AN IMPROVEMENT ON BROOKS' THEOREM

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ABSTRACT. We prove that $\chi(G) \leq \max\left\{\omega(G), \Delta_2(G), \frac{5}{6}(\Delta(G)+1)\right\}$ for every graph G with $\Delta(G) \geq 3$. Here Δ_2 is the parameter introduced by Stacho that gives the largest degree that a vertex v can have subject to the condition that v is adjacent to a vertex whose degree is at least as large as its own. This upper bound generalizes both Brooks' Theorem and the Ore-degree version of Brooks' Theorem.

1. Introduction

Brooks' Theorem [1] gives an upper bound on a graph's chromatic number in terms of its maximum degree and clique number.

Brooks' Theorem. *Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.*

In [6] Stacho introduced the graph parameter Δ_2 as the largest degree that a vertex v can have subject to the condition that v is adjacent to a vertex whose degree is at least as large as its own. He proved that for any graph G, the bound $\chi(G) \leq \Delta_2(G) + 1$ holds. Moreover, he proved that for any fixed $t \geq 3$, the problem of determining whether or not $\chi(G) \leq \Delta_2(G)$ for graphs with $\Delta_2(G) = t$ is *NP*-complete. It is tempting to think that an analogue of Brooks' Theorem like the following holds for Δ_2.

Tempting Thought. There exists t such that every graph with $\Delta_2 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_2\}$.

Unfortunately, using Lovász's ϑ parameter [2] which can be computed in polynomial time and has the property that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$ we see immediately that if $P \neq NP$, then the tempting thought cannot hold for any t. In the final section we give a construction showing that this is indeed the case whether or not $P \neq NP$. However, if we limit how far from $\Delta + 1$ our upper bound can stray, we can get a generalization of Brooks' Theorem involving Δ_2.

Main Theorem. *Every graph with $\Delta \geq 3$ satisfies*

$$\chi \leq \max\left\{\omega, \Delta_2, \frac{5}{6}(\Delta+1)\right\}.$$

In addition to generalizing Brooks' Theorem, this also generalizes the Ore-degree version of Brooks' Theorem as introduced by Kierstead and Kostochka in [3] and improved in [5].

Definition 1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) = d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) = \max_{xy \in E(G)} \theta(xy)$.

Note that $\Delta_2 \leq \left\lfloor \frac{\theta}{2} \right\rfloor \leq \Delta$. In [5] the following bound was proved. The graph O_5 exhibited in [3] shows that the $\theta \geq 10$ condition is best possible.

Ore Version of Brooks' Theorem. *Every graph with $\theta \geq 10$ satisfies $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Plainly, G is vertex critical. Thus $\delta(G) \geq \chi(G) - 1$. In particular, $\theta(G) \geq \delta(G) + \Delta(G) \geq \chi(G) + \Delta(G) - 1$. Hence $\Delta(G) \leq \chi(G)$. Applying the Main Theorem, we conclude $\Delta(G) \leq \chi(G) \leq \frac{5}{6}(\Delta(G) + 1)$ and hence $\Delta(G) \leq 5$. But then $\theta(G) = 10$ and we must have $\chi(G) \geq 6$. Now applying Brooks' Theorem gets the desired contradiction. □

In fact, a similar proof shows that a whole spectrum of generalizations hold.

Definition 2. For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lfloor \max_{xy \in E(G)} (1 - \epsilon) \min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rfloor.$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lfloor \frac{\theta}{2} \right\rfloor$ and $\Delta_0 = \Delta_2$.

Theorem 1. *For every $0 < \epsilon \leq 1$, there exists t_ϵ such that every graph with $\Delta_\epsilon \geq t_\epsilon$ satisfies*

$$\chi \leq \max\{\omega, \Delta_\epsilon\}.$$

It would be interesting to determine, for each ϵ, the smallest t_ϵ that works in Theorem 1. In the final section we give a simple construction showing that $t_\epsilon \geq 1 + \frac{2}{\epsilon}$. The Main Theorem implies $t_\epsilon < \frac{6}{\epsilon}$.

2. Rephrasing the problem

Definition 3. For a graph G and $r \geq 0$, let $G^{\geq r}$ be the subgraph of G induced on the vertices of degree at least r in G. Let $\mathcal{H}(G) = G^{\geq \chi(G)}$.

