# Painting Squares in $\Delta^2 - 1$ Shades

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**Abstract**

Cranston and Kim conjectured that if $G$ is a connected graph with maximum degree $\Delta$ and $G$ is not a Moore Graph, then $\chi_\ell(G^2) \leq \Delta^2 - 1$; here $\chi_\ell$ is the list chromatic number. We prove their conjecture; in fact, we show that this upper bound holds even for online list chromatic number.

## 1 Introduction

Graph coloring has a long history of upper bounds on a graph's chromatic number $\chi$ in terms of its maximum degree $\Delta$. A greedy coloring (in any order) gives the trivial upper bound $\chi \leq \Delta+1$. In 1941, Brooks [4] proved the following strengthening: If $G$ is a graph with maximum degree $\Delta \geq 3$ and clique number $\omega \leq \Delta$, then $\chi \leq \Delta$. In 1977, Borodin and Kostochka [3] conjectured the following further strengthening.

**Conjecture 1** (Borodin-Kostochka Conjecture [3])**.** *If $G$ is a graph with $\Delta \geq 9$ and $\omega \leq \Delta - 1$, then $\chi \leq \Delta - 1$.*

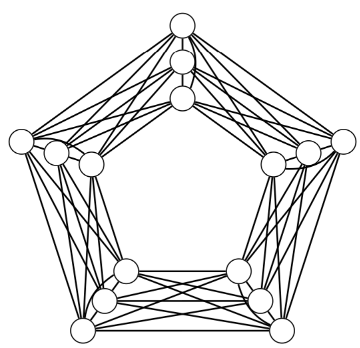

Figure 1: The hypothesis $\Delta \geq 9$ in the Borodin–Kostochka Conjecture is best possible.

If true, this conjecture is best possible in two senses. First, the condition $\Delta \geq 9$ cannot be dropped (or even weakened), as shown by the following graph (See Figure 1). Let $D_i$ induce a triangle for each $i \in \{1, \ldots, 5\}$; if $|i - j| \equiv 1 \pmod 5$, then add all edges between vertices of

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$D_i$ and $D_j$. This yields an 8-regular graph on 15 vertices with clique number 6 and chromatic number 8; it would be a counterexample to the conjecture if we weakened the hypothesis $\Delta \geq 9$. Similarly, even if we require $\omega \leq \Delta - 2$, we cannot conclude that $\chi \leq \Delta - 2$, as is show by the join of a clique and a 5-cycle. For each $\Delta \in \{3, \ldots, 8\}$, examples are known [7, 13] where $\omega \leq \Delta - 1$ but $\chi = \Delta$. Kostochka has informed us that already in 1977 when he and Borodin posed Conjecture 1, they believed the following stronger "list version" was true; however they omitted this version from their paper, and it appeared in print [7] only in 2013. We define the list chromatic number, denoted $\chi_\ell$, in Section 2 below.

**Conjecture 2** (Borodin-Kostochka Conjecture (list version))**.** *If $G$ is a graph with $\Delta \geq 9$ and $\omega \leq \Delta - 1$, then $\chi_\ell \leq \Delta - 1$.*

The purpose of this paper is to prove the following conjecture of Cranston and Kim [5]. In fact, we will prove this conjecture in the more general setting of online list coloring. It is easy to show, as we do below, that Conjecture 2 implies Conjecture 3.

**Conjecture 3** (Cranston-Kim [5])**.** *If $G$ is a connected graph with maximum degree $\Delta \geq 3$, and $G$ is not a Moore graph, then $\chi_\ell(G^2) \leq \Delta^2 - 1$.*

A Moore graph is a $\Delta$-regular graph $G$ on $\Delta^2 + 1$ vertices such that $G^2 = K_{\Delta^2+1}$; the sole example when $\Delta = 3$ is the Petersen graph. Hoffman and Singleton [12] famously proved that Moore graphs exist only when $\Delta \in \{2, 3, 7, 57\}$. When $\Delta \in \{2, 3, 7\}$ Moore graphs exist and are known to be unique, and when $\Delta = 57$ no Moore graph is known.

