obtain

$$\begin{aligned}
 p &= \frac{2b^{3/2}}{3c} \\
 &< 2p^{2/3} + \sum_{t=2}^n a_t \\
 &= 2p^{2/3} + |D_2(x) - D_{n+1}(x)| \\
 &\leq 2p^{2/3} + p - |P(x) + x| \\
 &< 2p^{2/3} + \frac{2p}{3} - 14,
 \end{aligned}$$

which is clearly false for all  $p$ . Hence, there is a  $t$  such that (10.24) is false.

Fix  $t$  throughout the rest of the proof so that

$$(10.28) \quad a_t a_{t-1} < b - ct.$$

Throughout the rest of the proof, let

$$(10.29) \quad D_t(x) = D(x),$$

for this value of  $t$  satisfying (10.28).

Thus, by (10.28), (10.29), (10.21), (10.5) and Props. 10.4, 10.3, and 10.2, and by (10.22) and (10.23),

$$\begin{aligned}
 b - ct + (p - |P(x)| - 1 - |D(x)|)(2t - 2) + 2|D(x)| \\
 &< b - ct + (2t - 2)\frac{p}{3} + 2\left(\frac{2p}{3}\right) \\
 &= p^{4/3} + \frac{2p}{3}.
 \end{aligned}$$

This proves Prop. 10.7.

Prop. 10.8 For any  $u_0 \in D(x)$ , the number of successions of the form  $(u, v)$ , with  $u, v \in D(x) - u_0$ , is at least

$$(|D(x)| - 1) \left( \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 \right) - p^{4/3} - \frac{2}{3}p.$$

Proof: By Prop. 10.3, the number of successions of the form  $(u, v)$ , with  $u \in D(x) - u_0$  and  $v \neq u_0$  is at least

$$|D(x) - u_0| \left( \frac{p}{3} + \max(2k, 3p^{1/3}) - 2 - |u_0| \right).$$

By Prop. 10.7, at most  $p^{4/3} + \frac{2p}{3}$  of these are not of the form with  $v \in D(x)$ . This implies the proposition.

Prop. 10.9 There are distinct vertices  $u_0, v_0$  in  $D(x)$  such that

$$(10.30) \quad |P(u_0) \cap D(x)| \geq |P(v_0) \cap D(x)| > \frac{p}{3} - 5.$$

Proof: Let  $u_0 \in D(x)$  be a vertex having the most predecessors in  $D(x)$ . Let  $v_0$  denote a vertex in  $D(x) - u_0$  having the most predecessors in  $D(x) - u_0$ . Clearly, the first inequality of (10.30) holds. Note that  $|P(v_0) \cap D(x)|$  is at least the average number of successions per vertex in  $D(x) - u_0$ , whence, by Prop. 10.8,

$$(10.31) \quad |P(v_0) \cap D(x)| \geq \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 - \frac{p^{4/3} + 2p/3}{|D(x)| - 1}$$

By Prop. 10.3, and since  $S(x) \subseteq D(x)$ ,

$$\begin{aligned}
 (10.32) \quad |D(x)| - 1 &\geq |S(x)| - 1 \\
 &\geq \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 \\
 &> \frac{p}{3}.
 \end{aligned}$$

By (10.31) and (10.32),

$$\begin{aligned}
 |P(v_0) \cap D(x)| &> \frac{p}{3} + \max(2k, 3p^{1/3}) - 3 - 3p^{1/3} - 2 \\
 &\geq \frac{p}{3} - 5.
 \end{aligned}$$

and hence, (10.30) holds.

Remarks: Vertices  $u_0$  and  $v_0$  satisfying Prop. 10.9 are chosen, and will remain fixed throughout the rest of the proof of Theorem 10.1.

Also, for the remainder of the proof, we shall use (10.1) and Props. 10.2 and 10.3 in their weaker form, without the term involving  $p^{1/3}$ . Thus, the inequalities of (10.1) and Props. 10.1 and 10.2 will be replaced by

$$\begin{aligned}
 \Delta(G^c) &\leq \frac{p}{3} - k, \\
 |P(v)| &\geq \frac{p}{3} + 2k - 2, \\
 |S(v)| &\geq \frac{p}{3} + 2k - 2,
 \end{aligned}$$

respectively.