We can rewrite the definition of Δ_2 as

$$\Delta_2(G) = \min\left\{r \geq 0 \mid G^{\geq r} \text{ is edgeless}\right\} - 1.$$

In particular we have the following.

Observation. For any graph G, $\chi(G) > \Delta_2(G)$ if and only if $\mathcal{H}(G)$ is edgeless.

This observation will allow us to prove our upper bound without worrying about Δ_2.

3. Proving the bound

We will use part of an algorithm of Mozhan [4]. The following is a generalization of his main lemma.

Definition 4. Let G be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-*partitioned coloring* of G we mean a proper coloring of G of the form

$$\left\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\right\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

Lemma 2. *Let G be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \dots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \dots, r_a)$-partitioned colorings of G pick one (call it π) minimizing*

$$\sum_{i=1}^{a} \left\| G\left[\bigcup_{j=1}^{r_i} L_{ij}\right]\right\|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

Proof. Let $1 \leq i \leq a$ such that $d_{Z_i(x)}(x) = r_i$. Put $Z_i = Z_i(x)$.

First suppose that $\Delta(Z_i) > r_i$. Take $y \in V(Z_i)$ with $d_{Z_i}(y) > r_i$ closest to x and let $x_1 x_2 \cdots x_t$ be a shortest $x - y$ path in Z_i. Plainly, for $k < t$, each x_k hits exactly one vertex in each color class besides its own. Thus we may recolor x_k with $\pi(x_{k+1})$ for $k < t$ and x_t with $\pi(x_1)$ to produce a new $\chi(G)$-coloring of G (this can be seen as a generalized Kempe chain). But we've moved a vertex (x_t) of degree $r_i + 1$ out of U_i while moving in a vertex (x_1) of degree r_i violating the minimality condition on π. This is a contradiction.

Thus $\Delta(Z_i) \leq r_i$. But $\chi(Z_i) = r_i + 1$, so Brooks' Theorem implies that Z_i is complete if $r_i \geq 3$ and Z_i is an odd cycle if $r_i = 2$. □

Definition 5. We call $v \in V(G)$ *low* if $d(v) = \chi(G) - 1$ and *high* otherwise.

Note that in Lemma 2, if $d_{Z_i(x)}(x) = r_i$ then we can *swap* x with any other $y \in Z_i(x)$ by changing π so that x is colored with $\pi(y)$ and y is colored with $\pi(x)$ to get another minimal $\chi(G)$-coloring of G.

Lemma 3. *Assume the same setup as Lemma 2 and that x is low. If $i \neq j$ such that $r_i \geq r_j \geq 3$ and a low vertex $w \in U_i \cap N(x)$ is adjacent to a low vertex $z \in U_j \cap N(x)$, then the low vertices in $(U_i \cup U_j) \cap N(x)$ are all universal in $G[(U_i \cup U_j) \cap N(x)]$.*

Proof. Suppose $i \neq j$ and a low vertex $w \in U_i \cap N(x)$ is adjacent to a low vertex $z \in U_j \cap N(x)$. Swap x with w to get a new minimal $\chi(G)$-coloring of G. Since w is low and adjacent to $z \in U_j \cap N(x)$, w is joined to $U_j \cap N(x)$ by Lemma 2. Similarly z is joined to $U_i \cap N(x)$. But now every low vertex in $U_i \cap N(x)$ is adjacent to the low vertex $z \in U_j \cap N(x)$ and is hence joined to $U_j \cap N(x)$. Similarly, every low vertex in $U_j \cap N(x)$ is joined to $U_i \cap N(x)$. Since both $U_i \cap N(x)$ and $U_j \cap N(x)$ induce cliques in G, the proof is complete. □

Theorem 4. *Fix $k \geq 2$ and let G be a vertex critical graph with $\chi(G) \geq \Delta(G) + 1 - k$. If $\Delta(G) + 1 \geq 6k$ and $\mathcal{H}(G)$ is edgeless then $G = K_{\chi(G)}$.*

Proof. Suppose that $\Delta(G) + 1 \geq 6k$ and $\mathcal{H}(G)$ is edgeless. Since $\Delta(G) + 1 \geq 6k$ we have $\chi(G) \geq 5k$ and thus we can find $r_1, \dots, r_{k+1}$ such that $r_1, r_2 \geq k + 1$, $r_i \geq 3$ for each $i \geq 3$ and $\sum_{i=1}^{k+1} r_i = \chi(G) - 1$. Note that $r_i \geq 3$ for each i since $k \geq 2$.

