AN IMPROVEMENT ON BROOKS’ THEOREM

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Abstract. We prove that \( \chi(G) \leq \max\{\omega(G), \Delta_2(G), \frac{5}{6}(\Delta(G) + 1)\} \) for every graph \( G \) with \( \Delta(G) \geq 3 \). Here \( \Delta_2 \) is the parameter introduced by Stacho that gives the largest degree that a vertex \( v \) can have subject to the condition that \( v \) is adjacent to a vertex whose degree is at least as large as its own. This upper bound generalizes both Brooks’ Theorem and the Ore-degree version of Brooks’ Theorem.

1. Introduction

Brooks’ Theorem [1] gives an upper bound on a graph’s chromatic number in terms of its maximum degree and clique number.

Brooks’ Theorem. Every graph with \( \Delta \geq 3 \) satisfies \( \chi \leq \max\{\omega, \Delta\} \).

In [6] Stacho introduced the graph parameter \( \Delta_2 \) as the largest degree that a vertex \( v \) can have subject to the condition that \( v \) is adjacent to a vertex whose degree is at least as large as its own. He proved that for any graph \( G \), the bound \( \chi(G) \leq \Delta_2(G) + 1 \) holds. Moreover, he proved that for any fixed \( t \geq 3 \), the problem of determining whether or not \( \chi(G) \leq \Delta_2(G) \) for graphs with \( \Delta_2(G) = t \) is NP-complete. It is tempting to think that an analogue of Brooks’ Theorem like the following holds for \( \Delta_2 \).

Tempting Thought. There exists \( t \) such that every graph with \( \Delta_2 \geq t \) satisfies \( \chi \leq \max\{\omega, \Delta_2\} \).

Unfortunately, using Lovász’s \( \vartheta \) parameter [2] which can be computed in polynomial time and has the property that \( \omega(G) \leq \vartheta(G) \leq \chi(G) \) we see immediately that if \( P \neq NP \), then the tempting thought cannot hold for any \( t \). In the final section we give a construction showing that this is indeed the case whether or not \( P \neq NP \). However, if we limit how far from \( \Delta + 1 \) our upper bound can stray, we can get a generalization of Brooks’ Theorem involving \( \Delta_2 \).

Main Theorem. Every graph with \( \Delta \geq 3 \) satisfies

\[ \chi \leq \max\{\omega, \Delta_2, \frac{5}{6}(\Delta + 1)\} \].

In addition to generalizing Brooks’ Theorem, this also generalizes the Ore-degree version of Brooks’ Theorem as introduced by Kierstead and Kostochka in [3] and improved in [5].

Definition 1. The Ore-degree of an edge \( xy \) in a graph \( G \) is \( \theta(xy) = d(x) + d(y) \). The Ore-degree of a graph \( G \) is \( \theta(G) = \max_{xy \in E(G)} \theta(xy) \).

Note that \( \Delta_2 \leq \left\lceil \frac{\theta}{2} \right\rceil \leq \Delta \). In [5] the following bound was proved. The graph \( O_5 \) exhibited in [3] shows that the \( \theta \geq 10 \) condition is best possible.
Ore Version of Brooks’ Theorem. Every graph with $\theta \geq 10$ satisfies $\chi \leq \max \{\omega, \left\lceil \frac{\theta}{2} \right\rceil \}$. 

Proof. Suppose the theorem is false and choose a counterexample $G$ minimizing $|G|$. Plainly, $G$ is vertex critical. Thus $\delta(G) \geq \chi(G) - 1$. In particular, $\theta(G) \geq \delta(G) + \Delta(G) \geq \chi(G) + \Delta(G) - 1$. Hence $\Delta(G) \leq \chi(G)$. Applying the Main Theorem, we conclude $\Delta(G) \leq \chi(G) \leq \frac{5}{6}(\Delta(G) + 1)$ and hence $\Delta(G) \leq 5$. But then $\theta(G) = 10$ and we must have $\chi(G) \geq 6$. Now applying Brooks’ Theorem gets the desired contradiction. 