Prop. 10.10 We have both

$$|S(x) \cap P(u_0)| > 4k - 8,$$

and

$$|S(x) \cap P(v_0)| > 4k - 8,$$

where  $u_0$  and  $v_0$  are the fixed vertices of Prop. 10.9.

Proof: Since the proofs are identical for  $u_0$  and  $v_0$ , we shall only state the proof for  $v_0$ . By Prop. 10.2,

$$(10.33) \quad |V(G) - P(x)| \leq \frac{2p}{3} - 2k + 2.$$

Since, by Prop. 10.4,

$$D(x) \subseteq V(G) - P(x) - x,$$

(10.33) gives

$$(10.34) \quad |D(x)| \leq \frac{2p}{3} - 2k + 1.$$

By (10.34), Prop. 10.9, Prop. 10.3, and  $S(x) \subseteq D(x)$ , we have

$$\begin{aligned} |S(x) \cap P(v_0) \cap D(x)| &\geq |S(x) \cap D(x)| + \\ &\quad |P(v_0) \cap D(x)| - |D(x)| \\ &> \left(\frac{p}{3} + 2k - 2\right) + \left(\frac{p}{3} - 5\right) - \left(\frac{2p}{3} - 2k + 1\right) \\ &\geq 4k - 8. \end{aligned}$$

Prop. 10.11 Suppose that  $v_1$  and  $v_2$  satisfy

$$(10.35) \quad v_1 \in S(v_0) - M(x)$$

and

$$(10.36) \quad v_2 \in S(v_1) - M(x, v_0).$$

Then  $v_2 \notin P(x) + x$ , and either  $v_0 = v_2$  or

$$|S(v_2) - (P(x) + x)| \geq \frac{p}{3} + 2k - 8.$$

A similar statement holds when  $v_0, v_1, v_2$  are replaced by  $u_0, u_1, u_2$ , respectively.

Proof: By Prop. 10.10, since  $k > 4$ , there is a vertex

$$x_1 \in S(x) \cap P(v_0) - M(v_1, v_2) - \{v_0, v_1, v_2\},$$

and  $x_1 \in S(x) \cap P(v_0)$  guarantees that  $x_1 \notin M(x, v_0)$  and  $v_0 \in S(x_1)$ . We claim that  $x, x_1, v_0, v_1, v_2$  is an alternating chain, or that  $v_0 = v_2$ . To see this, observe first that (10.2) holds for this sequence. Suppose next, that (10.3) fails for this sequence. By the choice of  $x_1$ ,  $x_1 \notin \{x, v_0, v_1, v_2\}$ . Thus, for (10.3) to fail, either  $x \in \{v_0, v_1\}$  or  $v_0 = v_2$ . If  $x \in \{v_0, v_1\}$ , then either  $x, x_1, v_0$  or  $x, x_1, v_0, v_1$  is a closed alternating chain, whence  $(x \ x_1 \ v_0)\pi$  or  $(x \ x_1 \ v_0 \ v_1)\pi$ , respectively, embeds  $H$  into  $G$ , contrary to the assumption that  $H$  is not a subgraph of  $G$ . If  $v_0 = v_2$ , then the proposition follows immediately, since  $v_0 \notin P(x) + x$ . Suppose, finally, that  $x, x_1, v_0, v_1, v_2$  is not an alternating chain because (10.4) fails. Thus, there are vertices

$$y_1, y_2 \in \{x, x_1, v_0, v_1, v_2\}$$

such that  $y_0 \in M(y_2)$ . By definition of successors,

$y_1$  and  $y_2$  cannot be consecutive vertices of  $x, x_1, v_0, v_1, v_2$ .

Since  $v_0 \in D(x)$ , we have  $v_0 \notin M(x)$ . By the definitions of  $v_1, v_2$ , and  $x_1$ , we exclude  $v_1 \in M(x)$ ,  $v_2 \in M(x)$ ,  $v_1 \in M(x_1)$ ,  $v_2 \in M(x_1)$  and  $v_2 \in M(v_0)$ . Thus, if  $v_0 \neq v_2$ , then (10.2),

(10.3), and (10.4) hold, and so  $x, x_1, v_0, v_1, v_2$  is an alternating chain.