In 2008 Cranston and Kim [5] proved Conjecture 3 when $\Delta = 3$, and suggested that a similar but more detailed approach might prove the whole conjecture. As mentioned above, it is easy to show that Conjecture 3 is implied by Conjecture 2. The key is the following easy lemma at the end of [5]: If $G$ is connected and is not a Moore graph and $G$ has maximum degree $\Delta \geq 3$, then $G^2$ has clique number at most $\Delta^2 - 1$. The proof is short once we have a result of Erdős, Fajtlowicz, and Hoffman [11] stating that a "near-Moore graph", i.e., a $\Delta$-regular graph such that $G^2 = K_{\Delta^2}$, exists only when $\Delta = 2$. For details, see the start of the proof of the Main Theorem.

We note that recently Conjecture 3 was generalized to higher powers. Let $M$ denote the maximum possible degree when a graph of maximum degree $k$ is raised to the $d$th power, i.e., vertices are adjacent in $G^d$ if they are distance at most $d$ in $G$. Miao and Fan [14] conjectured that if $G$ is connected and $G^d$ is not $K_{M+1}$, then we can save one color over the bound given by Brooks Theorem, i.e., $\chi(G^d) \leq M - 1$. This was proved by Bonamy and Bousquet [2] in the more general context of online list coloring.

The following conjecture is due to Wegner [20], in the late 1970's. It is a less well-known variant of Wegner's analogous conjecture when the class $\mathcal{G}_k$ is restricted to planar graphs.

**Conjecture 4** (Wegner [20])**.** *For each fixed $k$, let $\mathcal{G}_k$ denote the class of all graphs with maximum degree at most $k$ and form $\mathcal{G}_k^2$ by taking the square $G^2$ of each graph $G$ in $\mathcal{G}_k$. Now $\max_{H \in \mathcal{G}_k^2} \chi(H) = \max_{H \in \mathcal{G}_k^2} \omega(H)$.*

Wegner in fact posed a more general conjecture for all powers of $\mathcal{G}_k$; however, here we restrict our attention to Conjecture 4, specifically for small values of $k$. For each $H \in \mathcal{G}_k^2$, we have $\Delta(H) \leq k^2$, so Brooks' Theorem implies that $\chi(H) \leq k^2$ unless some component of $H$ is $K_{k^2+1}$. For $k = 1$ Wegner's Conjecture is trivial. For $k \in \{2, 3, 7\}$ it is easy; in each case $\mathcal{G}_k$ contains a Moore graph $G$, and letting $H = G^2$, we have $H = K_{k^2+1}$, so $\chi(H) = \omega(H) = k^2 + 1$. Thus, the first two open cases of Conjecture 4 are $k = 4$ and $k = 5$. Our Main Theorem shows that every graph $G$ in $\mathcal{G}_4$ satisfies $\chi_\ell(G^2) \leq 15$ and every graph $G$ in $\mathcal{G}_5$ satisfies $\chi_\ell(G^2) \leq 24$. Matching lower bounds are shown in Figure 2: we have $G_1 \in \mathcal{G}_4$ with $\omega(G_1^2) = 15$ and $G_2 \in G_5$ with $\omega(G_2^2) = 24$. Both graphs were discovered by Elspas ([9] and p. 14 of [15]) and are known to be the unique graphs $G$ with $\Delta \in \{4, 5\}$ and $G^2 = K_{\Delta^2-1}$. This confirms Wegner's Conjecture when $k = 4$ and $k = 5$.