Put $a = k + 1$. Of all $(r_1, r_2, \dots, r_a)$-partitioned colorings of G, pick one (call it π) minimizing

$$\sum_{i=1}^{a}\left\|G\left[\bigcup_{j=1}^{r_i} L_{ij}\right]\right\|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Throughout the proof we refer to a coloring that minimizes the above function as a *minimal* coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $C_i = \pi(U_i)$ (the colors used on U_i). For a minimal coloring γ of G, let $Z_{\gamma,i}(x)$ be the component of x in $G[\{x\} \cup \gamma^{-1}(C_i)]$. Note that $Z_i(x) = Z_{\pi,i}(x)$.

First suppose x is high. Since $a > k$ we have $1 \leq i \leq a$ such that $d_{Z_i(x)}(x) = r_i$. Thus $Z_i(x)$ is complete. Since $\mathcal{H}(G)$ is edgeless, each vertex in $Z_i(x) - x$ must be low. Hence we can swap x with a low vertex in U_i to get another minimal $\chi(G)$ coloring. Thus we may assume that x is low. Consider the following algorithm.

(1) Put $q_0(y) = 0$ for each $y \in V(G)$.
(2) Put $x_0 = x$, $\pi_0 = \pi$, $p_0 = 1$ and $i = 0$.
(3) Pick a low vertex $x_{i+1} \in Z_{\pi_i,p_i}(x_i) - x_i$ minimizing $q_i(x_{i+1})$. Swap x_{i+1} with x_i. Let π_{i+1} be the resulting coloring.
(4) If there exists $d \in \{3, \ldots, a\} - \{p_i\}$ with $\left|V(Z_{\pi_{i+1},d}(x_{i+1})) \cap \bigcup_{j=1}^{i} x_j\right| = 0$, then let $p_{i+1} = d$. Otherwise pick $p_{i+1} \in \{1, 2\} - \{p_i\}$.
(5) Put $q_i(x_i) = q_i(x_{i+1}) + 1$.
(6) Put $q_{i+1} = q_i$.
(7) Put $i = i + 1$.
(8) Goto (3).

Since G is finite we have a smallest t such that for $p = 1$ or $p = 2$ with $p \neq p_{t-1}$ we have $|\{y \in V(Z_{\pi_t,p}(x_t)) - \{x_t\} \mid q_t(y) = 1\}| = k$. Let $x_{t_1}, \ldots, x_{t_k}$ with $t_1 < t_2 \cdots < t_k$ be the vertices in $V(Z_{\pi_t,p}(x_t)) - \{x_t\}$ with $q_t(x_{t_j}) = 1$.

Swap x_t with x_{t_1} and note that x_{t_1} is low and adjacent to each of $x_{t_1+1}, \ldots, x_{t_k+1}$. Also note that $\{x_{t_1+1}, \ldots, x_{t_k+1}\}$ induces a clique in G since all those vertices are in U_p. By the condition in step (4) we see that $\{p_{t_1+1}, p_{t_2+1}, \ldots, p_{t_k+1}\} = \{1, \ldots, a\} - \{p\}$. Thus the low vertices in $\bigcup_{i \neq p} \pi_t^{-1}(C_i) \cap N(x_{t_1})$ are universal in $G\left[\bigcup_{i \neq p} \pi_t^{-1}(C_i) \cap N(x_{t_1})\right]$ by Lemma 3. Also since x_t is low and is joined to $\pi_t^{-1}(C_i) \cap N(x_{t_1})$ for each $i \neq p$, again applying Lemma 3 we get that the low vertices in $N(x_{t_1}) \cup \{x_{t_1}\}$ are universal in $G[N(x_{t_1}) \cup \{x_{t_1}\}]$.

Put $F = G[N(x_{t_1}) \cup \{x_{t_1}\}]$ and let S be the set of high vertices in F. Note that $|F| = \chi(G)$ and $|S| \leq k + 1$ since $\mathcal{H}(G)$ is edgeless. We will show that F is complete. Since all the low vertices in F are universal in F, it will suffice to show that $|S| \leq 1$.