If  $v_2 \in P(x) + x$ , then we are done, for  $(x \ x_1 \ v_0 \ v_1 \ v_2)\pi$  would be an embedding of  $H$  into  $G$ .

If there exists a vertex

$$v_3 \in S(v_2) \cap P(x) - M(v_0, v_1, x_1),$$

then  $x, x_1, v_0, v_1, v_2, v_3, x$  is a closed alternating chain, and we are done, for  $(x \ x_1 \ v_0 \ v_1 \ v_2 \ v_3)\pi$  embeds  $H$  into  $G$ . Otherwise, all members of  $S(v_2) \cap (P(x) + x)$  lie in  $M(v_0, v_1, x_1)$ , a set of at most 6 members. The number of successors of  $v_2$  outside of  $P(x) + x$  is therefore, by Prop. 10.3, at least  $\frac{p}{3} + 2k - 2 - 6$ . This proves Prop. 10.11.

Prop. 10.12 For any two vertices  $u$  and  $v$  in  $V(G)$ , the number of predecessors of  $u$  adjacent in  $G$  to  $v$  is at least  $3k - 3$ .

Proof: By Prop. 10.2,

$$|P(u)| \geq \frac{p}{3} + 2k - 2.$$

Since

$$\Delta(G^c) \leq \frac{p}{3} - k,$$

at most  $\frac{p}{3} - k + 1$  vertices of  $P(u)$  are adjacent in  $G^c$  to  $v$  (one is equal). This leaves at least

$$\left(\frac{p}{3} + 2k - 2\right) - \left(\frac{p}{3} - k + 1\right)$$

predecessors of  $u$  adjacent to  $v$ .

Prop. 10.13 For any two vertices  $u$  and  $v$  in  $V(G)$ , the number of successors of  $u$  that are adjacent in  $G$  to  $v$  is at least  $3k - 3$ .

Proof: Use the proof of Prop. 10.12, with Prop. 10.2 replaced by Prop. 10.3.

Prop. 10.14 Suppose that  $v_1$  satisfies (10.35). Let  $y_1, z_2$  be two vertices of  $G$  such that  $y_1$  is adjacent in  $G$  to all successors of  $v_1$ . Then the number of successors of  $v_1$  outside  $P(x) + x$  that are adjacent in  $G$  to  $z_2$  and  $y_1$  is at least  $3k - 7$ .

Proof: By the first conclusion of Prop. 10.11, at most 4 successors  $v_2$  of  $v_1$  lie in  $P(x) + x$  (namely,  $M(x, v_0)$ ), whence, by Prop. 10.3, at least

$$|S(v_1) - P(x) - x| \geq \frac{p}{3} + 2k - 6$$

successors of  $v_1$  lie outside  $P(x) + x$ . At most  $\frac{p}{3} - k + 1$  of these are not adjacent in  $G$  to  $z_2$ , by (10.1). All are adjacent to  $y_1$ , by hypothesis. This leaves at least  $3k - 7$  vertices.

Prop. 10.15 If  $v_2$  satisfies condition (10.36) of Prop. 10.11 and if  $v_2 \neq v_0$ , then

$$|S(v_2) \cap P(v_0) - (P(x) + x)| > 4k - 14.$$

Proof: By Prop. 10.11,

$$(10.37) \quad |S(v_2) - (P(x) + x)| \geq \frac{p}{3} + 2k - 8.$$

By Prop. 10.4 and 10.9,

$$(10.38) \quad |P(v_0) - (P(x) + x)| \geq |P(v_0) \cap D(x)| \\ > \frac{p}{3} - 5.$$

By (10.37), (10.38), and Prop. 10.2,

$$\begin{aligned} |S(v_2) \cap P(v_0) - (P(x) + x)| &\geq |S(v_2) - (P(x) + x)| \\ &\quad + |P(v_0) - (P(x) + x)| - |V(G) - (P(x) + x)| \\ &> (\frac{p}{3} + 2k - 8) + (\frac{p}{3} - 5) - p + \frac{p}{3} + 2k - 1 \\ &= 4k - 14. \end{aligned}$$