In fact, a similar proof shows that a whole spectrum of generalizations hold.

Definition 2. For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lceil \max_{xy \in E(G)} (1 - \epsilon) \min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rceil.$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lceil \frac{\theta}{2} \right\rceil$ and $\Delta_0 = \Delta_2$.

Theorem 1. For every $0 < \epsilon \leq 1$, there exists $t_\epsilon$ such that every graph with $\Delta_\epsilon \geq t_\epsilon$ satisfies $\chi \leq \max \{\omega, \Delta_\epsilon\}$.

It would be interesting to determine, for each $\epsilon$, the smallest $t_\epsilon$ that works in Theorem 1. In the final section we give a simple construction showing that $t_\epsilon \geq 1 + \frac{2}{\epsilon}$. The Main Theorem implies $t_\epsilon < \frac{6}{\epsilon}$.

2. REPHRASING THE PROBLEM

Definition 3. For a graph $G$ and $r \geq 0$, let $G^{\geq r}$ be the subgraph of $G$ induced on the vertices of degree at least $r$ in $G$. Let $H(G) = G^{\geq \chi(G)}$.

We can rewrite the definition of $\Delta_2$ as

$$\Delta_2(G) = \min \{r \geq 0 \mid G^{\geq r} \text{ is edgeless}\} - 1.$$

In particular we have the following.

Observation. For any graph $G$, $\chi(G) > \Delta_2(G)$ if and only if $H(G)$ is edgeless.

This observation will allow us to prove our upper bound without worrying about $\Delta_2$.

3. PROVING THE BOUND

We will use part of an algorithm of Mozhan [4]. The following is a generalization of his main lemma.

Definition 4. Let $G$ be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-partitioned coloring of $G$ we mean a proper coloring of $G$ of the form

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{ar_a}, L_{a_1}, L_{a_2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each $L_{ij}$ is a color class.
Lemma 2. Let $G$ be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of $G$ pick one (call it $\pi$) minimizing

$$\sum_{i=1}^a \left\| G \left[ \bigcup_{j=1}^{r} L_{ij} \right] \right\| .$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r} L_{ij}$ and let $Z_i(x)$ be the component of $x$ in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

Proof. Let $1 \leq i \leq a$ such that $d_{Z_i(x)}(x) = r_i$. Put $Z_i = Z_i(x)$.

First suppose that $\Delta(Z_i) > r_i$. Take $y \in V(Z_i)$ with $d_{Z_i}(y) > r_i$ closest to $x$ and let $x_1x_2 \cdots x_t$ be a shortest $x - y$ path in $Z_i$. Plainly, for $k < t$, each $x_k$ hits exactly one vertex in each color class besides its own. Thus we may recolor $x_k$ with $\pi(x_{k+1})$ for $k < t$ and $x_t$ with $\pi(x_1)$ to produce a new $\chi(G)$-coloring of $G$ (this can be seen as a generalized Kempe chain). But we've moved a vertex $(x_t)$ of degree $r_i + 1$ out of $U_i$ while moving in a vertex $(x_1)$ of degree $r_i$ violating the minimality condition on $\pi$. This is a contradiction.

Thus $\Delta(Z_i) \leq r_i$. But $\chi(Z_i) = r_i + 1$, so Brooks’ Theorem implies that $Z_i$ is complete if $r_i \geq 3$ and $Z_i$ is an odd cycle if $r_i = 2$. \hfill $\square$

Definition 5. We call $v \in V(G)$ low if $d(v) = \chi(G) - 1$ and high otherwise.

Note that in Lemma 2, if $d_{Z_i(x)}(x) = r_i$ then we can swap $x$ with any other $y \in Z_i(x)$ by changing $\pi$ so that $x$ is colored with $\pi(y)$ and $y$ is colored with $\pi(x)$ to get another minimal $\chi(G)$-coloring of $G$.

