An improvement on Brooks’ theorem

Landon Rabern

landon.rabern@gmail.com

February 28, 2011
An improvement on Brooks’ theorem

Landon Rabern

Outline

1 Introduction

2 Rephrasing the problem

3 Solving the rephrased problem
   Kierstead and Kostochka’s proof
   Problem solved
   Proof outline
   Mozhan’s lemma
   The recoloring algorithm

4 A spectrum of generalizations
   Generalizing maximum degree
   The generalized bound
   What about $\Delta_0$?

5 Further improvements
Theorem (Brooks 1941)

*Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$."

**Definition**

The *Ore-degree* of an edge $xy$ in a graph $G$ is $\theta(xy) = d(x) + d(y)$. The *Ore-degree* of a graph $G$ is $\theta(G) = \max_{xy \in E(G)} \theta(xy)$.

- every graph satisfies $\left\lfloor \frac{\theta}{2} \right\rfloor \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left\lfloor \frac{\theta}{2} \right\rfloor + 1$
An improvement on Brooks’ theorem

Landon Rabern

Introduction
Rephrasing the problem
Solving the rephrased problem
A spectrum of generalizations
Further improvements

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ satisfies $\chi \leq \max \{\omega, \lfloor \frac{\theta}{2} \rfloor \}$.

Kierstead and Kostochka [2] conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta = 9$, $\omega = 4$ and $\chi = 5$.

Figure: $O_5$, a counterexample with $\theta = 9$. 
Rephrasing the problem

- let $G$ be a critical graph with $\chi = \left\lfloor \frac{\theta}{2} \right\rfloor + 1$
- it follows that $G$ must satisfy $\theta \leq 2\chi - 1$
- if $\Delta < \chi$ we are done by Brooks’ theorem
- otherwise we have $\theta \geq \delta + \Delta \geq 2\chi - 1$ giving $\theta = 2\chi - 1$
- thus, $\chi = \Delta$ and no two vertices of max degree in $G$ can be adjacent
Definition

Let $G$ be a graph. The low vertex subgraph $L(G)$ is the graph induced on the vertices of degree $\chi(G) - 1$. The high vertex subgraph $H(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

Problem

Prove that $K_{\Delta(G)+1}$ is the only critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 6$ such that $H(G)$ is edgeless.
Kierstead and Kostochka’s proof

- take a minimal counterexample $G$ and use minimality to prove some structural properties
- $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$ by a result of Stiebitz [7]
- since $\mathcal{H}(G)$ is edgeless it has at most as many vertices as $\mathcal{L}(G)$ has components
- apply Alon and Tarsi’s algebraic list coloring theorem to an auxiliary bipartite graph
- do some counting and get a contradiction
- it only works for $\theta \geq 12$
In [5] we solved the problem in a more general fashion.

**Theorem (Rabern 2010)**

\[ K_{\Delta(G)+1} \text{ is the only critical graph } G \text{ with } \chi(G) \geq \Delta(G) \geq 6 \]

and \( \omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2 \).

Setting \( \omega(\mathcal{H}(G)) = 1 \) proves the conjecture of Kierstead and Kostochka.
Proof outline

- take a minimal counterexample $G$ and use minimality to prove some structural properties
- run a carefully chosen recoloring algorithm to construct a large "dense" subgraph $H$
- inductively $\Delta - 1$ color $G - H$
- use minimality of $G$ to show that the $\Delta - 1$ coloring can be completed to $H$
Partitioned colorings

Definition

Let $G$ be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-partitioned coloring of $G$ we mean a proper coloring of $G$ of the form

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$ 

Here $\{x\}$ is a singleton color class and each $L_{ij}$ is a color class.
Mozhan’s Lemma

Lemma (Mozhan 1983)

Let $G$ be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of $G$ pick one minimizing

$$\sum_{i=1}^{a} \left| E\left(G \left[ \bigcup_{j=1}^{r_i} L_{ij} \right] \right) \right| .$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of $x$ in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$. 
The recoloring algorithm

- take a \((\lceil \frac{\Delta - 1}{2} \rceil, \lceil \frac{\Delta - 1}{2} \rceil)\)-partitioned coloring minimizing the above function
- prove that we may assume that \(x\) is a low vertex
- by Mozhan’s lemma, the neighborhood of \(x\) in each part induces a clique or an odd cycle
- swap \(x\) with a low vertex \(x_1\) in the right part
- swap \(x_1\) with a low vertex \(x_2\) in the left part
- continue swapping back and forth until you wrap around
- use the fact that you wrapped around to show that there are many edges between the two induced cliques (odd cycles)
- we have now constructed the desired large “dense” subgraph
Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lfloor \max_{xy \in E(G)} (1 - \epsilon) \min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rfloor.$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lfloor \frac{\theta}{2} \right\rfloor$. 
The generalized bound