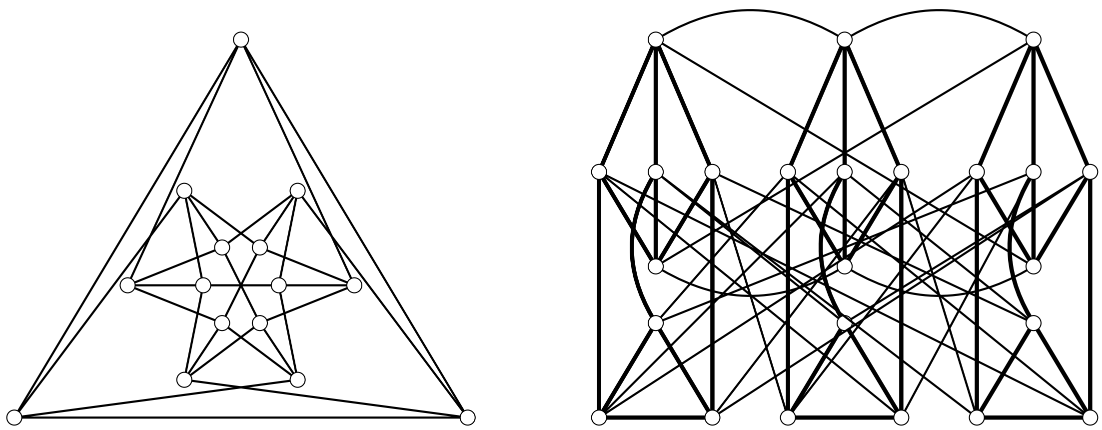


Figure 2: On the left is a 4-regular graph $G_1$ such that $G_1^2 = K_{15}$.
On the right is a 5-regular graph $G_2$ such that $G_2^2 = K_{24}$.

Rather than coloring, or even list coloring, this paper is about *online list coloring*, a generalization introduced in 2009 by Schauz [16] and Zhu [22], and the *online list chromatic number*, $\chi_p$, also called the *paint number*. We give the definition in Section 2, but for now if you are unfamiliar with $\chi_p$, you can substitute $\chi_\ell$ (or even $\chi$) and the Main Theorem remains true. Our main result is the following.

**Main Theorem.** *If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $G$ is not the Peterson graph, the Hoffman-Singleton graph, or a Moore graph with $\Delta = 57$, then $\chi_p(G^2) \leq \Delta^2 - 1$.*

We conclude this section with the following conjecture, which generalizes our Main Theorem as well as Conjecture 2.

**Conjecture 5** (Borodin-Kostochka Conjecture (online list coloring version))**.** *If $G$ is a graph with $\Delta \geq 9$ and $\omega \leq \Delta - 1$, then $\chi_p \leq \Delta - 1$.*

The structure of the paper is as follows. In Section 2 we give background and definitions. In Section 3, we prove the Main Theorem, subject to a number of lemmas about forbidden subgraphs in a minimal counterexample. In Section 4 we prove the lemmas that we deferred in Section 3. Finally, in Section 5, we generalize the online list chromatic number to the Alon-Tarsi number, and extend our Main Theorem to that setting.

## 2 Preliminaries

Here we give definitions and background. Most of our terminology and notation is standard. We write $A \setminus B$ for $A \cap \overline{B}$. If $H$ is a subgraph of $G$, then $G \setminus H$ means $G[V(G) \setminus V(H)]$, that is $G$ with the vertices of $H$ deleted. For graphs $G$ and $H$, the *join* $G \vee H$ is formed from the disjoint union of $G$ and $H$ by adding all edges with one endpoint in each of $V(G)$ and $V(H)$. For any undefined terms, see West [21].

A *list size assignment* $f : V(G) \to \mathbb{Z}^+$ assigns to each vertex in $G$ a list size. An *$f$-assignment* $L$ assigns to each vertex $v$ a subset of the positive integers $L(v)$ with $|L(v)| = f(v)$. An $L$-coloring is a proper coloring $\phi$ such that $\phi(v) \in L(v)$ for all $v$. A graph $G$ is *$f$-list colorable* (or *$f$-choosable*) if $G$ has an $L$-coloring for every $f$-assignment $L$. In particular, we are interested in the case where $f(v) = k$ for all $v$ and some constant $k$. The *list chromatic number* of $G$ or *choice number* of $G$, denoted $\chi_\ell(G)$, or simply $\chi_\ell$ when $G$ is clear from context, is the minimum $k$ such that $G$ is $k$-choosable. List coloring was introduced by Vizing [19] and Erdős, Rubin, and Taylor [10] in the 1970s. Both groups proved the following extension of Brooks' Theorem. If $G$ is a graph with maximum degree $\Delta \geq 3$ and clique number $\omega \leq \Delta$, then $\chi_\ell \leq \Delta$.

The next idea we need came about 30 years later. In 2009, Schauz [16] and Zhu [22] independently introduced the notion of online list coloring. This is a variation of list coloring, in which the list sizes are determined (each vertex $v$ gets $f(v)$ colors), but the lists themselves are provided online by an adversary.