Lemma 3. Assume the same setup as Lemma 2 and that $x$ is low. If $i \neq j$ such that $r_i \geq r_j \geq 3$ and a low vertex $w \in U_i \cap N(x)$ is adjacent to a low vertex $z \in U_j \cap N(x)$, then the low vertices in $(U_i \cup U_j) \cap N(x)$ are all universal in $G[(U_i \cup U_j) \cap N(x)]$.

Proof. Suppose $i \neq j$ and a low vertex $w \in U_i \cap N(x)$ is adjacent to a low vertex $z \in U_j \cap N(x)$. Swap $x$ with $w$ to get a new minimal $\chi(G)$-coloring of $G$. Since $w$ is low and adjacent to $z \in U_j \cap N(x)$, $w$ is joined to $U_j \cap N(x)$ by Lemma 2. Similarly $z$ is joined to $U_i \cap N(x)$. But now every low vertex in $U_i \cap N(x)$ is adjacent to the low vertex $z \in U_j \cap N(x)$ and is hence joined to $U_j \cap N(x)$. Similarly, every low vertex in $U_j \cap N(x)$ is joined to $U_i \cap N(x)$. Since both $U_i \cap N(x)$ and $U_j \cap N(x)$ induce cliques in $G$, the proof is complete. \hfill $\square$

Theorem 4. Fix $k \geq 2$ and let $G$ be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - k$. If $\Delta(G) + 1 \geq 6k$ and $\mathcal{H}(G)$ is edgeless then $G = K_{\chi(G)}$.

Proof. Suppose that $\Delta(G) + 1 \geq 6k$ and $\mathcal{H}(G)$ is edgeless. Since $\Delta(G) + 1 \geq 6k$ we have $\chi(G) \geq 5k$ and thus we can find $r_1, \ldots, r_{k+1}$ such that $r_1, r_2 \geq k + 1$, $r_i \geq 3$ for each $i \geq 3$ and $\sum_{i=1}^{k+1} r_i = \chi(G) - 1$. Note that $r_i \geq 3$ for each $i$ since $k \geq 2$.

Put $a = k + 1$. Of all $(r_1, r_2, \ldots, r_a)$-partitioned colorings of $G$, pick one (call it $\pi$) minimizing
\begin{align*}
\sum_{i=1}^{a} \left\| G \left[ \bigcup_{j=1}^{r_i} L_{ij} \right] \right\| .
\end{align*}

Remember that \{x\} is a singleton color class in the coloring. Throughout the proof we refer to a coloring that minimizes the above function as a \textit{minimal} coloring. Put \(U_i = \bigcup_{j=1}^{r_i} L_{ij}\) and let \(C_i = \pi(U_i)\) (the colors used on \(U_i\)). For a minimal coloring \(\gamma\) of \(G\), let \(Z_{x,i}(x)\) be the component of \(x\) in \(G[\{x\} \cup \gamma^{-1}(C_i)]\). Note that \(Z_i(x) = Z_{x,i}(x)\).

First suppose \(x\) is high. Since \(a > k\) we have \(1 \leq i \leq a\) such that \(d_{Z_i(x)}(x) = r_i\). Thus \(Z_i(x)\) is complete. Since \(\mathcal{H}(G)\) is edgeless, each vertex in \(Z_i(x) - x\) must be low. Hence we can swap \(x\) with a low vertex in \(U_i\) to get another minimal \(\chi(G)\) coloring. Thus we may assume that \(x\) is low. Consider the following algorithm.

