Improving Brooks' theorem

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Outline

A prison problem

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You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?

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- plainly, if there is a group of six inmates who have all fought one another, then you cannot complete your task
- is this simple necessary condition sufficient?

Greedy coloring

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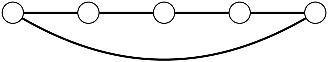

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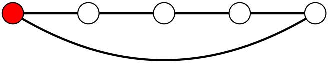

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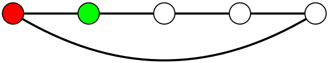

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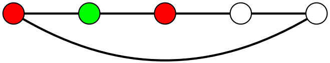

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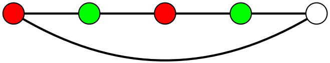

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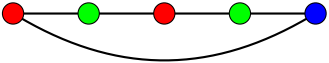

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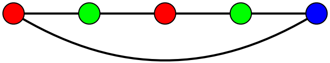

- if G has maximum degree k, then v_i has at most k colored neighbors, so greedy coloring uses at most $k+1$ colors

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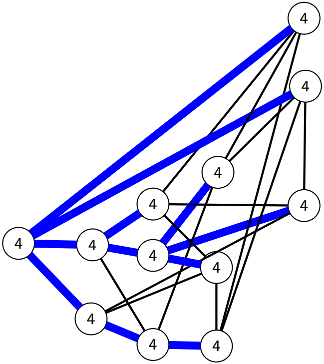
Theorem (Brooks 1941)

Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



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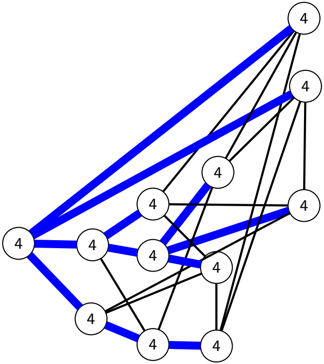

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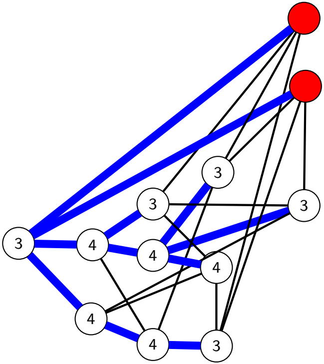

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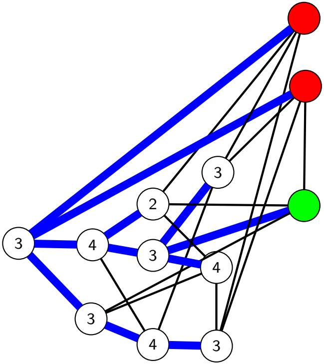

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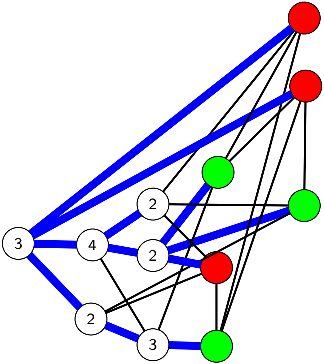

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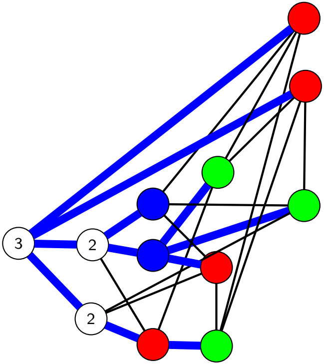

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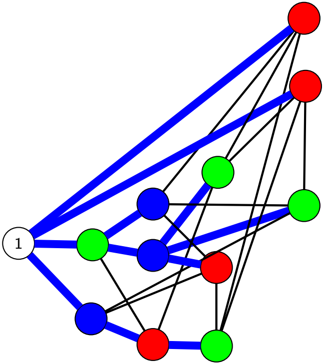

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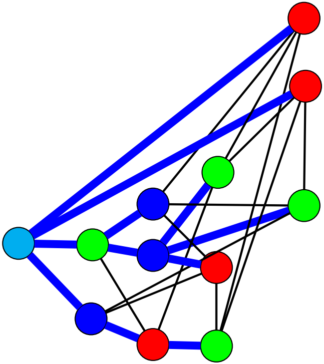

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The Ore-degree

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- every graph satisfies $\left\lfloor \frac{\theta}{2} \right\rfloor \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left\lfloor \frac{\theta}{2} \right\rfloor + 1$

Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ *satisfies* $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.