We consider a game between two players, *Lister* and *Painter*. In round 1, Lister presents the set of all vertices whose lists contain color 1. Painter must then use color 1 on some independent subset of these vertices, and cannot change this set in the future. In each subsequent round $k$, Lister chooses some subset of the uncolored vertices to contain color $k$ in their lists, and Painter chooses some independent subset of these vertices to receive color $k$. Painter wins if he succeeds in painting all vertices. Alternatively, Lister wins if he includes a vertex $v$ among those presented on each of $f(v)$ rounds, but Painter never paints $v$.

A graph is *online $k$-list colorable* (or *$k$-paintable*) if Painter can win whenever $f(v) = k$ for all $v$. The minimum $k$ such that a graph $G$ is online $k$-list colorable is its *online list chromatic number*, or *paint number*, denoted $\chi_p$. A graph is *$d_1$-paintable* if it is paintable when $f(v) = d(v) - 1$ for each vertex $v$. In [6], the authors introduced *$d_1$-choosable* graphs, which are the list-coloring analogue. Interest in $d_1$-paintable graphs owes to the fact that none can be induced subgraphs of a minimal graph with maximum degree $\Delta$ that is not $(\Delta - 1)$-paintable. In particular, if $G$ is a minimal counterexample to our Main Theorem, then $G^2$ contains no induced $d_1$-paintable subgraph.

**Lemma 1.** *Let $G$ be a graph with maximum degree $\Delta$ and $H$ be an induced subgraph of $G$ that is $d_1$-paintable. If $G \setminus H$ is $(\Delta - 1)$-paintable, then $G$ is $(\Delta - 1)$-paintable.*

*Proof.* Let $G$ and $H$ satisfy the hypotheses. We give an algorithm for Painter to win the online coloring game when $f(v) = \Delta - 1$ for all $v$. Painter will simulate playing two games simultaneously: a game on $G \setminus H$ with $f(v) = \Delta - 1$ and a game on $H$ with $f(v) = d_H(v) - 1$. Let $S_k$ denote the set of vertices presented by Lister on round $k$. Painter first plays round $k$ of the game on $G \setminus H$, pretending that Lister listed the vertices $S_k \setminus H$. Let $I_k$ denote the independent set of these that Painter chooses to color $k$.

Let $S'_k = (S_k \cap V(H)) \setminus I_k$, the vertices of $H$ that are in $S_k$ and have no neighbor in $I_k$. Now Painter plays round $k$ of the game on $H$, pretending that Lister listed $S'_k$. Each vertex in $V(G\setminus H)$ will clearly be listed $\Delta-1$ times. Consider a vertex $v$ in $V(H)$. It will appear in $S_k \setminus S'_k$ for at most $d_G(v) - d_H(v)$ rounds. So $v$ will appear in $S'_k$ for at least $(\Delta - 1) - (d_G(v) - d_H(v)) \geq d_H(v) - 1$ rounds. Now Painter will win both simulated games, and thus win the actual game on $G$. □

When the graph $G$ in Lemma 1 is a square, we immediately get that $G \setminus H$ is $(\Delta-1)$-paintable, as we note in the next lemma.

**Lemma 2.** *Let $G$ be a graph with maximum degree $\Delta$ and let $H$ be an induced subgraph of $G^2$. If $H$ is $d_1$-paintable, then $G^2$ is $d_1$-paintable. If there exists $v$ with $d_{G^2}(v) < \Delta^2 - 1$, then $G^2$ is $(\Delta^2 - 1)$-paintable.*

*Proof.* We prove the first statement first. Let $V = V(G)$ and $V_1 = V(H)$. Clearly a graph is $d_1$-paintable only if each component is. So we assume that $G^2[V_1]$ is connected. For simplicity, we assume also that $G[V_1]$ is connected. If not, then some vertex $v$ has neighbors in two or more components of $G[V_1]$. We simply add $v$ to $V_1$, since we can color $v$ first (when it still has at least two uncolored neighbors).

Form $G'$ from $G$ by contracting $G[V_1]$ to a single vertex $r$. Let $T$ be a spanning tree in $G'$ rooted at $r$. Let $\sigma$ be an ordering of the vertices of $G \setminus H$ by nonincreasing distance in $T$ from $r$.

Each time that Lister presents a list of vertices, Painter chooses a maximal independent subset of them, by greedily adding vertices in order $\sigma$. Each vertex $v \in V \setminus V_1$ is followed in $\sigma$ by the first two vertices on a path in $T$ from $v$ to $r$. Thus $v$ will be colored. We now combine strategies for $G^2 \setminus H$ and $H$ as in the proof of Lemma 1.