Suppose otherwise that we have different $w, z \in S$. Then w and z are non-adjacent since $\mathcal{H}(G)$ is edgeless. Color $G - F$ with $\chi(G) - 1$ colors. This leaves a list assignment L on F with $|L(v)| \geq d_F(v) - k$ for each $v \in V(F)$. Thus $|L(w)| + |L(z)| \geq d_F(w) + d_F(z) - 2k \geq 2(|F| - |S|) - 2k \geq 2(\Delta(G) - 2k) - 2k = 2\Delta(G) - 6k$. Since $\Delta(G) + 1 \geq 6k$ and $k \geq 2$, we have $|L(w)| + |L(z)| \geq 2\Delta(G) - 6k \geq \Delta(G) + 1 - k$. Hence we have $c \in L(w) \cap L(z)$. Color both w and z with c to get a new list assignment L' on $F' = F - \{w, z\}$. Put $A = G[S - \{w, z\}]$. Then we can complete the coloring to A since for any $v \in V(A)$ we have $|L'(v)| \geq d_{F'}(v) - k \geq d_A(v) + |F| - |S| - k \geq d_A(v) + \Delta(G) - 3k \geq d_A(v) + 1$. Let J be the resulting list assignment on $B = F - S$. Since the vertices in B are all low and they each have a pair of neighbors that received the same color (w and z) we have $|J(v)| \geq d_B(v) + 1$ for each $v \in V(B)$. Hence we can complete the $\chi(G) - 1$ coloring to all of F. This is a contradiction. □

The $k = 1$ case was dealt with in [5]. The proof is similar but complicated by having to deal with odd cycles instead of just cliques. There the following was proved.

Corollary 5. *$K_{\chi(G)}$ is the only critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.*

Now the proof of the Main Theorem is almost immediate.

Proof of Main Theorem. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Plainly, G is vertex critical. Let $k = \Delta(G) + 1 - \chi(G)$. Note that $k \geq 1$ by Brooks' Theorem. Since $\chi(G) > \Delta_2(G)$, we know by our observation above that $\mathcal{H}(G)$ is edgeless. Also, since $\chi(G) > \frac{5}{6}(\Delta(G) + 1)$ we have $\Delta(G) + 1 - k = \chi(G) \geq 5k + 1$. If $k \geq 2$ we have a contradiction by Theorem 4. If $k = 1$ we have a contradiction by Corollary 5. □

4. A simple construction

Let F_n be the graph formed from the disjoint union of $K_n - xy$ and K_{n-1} by joining $\left\lfloor \frac{n-1}{2} \right\rfloor$ vertices of the K_{n-1} to x and the other $\left\lceil \frac{n-1}{2} \right\rceil$ vertices of the K_{n-1} to y. It is easily verified that for $n \geq 4$ we have $\chi(F_n) = n > \omega(F_n)$, $\Delta(F_n) = \left\lceil \frac{n-1}{2} \right\rceil + n - 2$ and $\mathcal{H}(G)$ is edgeless (and nonempty). Moreover, $\Delta_\epsilon(F_n) = \left\lfloor (1-\epsilon)(n-1) + \epsilon\left(\left\lceil \frac{n-1}{2} \right\rceil + n - 2\right) \right\rfloor = \left\lfloor n - 1 - \epsilon + \epsilon \left\lceil \frac{n-1}{2} \right\rceil \right\rfloor$. For $0 < \epsilon \leq 1$, choose $n_\epsilon \in \mathbb{N}$ maximal such that $\left\lceil \frac{n_\epsilon - 1}{2} \right\rceil < 1 + \frac{1}{\epsilon}$. Then $\Delta_\epsilon(F_{n_\epsilon}) = n_\epsilon - 1$. Hence in Theorem 1, we must have $t_\epsilon \geq n_\epsilon$. By maximality, n_ϵ must be odd. Thus

$$n_\epsilon = \begin{cases} 1 + \frac{2}{\epsilon} & \text{if } \frac{1}{\epsilon} \in \mathbb{N} \\ 3 + 2\left\lfloor \frac{1}{\epsilon} \right\rfloor & \text{if } \frac{1}{\epsilon} \notin \mathbb{N}. \end{cases}$$

In particular, $t_\epsilon \geq n_\epsilon \geq 1 + \frac{2}{\epsilon}$ for all $0 < \epsilon \leq 1$. Additionally, we see that t_0 does not exist; that is, the tempting thought is false.

References

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