Prop. 10.16 For appropriate vertices  $x_1 \in S(x)$ ,

$u_1 \in S(u_0) - M(x)$ ,  $v_1 \in S(v_0) - M(x)$ , and  $z_1 \in V(G)$ , one of the following six cases holds:

$$(10.39) \quad x_1 \in M(v_1);$$

$$(10.40) \quad x_1 \in M(u_1);$$

$$(10.41) \quad v_1 \in M(u_1);$$

$$(10.42) \quad M(z_1) = \{x_1, v_1\};$$

$$(10.43) \quad M(z_1) = \{x_1, u_1\};$$

$$(10.44) \quad M(z_1) = \{u_1, v_1\}.$$

Proof: Let

$$\begin{aligned} X &= (S(x) \cap (S(u_0) - M(x))) \cup (S(x) \cap (S(v_0) - M(x))) \\ &\quad \cup ((S(u_0) - M(x)) \cap (S(v_0) - M(x))), \end{aligned}$$

and let

$$X' = S(x) \cup S(u_0) \cup S(v_0) - M(x).$$

Note that the definitions of  $S(x)$ ,  $X$ , and  $X'$  imply that these sets are disjoint from  $M(x)$ . If (10.39), (10.40), and (10.41) are false, then for any vertex  $z \in X$ ,

$$M(z) \subseteq V(G) - X'.$$

If also, (10.42), (10.43), and (10.44) are false, then the sets  $M(z)$ , where  $z$  runs over  $X$ , are disjoint sets of 2 elements contained in  $V(G) - X'$ . Hence,

$$(10.45) \quad |V(G) - X'| \geq 2|X|.$$

By Prop. 10.3, we have

$$\begin{aligned} |S(x) - X| + |X| &\geq |S(x)| \\ &\geq \frac{p}{3} + 2k - 2 \\ |S(u_0) - M(x) - X| + |X| &\geq |S(u_0) - M(x)| \\ &\geq \frac{p}{3} + 2k - 4, \\ |S(v_0) - M(x) - X| + |X| &\geq |S(v_0) - M(x)| \\ &\geq \frac{p}{3} + 2k - 4, \end{aligned}$$

whence,

$$(10.46) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X| + 3|X| \geq p + 6k - 10.$$

We also have

$$(10.47) \quad |S(x) - X| + |S(u_0) - M(x) - X| + |S(v_0) - M(x) - X| + |X| = |X'|.$$

We combine (10.46) and (10.47) to obtain

$$p + 6k - 10 \leq |X'| + 2|X|,$$

whence

$$p - |X'| + 6k - 10 \leq 2|X|,$$

and so, since  $6k > 10$ ,

$$|V(G) - X'| < 2|X|,$$

in contradiction with (10.45). Prop. 10.16 follows.

In the two lemmas below, we define for vertices of  $G$ ,

$$X = \{x_0, x_1, \dots, x_n\};$$

$$V = \{v_0, v_1, \dots, v_m\};$$

$$\alpha = (x_0 \ x_1 \ \dots \ x_n);$$

$$\beta = (v_0 \ v_1 \ \dots \ v_m).$$

Let  $G + \{x_i, v_j\}$  denote the graph obtained from  $G$  by adding to  $E(G)$  the edge  $\{x_i, v_j\}$ , where  $x_i, v_j \in V(G)$ .

Lemma 10.17 Let  $x_0, x_1, \dots, x_n, x_0$  be a closed alternating chain in  $G + \{x_2, v_1\}$ , and let  $v_0, v_1, \dots, v_m, v_0$  be a closed alternating chain in  $G + \{x_1, v_2\}$ . If

$$(10.48) \quad x_1 \in M(v_1),$$

$$(10.49) \quad x \in X,$$

$$(10.50) \quad \{x_2, v_2\} \in E(G),$$

and if

$$(10.51) \quad V \cap (M(X) \cup X) = v_1,$$

then  $\beta\alpha\pi$  is an embedding of  $H$  into  $G$ .

Proof: By (10.48), there is an edge  $e'$  of  $H$  mapped by  $\pi$  to  $v_1, x_1$ . Recall that  $e$  is the edge of  $H$  not mapped into  $E(G)$  by  $\pi$  and that  $x \in \pi(e)$ .