Now we prove the second statement, which has a similar proof. Suppose there exists $v$ with $d_{G^2}(v) < \Delta^2 - 1$. As before we order the vertices by nonincreasing distance in some spanning tree $T$ from $v$, and we put $v$ and some neighbor $u$ last in $\sigma$. The difference now is that even for $u$ and $v$ we are given $\Delta^2 - 1$ colors. Since $d_{G^2}(v) < \Delta^2 - 1$, either (i) $v$ lies on a 3-cycle or 4-cycle or else (ii) $d_G(v) < \Delta$ or $v$ has some neighbor $u$ with $d_G(u) < \Delta$; in Case (ii), by symmetry we assume $d_G(v) < \Delta$. In Case (i), $d_{G^2}(u) \leq \Delta^2 - 1$ for some neighbor $u$ of $v$ on the short cycle and by assumption $d_{G^2}(v) < \Delta^2 - 1$; so the two final vertices of $\sigma$ are $u$ and $v$. In Case (ii), we again have $d_{G^2}(v) < \Delta^2 - 1$ and $d_{G^2}(u) \leq \Delta^2 - 1$, so again $u$ and $v$ are last in $\sigma$. □

The previous lemma implies that $\Delta^2 - 1 \leq d_{G^2}(v) \leq \Delta^2$ for every vertex $v$ in a graph $G$ such that $G^2$ is not $(\Delta^2 - 1)$-paintable. A vertex $v$ is *high* if $d_{G^2}(v) = \Delta^2$, and otherwise it is *low*. The proof of Lemma 2 proves something slightly more general, which we record in the following corollary.

**Corollary 3.** *Let $G$ be a graph with maximum degree $\Delta$ and let $H$ be an induced subgraph of $G^2$. Let $f(v) = d(v) - 1$ for each high vertex of $G^2$ and $f(v) = d(v)$ for each low vertex. If $H$ is $f$-paintable, then $G^2$ is $(\Delta^2 - 1)$-paintable.*

Now we will introduce the Alon-Tarsi Theorem, but we need a few definitions first. Let $G$ be a graph and let $\vec{D}$ be a digraph arising by orienting the edges of $G$. A *circulation* is a subgraph of $\vec{D}$ in which each vertex has equal indegree and outdegree; circulations are also called eulerian subgraphs. The parity of a circulation is the parity of its number of edges. For a digraph $\vec{D}$, let $EE(\vec{D})$ (resp. $EO(\vec{D})$) denote the set of circulations that are even (resp. odd).

**Theorem A** (Alon and Tarsi [1])**.** *For a digraph $\vec{D}$, if $|EE(\vec{D})| \neq |EO(\vec{D})|$, then $\vec{D}$ is $f$-choosable, where $f(v) = 1 + d_{\vec{D}}(v)$ for all $v$.*

The proof that Alon and Tarsi gave was algebraic and not constructive. In their paper, they asked for a combinatorial proof. This was provided by Schauz [18], in the more general setting of paintability. His proof relies on an elaborate inductive argument. The argument does yield a constructive algorithm, although in general it may run in exponential time. In [17], Schauz proved an online version of the combinatorial nullstellensatz from which the paintability version of Alon and Tarsi's theorem can also be derived.

**Theorem B** (Schauz [18])**.** *For a digraph $\vec{D}$, if $|EE(\vec{D})| \neq |EO(\vec{D})|$, then $\vec{D}$ is $f$-paintable, where $f(v) = 1 + d_{\vec{D}}(v)$ for all $v$.*

Our main result relies heavily on forbidding $d_1$-paintable subgraphs. For many of the smaller $d_1$-paintable graphs that we need, we give direct proofs. However, for some of the larger $d_1$-paintable graphs, particularly the classes of unbounded size, our proofs of $d_1$-paintability use Theorem B.

# 3 Proof of Main Theorem

In this section we prove our main result, subject to a number of lemmas on forbidden subgraphs, which we defer to the next section. We typically prove that a subgraph is forbidden by showing that it is $d_1$-paintable. If a copy of a subgraph $H$ in $G^2$ contains low vertices, then this configuration is reducible as long as $H$ is $f$-paintable, where $f(v) = d_H(v) - 1$ for each high vertex $v$ and $f(w) = d_H(w)$ for each low vertex $w$. For many of the graphs, we give an explicit winning strategy for Painter. In contrast, for some of the graphs, particularly those of unbounded size, we don't give explicit winning strategies. Instead, we show that they are $d_1$-paintable via Schauz's extension of the Alon-Tarsi Theorem (Theorem B).