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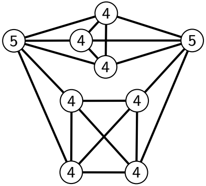

Figure: O_5, a counterexample with $\theta = 9$.

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Problem

Prove that $K_{\Delta(G)+1}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.

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- using these it is basically just a counting argument
- unfortunately, it only works for $\Delta \geq 7$

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Theorem (Rabern 2010)

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- setting $\omega(\mathcal{H}(G)) = 1$ proves Kierstead and Kostochka's conjecture
- equivalently, as long as there is no group of six inmates who have all fought one another, you (the warden) can complete your inmate-cell-assignment task

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- run the following recoloring algorithm to construct H

Partitioned colorings

Definition

Let G be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a *$(r_1, \ldots, r_a)$-partitioned coloring* of G we mean a proper coloring of G of the form

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

Mozhan's Lemma

Lemma (Mozhan 1983)

Let G be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of G pick one minimizing

$$\sum_{i=1}^{a} \left\| G\left[\bigcup_{j=1}^{r_i} L_{ij}\right] \right\|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i := \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

The recoloring algorithm

- take a $\left(\left\lfloor\frac{\Delta-1}{2}\right\rfloor, \left\lceil\frac{\Delta-1}{2}\right\rceil\right)$-partitioned coloring minimizing the above function

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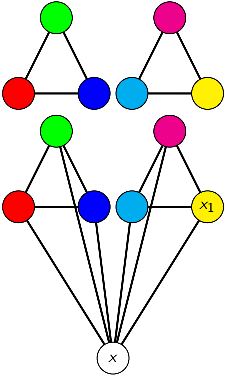
- take a $\left(\left\lfloor \frac{\Delta - 1}{2} \right\rfloor, \left\lceil \frac{\Delta - 1}{2} \right\rceil\right)$-partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex
- by Mozhan's lemma, the neighborhood of x in each part induces a clique or an odd cycle

The recoloring algorithm



- swap x with a low vertex x_1 in the right part

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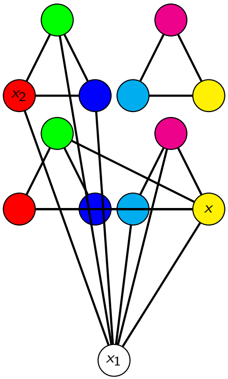

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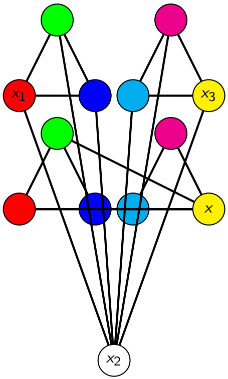

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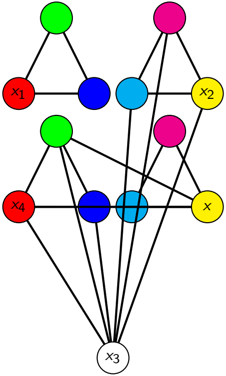

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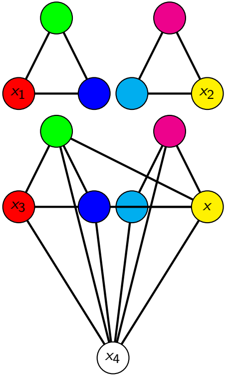

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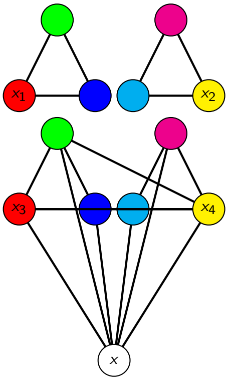

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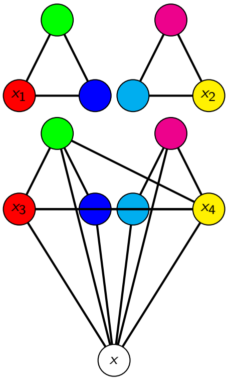