(1) Put \(q_0(y) = 0\) for each \(y \in V(G)\).
(2) Put \(x_0 = x\), \(\pi_0 = \pi\), \(p_0 = 1\) and \(i = 0\).
(3) Pick a low vertex \(x_{i+1} \in Z_{\pi_i, p_i}(x_i) - x_i\) minimizing \(q_i(x_{i+1})\). Swap \(x_{i+1}\) with \(x_i\). Let \(\pi_{i+1}\) be the resulting coloring.
(4) If there exists \(d \in \{3, \ldots, a\} - \{p_i\}\) with \(\left| V(Z_{\pi_{i+1}, d}(x_{i+1})) \right| \cap \bigcup_{j=1}^{i} x_j = 0\), then let \(p_{i+1} = d\). Otherwise pick \(p_{i+1} \in \{1, 2\} - \{p_i\}\).
(5) Put \(q_i(x_i) = q_i(x_{i+1}) + 1\).
(6) Put \(q_{i+1} = q_i\).
(7) Put \(i = i + 1\).
(8) Goto (3).

Since \(G\) is finite we have a smallest \(t\) such that for \(p = 1\) or \(p = 2\) with \(p \neq p_{t-1}\) we have \(\left| \{y \in V(Z_{\pi_t, p_t}(x_t)) - \{x_t\} \mid q_t(y) = 1\} \right| = k\). Let \(x_{t_1}, \ldots, x_{t_k}\) with \(t_1 < t_2 \cdots < t_k\) be the vertices in \(V(Z_{\pi_{t_1}, p_{t_1}}(x_{t_1})) - \{x_t\} \) with \(q_t(x_{t_1}) = 1\).

Swap \(x_t\) with \(x_{t_1}\) and note that \(x_{t_1}\) is low and adjacent to each of \(x_{t_1+1}, \ldots, x_{t_k+1}\). Also note that \(\{x_{t_1+1}, \ldots, x_{t_k+1}\}\) induces a clique in \(G\) since all those vertices are in \(U_p\). By the condition in step (4) we see that \(\{p_{t_1+1}, p_{t_2+1}, \ldots, p_{t_k+1}\} = \{1, \ldots, a\} - \{p\}\). Thus the low vertices in \(\bigcup_{i \neq p} \pi_t^{-1}(C_i) \cap N(x_{t_1})\) are universal in \(G\left[ \bigcup_{i \neq p} \pi_t^{-1}(C_i) \cap N(x_{t_1}) \right]\) by Lemma 3. Also since \(x_{t_1}\) is low and is joined to \(\pi_t^{-1}(C_i) \cap N(x_{t_1})\) for each \(i \neq p\), again applying Lemma 3 we get that the low vertices in \(N(x_{t_1}) \cup \{x_{t_1}\}\) are universal in \(G[N(x_{t_1}) \cup \{x_{t_1}\}]\).

Put \(F = G[N(x_{t_1}) \cup \{x_{t_1}\}]\) and let \(S\) be the set of high vertices in \(F\). Note that \(|F| = \chi(G)|S| \leq k + 1\) since \(\mathcal{H}(G)\) is edgeless. We will show that \(F\) is complete. Since all the low vertices in \(F\) are universal in \(F\), it will suffice to show that \(|S| \leq 1\).

Suppose otherwise that we have different \(w, z \in S\). Then \(w\) and \(z\) are non-adjacent since \(H(G)\) is edgeless. Color \(G - F\) with \(\chi(G) - 1\) colors. This leaves a list assignment \(L\) on \(F\) with \(|L(v)| \geq d_F(v) - k\) for each \(v \in V(F)\). Thus \(|L(w)| + |L(z)| \geq d_F(w) + d_F(z) - 2k \geq 2(|F| - |S|) - 2k \geq 2(\Delta(G) - 2k) - 2k = 2\Delta(G) - 6k\). Since \(\Delta(G) + 1 \geq 6k\) and \(k \geq 2\), we have \(|L(w)| + |L(z)| \geq 2\Delta(G) - 6k \geq \Delta(G) + 1 - k\). Hence we have \(c \in L(w) \cap L(z)\). Color both \(w\) and \(z\) with \(c\) to get a new list assignment \(F' = F - \{w, z\}\). Put \(A = G[S - \{w, z\}]\). Then we can complete the coloring to \(A\) since for any \(v \in V(A)\) we have \(|L'(v)| \geq d_{F'}(v) - k \geq d_A(v) + |F| - |S| - k \geq d_A(v) + \Delta(G) - 3k \geq d_A(v) + 1\). Let \(J\) be the
resulting list assignment on \( B = F - S \). Since the vertices in \( B \) are all low and they each have a pair of neighbors that received the same color \((w \text{ and } z)\) we have \( |J(v)| \geq d_B(v) + 1 \) for each \( v \in V(B) \). Hence we can complete the \( \chi(G) - 1 \) coloring to all of \( F \). This is a contradiction.