**Main Theorem.** *If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $G$ is not the Peterson graph, the Hoffman-Singleton graph, or a Moore graph with $\Delta = 57$, then $\chi_p(G^2) \leq \Delta^2 - 1$.*

*Proof.* Let $G$ be a connected graph with maximum degree $\Delta \geq 3$, other than the graphs excluded in the Main Theorem. Assume that $G^2$ is not $(\Delta^2 - 1)$-paintable. By Lemma 2, if there exists $v \in V(G)$ with $d_{G^2}(v) < \Delta^2 - 1$, then $G^2$ is $(\Delta^2 - 1)$-paintable. So $G$ is $\Delta$-regular and has girth at least 4. Further, no vertex of $G$ lies on two or more 4-cycles. It will be helpful in what follows to show that $\omega(G^2) \leq \Delta^2 - 1$.

Clearly $\Delta(G^2) \leq \Delta^2$. Further, $\omega(G^2) = \Delta^2 + 1$ only if $G^2 = K_{\Delta^2+1}$. Hoffman and Singleton [12] showed this is possible only if $\Delta \in \{2, 3, 7, 57\}$; such a graph $G$ is called a Moore graph. When $\Delta \in \{2, 3, 7\}$, the unique realizations are the 5-cycle, the Peterson graph, and the Hoffman-Singleton graph. When $\Delta = 57$, no realization is known. These are precisely the graphs excluded from the theorem. Now we consider the case $\omega(G^2) = \Delta^2$. Erdős, Fajtlowicz, and Hoffman [11] showed that the only graph $H$ such that $H^2 = K_{\Delta(H)^2}$ is $C_4$. Cranston and Kim noted that if $H^2$ is not a clique on at least $\Delta^2$ vertices, then in fact $\omega(H^2) \leq \Delta^2 - 1$. For completeness, we reproduce the details.

Suppose that $\omega(G^2) = \Delta^2$, and let $U$ be the vertices of a maximum clique in $G^2$. The result of Erdős, Fajtlowicz, and Hoffman implies that $U$ is not all of $V$. Choose $v, w \in V$ with $v \in U$, $w \notin U$ and $v$ adjacent to $w$. Since $d_{G^2}(v) = \Delta^2$ and $w \notin U$, every neighbor of $w$ must be in $U$. Applying the same logic to these neighbors, every vertex within distance 2 of $w$ must be in $U$. But now we can add $w$ to $U$ to get a larger clique in $G^2$. This contradiction implies that in fact $\omega(G^2) \leq \Delta^2 - 1$.

Two vertices are *linked* if they are adjacent in $G^2$, and otherwise they are *unlinked*. When we write that vertices are adjacent or nonadjacent, we mean in $G$; otherwise we write linked or unlinked. We write $v \leftrightarrow w$ if $v$ and $w$ are adjacent, and $v \not\leftrightarrow w$ otherwise.

**Case 1: $G$ has girth 4**

Let $C$ be a 4-cycle with vertices $v_1, \ldots, v_4$, and let $\mathcal{C} = V(C)$. It is helpful to note that every $v_i$ is low. We need two lemmas. These were first proved in [8] for list coloring, and we generalize them to online list coloring in Lemmas 5 and 6. The following two configurations in $G^2$ are reducible: (A) $K_4 \vee \overline{K_2}$ where some vertex $w \in V(K_4)$ is low and (B) $K_3 \vee \overline{K_2}$ where some vertices $w \in V(K_3)$ and $x \in V(\overline{K_2})$ are both low.

Note that $G^2[\mathcal{C}] \cong K_4$. This implies that every $w$ adjacent to some $v_i \in \mathcal{C}$ must be linked to all of $\mathcal{C}$. Suppose not, and let $w$ be adjacent to $v_1$ and not linked to $v_3$. Now $G^2[\mathcal{C} \cup \{w\}] \cong K_3 \vee \overline{K_2}$, and every $v_i$ is low; this is (B), which is forbidden. Now suppose that $w_1$ and $w_2$ are vertices adjacent to $v_i$ and $v_j$, respectively. We must have $w_1$ linked to $w_2$, since otherwise $G^2[\mathcal{C} \cup \{w_1, w_2\}]$ is (A), which is forbidden.