**Theorem (Rabern 2010)**

For every $0 < \epsilon \leq 1$, there exists $t_\epsilon$ such that every graph with $\Delta_\epsilon \geq t_\epsilon$ satisfies

$$\chi \leq \max\{\omega, \Delta_\epsilon\}.$$

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each $\epsilon$, the smallest $t_\epsilon$ that works
- that $t_1 = 3$ is smallest is Brooks’ theorem
- the graph $O_5$ shows that $t_\epsilon = 6$ is smallest for $\frac{1}{2} \leq \epsilon < 1$
- we will see below that if $P \neq NP$, then $t_0$ does not exist and hence $t_\epsilon \to \infty$ as $\epsilon \to 0$
An improvement on Brooks' theorem

Landon Rabern

Introduction
Rephrasing the problem
Solving the rephrased problem
A spectrum of generalizations
Generalizing maximum degree
The generalized bound
What about $\Delta_0$?
Further improvements

What about $\Delta_0$?

- the above proofs only work for $\epsilon > 0$
- what happens when $\epsilon = 0$?
- the parameter $\Delta_0$ has already been investigated by Stacho [6] under the name $\Delta_2$

Definition (Stacho 2001)

For a graph $G$ define

$$\Delta_2(G) = \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$
Facts about $\Delta_2$

- $\Delta_2 = \Delta_0$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_2 + 1$
- for any fixed $t \geq 3$, the problem of determining whether or not $\chi(G) \leq \Delta_2(G)$ for graphs with $\Delta_2(G) = t$ is $NP$-complete (see [6])
A tempting thought

There exists $t$ such that every graph with $\Delta_2 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_2\}$.

- unfortunately, the tempting thought cannot hold for any $t$ if $P \neq NP$
- to show this, we use Lovász’s $\vartheta$ parameter [1] which can be approximated in polynomial time and has the property that $\omega(G) \leq \vartheta(G) \leq \chi(G)$
A polynomial-time algorithm

- assume the tempting thought holds for some \( t \geq 3 \)
- take any arbitrary graph with \( \Delta_2 \geq t \)
- first, compute \( \Delta_2 \) in polynomial time
- second, compute \( x \) such that \( x - \frac{1}{2} < \vartheta < x + \frac{1}{2} \) in polynomial time
  - if \( x \geq \Delta_2 + \frac{1}{2} \), then \( \chi \geq \vartheta > \Delta_2 \) and hence \( \chi = \Delta_2 + 1 \)
  - if \( x < \Delta_2 + \frac{1}{2} \), then \( \omega \leq \vartheta < \Delta_2 + 1 \), and hence \( \omega \leq \Delta_2 \)
- now, \( \chi \leq \max\{\omega, \Delta_2\} \leq \Delta_2 \)
- we just gave a polynomial time algorithm to determine whether or not \( \chi \leq \Delta_2 \) for graphs with \( \Delta_2 \geq t \)
- this is impossible unless \( P = NP \)
What we can prove about $\Delta_0$ (aka $\Delta_2$)

**Theorem (Rabern 2010)**

*Every graph with $\Delta \geq 3$ satisfies*

\[ \chi \leq \max \left\{ \omega, \Delta_2, \frac{5}{6}(\Delta + 1) \right\}. \]

- the proof uses a recoloring algorithm similar to the above
- actually, all the above results about $\Delta_\varepsilon$ follow from this result
In joint work with Kostochka and Stiebitz [3] similar techniques were used to improve the bounds further. We give some highlights.

**Theorem (Kostochka, Rabern and Stiebitz 2010)**

*Every graph with $\theta \geq 8$, except $O_5$, satisfies $\chi \leq \max \{\omega, \lfloor \theta/2 \rfloor \}$.***

**Theorem (Kostochka, Rabern and Stiebitz 2010)**

*Every graph satisfies*

$$\chi \leq \max \left\{ \omega, \Delta, \frac{3}{4}(\Delta + 2) \right\}.$$
An improvement on Brooks' theorem

Introduction

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

Further improvements


A.V. Kostochka, L. Rabern, and M. Stiebitz, *Graphs with chromatic number close to maximum degree*, Submitted.