By hypothesis and by (10.49),  $\alpha\pi$  embeds  $H$  into  $G + \{x_2, v_1\}$  and maps  $e'$  to  $\{x_2, v_1\}$ . Also, by hypothesis,

$\beta\pi$  embeds  $H - e$  into  $G + \{x_1, v_2\}$ , and maps  $e'$  to  $\{x_1, v_2\}$ . By (10.51),  $e'$  is the only edge of  $H$  affected by both  $\alpha$  and  $\beta$ . Hence,  $\beta\alpha\pi$  embeds  $H - e'$  into  $G$ , and since

$$\beta\alpha\pi(e) = \{x_2, v_2\} \in E(G),$$

by (10.50),  $\beta\alpha\pi$  embeds  $H$  into  $G$ .

Lemma 10.18 Let  $x_0, x_1, \dots, x_n, x_0$  be a closed alternating chain in  $G + \{x_2, z_1\}$ , and let  $v_0, v_1, \dots, v_m, v_0$  be a closed alternating chain in  $G + \{v_2, z_1\}$ . Also, let  $Z = \{z_1, z_2\}$ , and let  $\gamma = (z_1 \ z_2)$ . If

$$(10.52) \quad M(z_1) = \{v_1, x_1\},$$

$$(10.53) \quad x \in X,$$

$$(10.54) \quad \{v_2, z_2\}, \{x_2, z_2\} \in E(G),$$

$$(10.55) \quad (V \cup Z) \cap (M(X) \cup X) = z_1,$$

$$(10.56) \quad (X \cup Z) \cap (M(V) \cup V) = z_1,$$

and if

$$(10.57) \quad z_2 \in P(z_1),$$

then  $\gamma\beta\alpha\pi$  embeds  $H$  into  $G$ .

Proof: By (10.52), there are edges  $e_1, e_2$ , respectively, mapped by  $\pi$  to  $\{x_1, z_1\}$  and  $\{z_1, v_1\}$ . Recall that  $e$  is the only edge of  $H$  not mapped into  $E(G)$  by  $\pi$ , and that  $x \in \pi(e)$ .

By the first hypothesis, and by (10.53),  $\alpha\pi$  embeds  $H$  into  $G + \{x_2, z_1\}$ , with  $e_1$  mapped to  $\{x_2, z_1\}$ . Also,

by hypothesis,  $\beta\pi$  embeds  $H - e$  into  $G + \{v_2, z_1\}$ . By (10.55) and (10.56), no edge of  $H$  is affected by both  $\alpha$  and  $\beta$ . Therefore,  $\beta\pi$  embeds  $H$  into  $G + \{x_2, z_1\} + \{v_2, z_1\}$ , with  $e_1, e_2$  mapped to  $\{x_2, z_1\}, \{v_2, z_1\}$ , respectively. By (10.55) and (10.56),  $e_1$ , respectively  $e_2$ , is the only edge affected by both  $\alpha$  and  $\gamma$ , respectively  $\beta$  and  $\gamma$ . Hence, (10.57) ensures that  $\gamma\beta\alpha\pi$  embeds  $H - \{e_1, e_2\}$  into  $G$ , and since  $\gamma\beta\alpha\pi$  maps  $e_1$  and  $e_2$  to  $\{x_2, z_2\}$  and  $\{v_2, z_2\}$ , (10.54) ensures that  $\gamma\beta\alpha\pi$  embeds  $H$  into  $G$ .

Remark: In the six cases of Prop. 10.16 which we consider below, we shall verify the hypotheses of either Lemma 10.17 or Lemma 10.18, whence we conclude that  $H$  is a subgraph of  $G$ . We shall construct the desired alternating chains one vertex at a time. Each time another vertex is chosen, we take care to ensure that (10.51) or both (10.55) and (10.56) hold, although we shall not say so explicitly. As chains are constructed, we select vertices which satisfy (10.2), (10.3), and (10.4). Again, we do not refer to these three conditions explicitly. The other conditions of the lemmas will be verified explicitly in each of the six cases.