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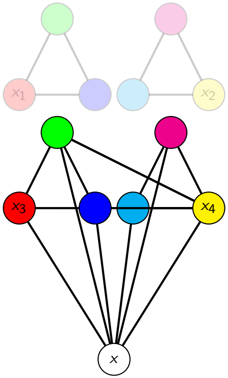

- use the fact that you wrapped around to show that there are many edges between the two cliques
- we have now constructed the desired large "dense" subgraph

Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lfloor \max_{xy \in E(G)} (1-\epsilon)\min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rfloor .$$

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Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lfloor \frac{\theta}{2} \right\rfloor$.

The generalized bound

Theorem (Rabern 2010)

For every $0 < \epsilon \leq 1$, there exists t_ϵ such that every graph with $\Delta_\epsilon \geq t_\epsilon$ satisfies $\chi \leq \max\{\omega, \Delta_\epsilon\}$.

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- the proof uses a recoloring algorithm similar to the above

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- best known general bounds, $\frac{2}{\epsilon} + 1 \leq t_\epsilon \leq \frac{4}{\epsilon} + 2$

The lower bound on t_ϵ

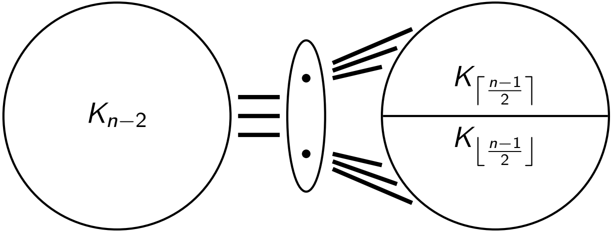

Figure: The graph O_n.

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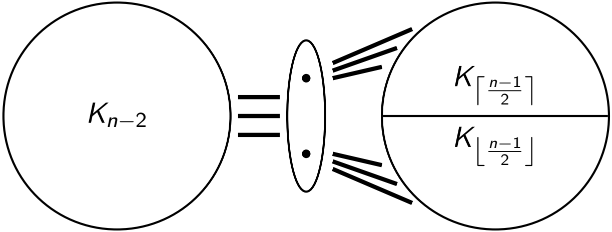

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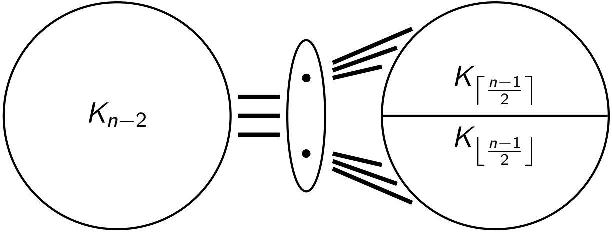

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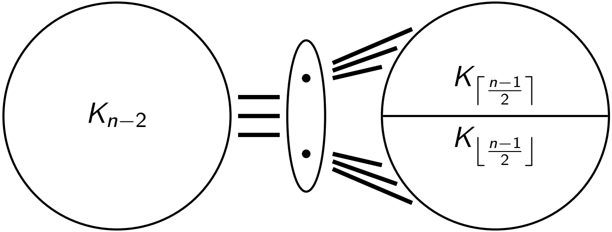

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Definition (Stacho 2001)

For a graph G define

$$\Delta_0(G) := \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$

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- for any fixed $t \geq 3$, the problem of determining whether or not $\chi(G) \leq \Delta_0(G)$ for graphs with $\Delta_0(G) = t$ is *NP*-complete (Stacho 2001)

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- we use Lovász's ϑ parameter which can be appoximated in polynomial time and has the property that $\omega(G) \leq \vartheta(G) \leq \chi(G)$

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- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_0$ for graphs with $\Delta_0 \geq t$
- this is impossible unless $P{=}NP$

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- actually, all the above results about Δ_ϵ follow from this result

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Every graph satisfies

$$\chi \leq \max\left\{\omega, \Delta_0, \frac{3}{4}(\Delta + 2)\right\}.$$

Conjecture

Every graph satisfies

$$\chi \leq \max\left\{\omega, \Delta_0, \frac{2\Delta + 5}{3}\right\}.$$

The examples O_n above show that this would be tight.

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