The \( k = 1 \) case was dealt with in [5]. The proof is similar but complicated by having to deal with odd cycles instead of just cliques. There the following was proved.

**Corollary 5.** \( K_{\chi(G)} \) is the only critical graph \( G \) with \( \chi(G) \geq \Delta(G) \geq 6 \) such that \( \mathcal{H}(G) \) is edgeless.

Now the proof of the Main Theorem is almost immediate.

**Proof of Main Theorem.** Suppose the theorem is false and choose a counterexample \( G \) minimizing \( |G| \). Plainly, \( G \) is vertex critical. Let \( k = \Delta(G) + 1 - \chi(G) \). Note that \( k \geq 1 \) by Brooks' Theorem. Since \( \chi(G) > \Delta_2(G) \), we know by our observation above that \( \mathcal{H}(G) \) is edgeless. Also, since \( \chi(G) > \frac{5}{6}(\Delta(G) + 1) \) we have \( \Delta(G) + 1 - k = \chi(G) \geq 5k + 1 \). If \( k \geq 2 \) we have a contradiction by Theorem 4. If \( k = 1 \) we have a contradiction by Corollary 5. \( \square \)

### 4. A simple construction

Let \( F_n \) be the graph formed from the disjoint union of \( K_n - xy \) and \( K_{n-1} \) by joining \( \lfloor \frac{n-1}{2} \rfloor \) vertices of the \( K_{n-1} \) to \( x \) and the other \( \lceil \frac{n-1}{2} \rceil \) vertices of the \( K_{n-1} \) to \( y \). It is easily verified that for \( n \geq 4 \) we have \( \chi(F_n) = n > \omega(F_n) \), \( \Delta(F_n) = \lfloor \frac{n-1}{2} \rfloor + n - 2 \) and \( \mathcal{H}(G) \) is edgeless (and nonempty). Moreover, \( \Delta_\epsilon(F_n) = \lfloor (1 - \epsilon)(n - 1) + \epsilon \left( \left\lfloor \frac{n-1}{2} \right\rfloor + n - 2 \right) \rfloor = \lfloor n - 1 - \epsilon + \epsilon \left\lfloor \frac{n-1}{2} \right\rfloor \rfloor \). For \( 0 < \epsilon \leq 1 \), choose \( n_\epsilon \in \mathbb{N} \) maximal such that \( \left\lfloor \frac{n_\epsilon}{2} \right\rfloor < 1 + \frac{1}{\epsilon} \). Then \( \Delta_\epsilon(F_{n_\epsilon}) = n_\epsilon - 1 \). Hence in Theorem 1, we must have \( t_\epsilon \geq n_\epsilon \). By maximality, \( n_\epsilon \) must be odd. Thus

\[
n_\epsilon = \begin{cases} 
1 + 2 \left\lfloor \frac{1}{\epsilon} \right\rfloor & \text{if } \frac{1}{\epsilon} \in \mathbb{N} \\
3 + 2 \left\lfloor \frac{1}{\epsilon} \right\rfloor & \text{if } \frac{1}{\epsilon} \notin \mathbb{N}.
\end{cases}
\]

In particular, \( t_\epsilon \geq n_\epsilon \geq 1 + \frac{2}{\epsilon} \) for all \( 0 < \epsilon \leq 1 \). Additionally, we see that \( t_0 \) does not exist; that is, the tempting thought is false.

**References**