Suppose (10.39) holds. We shall apply Lemma 10.17 to show that  $(v_0 v_1 v_2 v_3)^n (x x_1 x_2)^n$  embeds  $H$  into  $G$ , for vertices  $v_2, v_3, x_2$  defined below. We must verify the hypotheses of the lemma. If  $x_1 \in M(v_0)$ , then we pick  $v_1$  to equal  $v_0$ . This is in compliance with (10.39). Already by (10.39), we have (10.48), and clearly we have (10.49). By Prop. 10.5, there exists  $w_1 \in V(G)$  such that

$$M(x_1) = \{v_1, w_1\}.$$

Define the set

$$T_1 = \{v_0, v_1, w_1, x, x_1\} \cup M(v_0, v_1, x, x_1).$$

Since  $x_1, v_1$ , and  $w_1$  are counted twice,

$$|T_1| \leq 10.$$

By Prop. 10.12, since

$$3k - 3 > 10 \geq |T_1|,$$

a vertex  $x_2 \in P(x) - T_1$  exists adjacent in  $G$  to  $w_1$ . Thus  $x_2$  is a successor of  $x_1$  in  $G + \{x_2, v_1\}$ , whence  $x, x_1, x_2, x$  is the desired alternating closed chain in  $G + \{x_2, v_1\}$ .

Let

$$T_2 = T_1 \cup (M(x_2) + x_2).$$

Hence,

$$|T_2| \leq 13.$$

By Prop. 10.13, since

$$3k - 3 > 13 \geq |T_2|,$$

a vertex  $v_2 \in S(v_1) - T_2$  exists adjacent in  $G$  to  $x_2$ , in

accordance with (10.50). Let

$$T_3 = T_2 \cup (M(v_2) + v_2).$$

Thus,

$$|T_3| \leq 16.$$

If  $v_2 \in P(v_0) + v_0$ , then let  $v_3 = v_2$ . Otherwise, by Prop. 10.15, since

$$4k - 14 > 16 \geq |T_3|,$$

there is a vertex

$$v_3 \in S(v_2) \cap P(v_0) - (P(x) + x) - T_3.$$

Thus,  $v_0, v_1, v_2, v_3, v_0$  is a closed alternating chain in  $G = G + \{v_2, x_1\}$ , as desired. We have chosen the vertices  $v_2, v_3, x_2$  so that (10.51) holds, whence Lemma 10.17 may be applied.

Suppose (10.40) holds. This case proceeds as does the previous case with (10.39), but with  $v$  replaced by  $u$ , and so we omit the proof.

Suppose (10.41) holds. We shall apply Lemma 10.17 to show that  $(v_0 v_1 v_2 v_3)^n (u_0 u_1 x_2 x_3 x_4)^n \pi$  embeds  $H$  into  $G$ , for vertices  $v_2, v_3, x_2, x_3, x_4$  defined below. We must verify the hypotheses of the lemma. Let

$$x_3 = x.$$

Thus, (10.49) holds. We shall apply Lemma 10.17 with  $u_1$  corresponding to  $x_1$  of the lemma, whence by (10.41), (10.48) holds. If  $v_1 \in M(u_0)$ , then let  $u_1 = u_0$  in this argument. Define the set

$$T_1 = \{u_0, v_0, x\} \cup M(u_0, u_1, v_0, v_1, x).$$

By (10.41),  $\{u_1, v_1\} \in T_1$ , and so by Prop. 10.10, since

$$4k - 8 > 13 \geq |T_1|,$$

there is a vertex  $x_4 \in S(x) \cap P(u_0) - T_1$ . By Prop. 10.5,

there exists  $w_1 \in V(G)$  such that

$$M(u_1) = \{v_1, w_1\}.$$

Define

$$T_2 = T_1 \cup (M(x_4) + x_4).$$

By Prop. 10.12, since

$$3k - 3 > 16 \geq |T_2|,$$

there is a vertex  $x_2 \in P(x) - T_2$  adjacent in  $G$  to  $w_1$ .

Observe that  $u_0, u_1, x_2, x_3, x_4, u_0$  is a closed alternating chain in  $G + \{v_1, x_2\}$ . Thus, we have the first desired chain of Lemma 10.17. Define

$$T_3 = T_2 \cup (M(x_2) + x_2).$$

By Prop. 10.13, since

$$3k - 3 > 19 \geq |T_3|,$$

a vertex  $v_2 \in S(v_1) - T_3$  exists adjacent in  $G$  to  $x_2$ , in compliance with (10.50). If  $v_2 \in P(v_0) + v_0$ , then let  $v_3 = v_2$ . Otherwise, by Prop. 10.15, since

$$4k - 14 \geq 19 \geq |T_3|,$$

a vertex

$$v_3 \in S(v_2) \cap P(v_0) - T_3 - (P(x) + x)$$

exists. Since  $v_3 \in S(v_2)$ , we have  $v_3 \notin M(v_2)$ . Thus,

$v_0, v_1, v_2, v_3, v_0$  is a closed alternating chain in  $G + \{x_1, v_2\}$  that satisfies the conditions of the first chain lemma.

By the lemma,  $H$  is a subgraph of  $G$ .

Suppose (10.42) holds. We shall apply Lemma 10.18 to show that  $(z_1 z_2)(v_0 v_1 v_2 v_3)^n (x x_1 x_2)^n$  embeds  $H$  into  $G$ . Thus, we already have (10.52) and (10.53).

Without loss of generality, we may assume in this case that  $x_1 \notin M(v_0)$ , for otherwise, we would use the argument associated with (10.39). By Prop. 10.5, and by (10.42), there is a vertex  $w_1 \in V(G)$  such that

$$M(x_1) = \{w_1, z_1\}.$$

If the vertex of  $H$  mapped to  $x_1$  lies in a triangular component of  $H$ , then  $w_1 = v_1$  and the triangle is mapped to the vertices  $x_1, v_1, z_1$ , and hence (10.39) holds.

Hence, without loss of generality, we may assume that  $w_1, x_1, z_1, v_1$  are distinct vertices in the image of a path of  $H$ . Thus, for the vertices of  $V, X$ , and  $Z$  already selected, (10.55) and (10.56) hold. Define

$$T_1 = \{v_0, w_1, x\} \cup M(v_0, v_1, x, z_1).$$

Note that

$$v_1, x_1, z_1 \in M(v_1, z_1) \subseteq T_1,$$

and that

$$|T_1| \leq 11.$$

By Prop. 10.12, since

$$3k - 3 > 11 \geq |T_1|,$$

a vertex  $x_2 \in P(x) - T_1$  exists adjacent in  $G$  to  $w_1$ . Thus,

$x, x_1, x_2, x$  is a closed alternating chain in  $G + \{x_2, z_1\}$ , in compliance with Lemma 10.18. Define

$$T_2 = T_1 \cup (M(x_2) + x_2).$$

By Prop. 10.12, since

$$3k - 3 > 14 \geq |T_2|,$$

a vertex  $z_2 \in P(z_1) - T_2$  exists adjacent in  $G$  to  $x_2$ , in accordance with (10.57) and the second part of (10.54).

Define

$$T_3 = T_2 \cup (M(z_2) + z_2).$$

By Prop. 10.5 and (10.42), there exists  $y_1 \in V(G)$  such that

$$M(v_1) = \{y_1, z_1\}.$$

Since  $y_1 \in M(v_1)$ ,  $y_1$  is adjacent in  $G$  to all successors of  $v_1 \in S(v_0) - M(x)$ . By Prop. 10.14, since

$$3k - 7 > 17 \geq |T_3|,$$

a vertex  $v_2 \notin (P(x) + x) \cup T_3$  exists adjacent to  $y_1$  and  $z_2$ .

Hence, (10.54) holds. Define

$$T_4 = T_3 \cup (M(v_2) + v_2).$$

If  $v_2 \in P(v_0) + v_0$ , then let  $v_3 = v_2$ . Otherwise, by Prop. 10.15, since

$$4k - 14 \geq 18 \geq |T_4|,$$

a vertex  $v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x)$  exists. Thus,  $v_0, v_1, v_2, v_3, v_0$  is a closed alternating chain in  $G + \{v_2, z_1\}$ , as required by Lemma 10.18. Since we have verified the requirements of the lemma,  $H$  is a subgraph of  $G$ .

Suppose (10.43) holds. This case proceeds just like the proceeding case, except that  $u$  is substituted for  $v$ . Thus, we omit the details.

Suppose (10.44) holds. We apply Lemma 10.18 to show that  $(z_1 \ z_2)(v_0 \ v_1 \ v_2 \ v_3)'(u_0 \ u_1 \ x_2 \ x_3 \ x_4)'\pi$  embeds  $H$  into  $G$ , for vertices  $v_2, v_3, x_2, x_3, x_4, z_2$  defined below. Let

$$x_3 = x.$$

Thus, we have (10.53), and by (10.44) with  $u_1$  equal to  $x_1$  of Lemma 10.18, we have (10.52). If  $v_1 \in M(u_0)$ , then let  $u_1 = u_0$  in this argument. Let

$$T_1 = \{u_0, v_0, x\} \cup M(u_0, u_1, v_0, v_1, x, z_1).$$

Note that by (10.44),

$$\{u_1, v_1\} = M(z_1) \subseteq T_1,$$

and that since  $z_1 \in M(u_1) \cap M(v_1)$  is twice counted,

$$|T_1| \leq 14.$$

By Prop. 10.10, since

$$4k - 8 \geq 14 \geq |T_1|,$$

a vertex  $x_4 \in S(x) \cap P(u_0) - T_1$  exists. By Prop. 10.5 and (10.44), there exists a vertex  $w_1 \in V(G)$  such that

$$M(u_1) = \{z_1, w_1\}.$$

Let

$$T_2 = T_1 \cup (M(x_4) + x_4).$$

If the vertex of  $H$  mapped to  $z_1$  lies in a triangular component of  $H$ , then  $v_1 = w_1$ , and the triangle is embedded onto  $z_1, u_1, v_1$ , and (10.41) holds. Hence, without loss of generality, we may assume that  $w_1, u_1, z_1, v_1$  are distinct successive vertices in the image of a path of  $H$ . Thus, for the vertices already selected, (10.55) and (10.56) hold.

By Prop. 10.12, since

$$3k - 3 > 17 \geq |T_2|,$$

a vertex  $x_2 \in P(x) - T_2$  exists adjacent in  $G$  to  $w_1$ .

Observe that  $u_0, u_1, x_2, x_3, x_4, u_0$  is thus a closed alternating chain in  $G + \{x_2, z_1\}$ . Thus, we have the first of the alternating chains of Lemma 10.18. Define

$$T_3 = T_2 \cup (M(x_2) + x_2).$$

By Prop. 10.12, since

$$3k - 3 > 20 \geq |T_3|,$$

a vertex  $z_2 \in P(z_1) - T_3$  exists adjacent in  $G$  to  $x_2$ .

Hence, the second part of (10.54) holds, and (10.57) holds. By Prop. 10.5, and by (10.44), there is a vertex  $y_1 \in V(G)$  such that

$$M(v_1) = \{y_1, z_1\}.$$

Since  $y_1 \in M(v_1)$ ,  $y_1$  is adjacent in  $G$  to all successors of  $v_1 \in S(v_0) - M(x)$ . Let

$$T_4 = T_3 \cup M(z_2).$$

By Prop. 10.14, since

$$3k - 7 > 19 \geq |T_4 - \{v_0, x, y_1\}|,$$

a vertex

$$v_2 \notin (P(x) + x) \cup (T_4 - \{v_0, x, y_1\})$$

exists adjacent in  $G$  to  $y_1$  and  $z_2$ . This verifies (10.54).

If  $v_2 \in P(v_0) + v_0$ , then let  $v_3 = v_2$ . Otherwise, since

$$4k - 14 \geq 19 \geq |T_4 - (M(v_0) + v_0)|,$$

Prop. 10.15 implies that there is a vertex

$$v_3 \in S(v_2) \cap P(v_0) - T_4 - (P(x) + x).$$

Thus,  $v_0, v_1, v_2, v_3, v_0$  is a closed alternating chain in  $G + \{v_2, z_1\}$ . The other conditions of Lemma 10.18 may

be readily verified. Thus,  $H$  is a subgraph of  $G$ .

This completes the proof of Theorem 10.1.

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