Now let $x$ be a vertex at distance 2 from $v_1$ and not adjacent to any $v_i$; let $w_1$ be a common neighbor of $v_1$ and $x$. Since $w_1$ is linked to $v_3$, they have a common neighbor $w_3$. Now $x$ is linked to $v_1$, $w_1$, and $w_3$. To avoid configuration (B), $x$ must be linked to all of $\mathcal{C}$. Thus, all vertices within distance 2 of $v_1$ must be linked to all of $\mathcal{C}$. Now every pair of vertices $x$ and $y$ that are both within distance 2 of $v_1$ must be linked; otherwise $G^2[\mathcal{C} \cup \{x, y\}]$ is (A). So the vertices within distance 2 of $v_1$ induce in $G^2$ a clique of size $\Delta^2$, which contradicts that $\omega(G^2) \leq \Delta^2 - 1$.

**Case 2: $G$ has girth at least 5**

Let $g$ denote the girth of $G$. First suppose that $g = 6$, and let $U$ be the vertices of a 6-cycle. Note that $G^2[U] = C_6^2$, since girth 6 implies there are no extra edges. Since $C_6^2$ is $d_1$-paintable, by Lemma 9, we are done by Lemma 2.

Suppose $g = 7$. Let $U$ denote the vertices of some 7-cycle in $G$, with a pendant edge at a single vertex of the cycle. Because $G$ has girth 7, $G^2[U]$ has only the edges guaranteed by its definition. We show in Lemma 17 that $G^2[U]$ is $d_1$-paintable. So again, we are done by Lemma 2.

Suppose instead that $g \geq 8$. Let $U = \{v_1, \ldots, v_g, w_1, w_5\}$ be the vertices of some $g$-cycle in $G$ together with pendant edges $v_1w_1$ and $v_5w_5$. If $g \geq 9$, then $G^2[U]$ has only the edges guaranteed by its definition. If $g = 8$, then $G^2[U]$ has the edges guaranteed by its definition as well as possibly the extra edge $w_1w_5$. For each girth $g$ at least 8, we show in Lemma 18 and Lemma 20 that $G^2[U]$ is $d_1$-paintable. So again, we are done by Lemma 2.

Now we consider girth 5. Our approach is similar to that for girth 4, but we must work harder since we don't necessarily have any low vertices. Let $C$ be a 5-cycle with vertices $v_1, \ldots, v_5$. Let $k = \Delta - 2$. For each $i$, let $V_i$ denote the neighbors of $v_i$ not on $C$. Let $\mathcal{C} = V(C)$ and let $\mathcal{D} = \cup_{i=1}^{5} V_i$. Each vertex of $\mathcal{D}$ is linked to either 5, 4, or 3 vertices of $\mathcal{C}$. We call these $B_0$-vertices, $B_1$-vertices, and $B_2$-vertices, respectively (a $B_i$-vertex is unlinked to $i$ vertices of $\mathcal{C}$). We will consider four possibilities for the number and location of each type of vertex. In each case we find a $d_1$-paintable subgraph. Let $L$ denote the subgraph $G[\mathcal{D}]$. Since $G$ has girth 5, we have $\Delta(L) \leq 2$. Each vertex $w$ with $d_L(w) = 2 - i$ is a $B_i$-vertex (for $i \in \{0, 1, 2\}$).

Suppose that $G$ has two $B_1$-vertices $w_1$ and $w_2$ and they are unlinked with distinct vertices in $\mathcal{C}$. Let $H = G^2[\mathcal{C} \cup \{w_1, w_2\}]$. If $w_1$ and $w_2$ are linked, then $H = K_3 \vee C_4 \supset K_2 \vee C_4$, which is $d_1$-paintable, by Lemma 10. If instead $w_1$ and $w_2$ are unlinked, then $H = K_3 \vee P_4$, which is also $d_1$-paintable, by Lemma 11. So we assume that all $B_1$-vertices are unlinked with the same vertex $v \in \mathcal{C}$. As a result, each $B_1$-vertex is an endpoint of a path of length 3 (mod 5) in $L$, for otherwise the two endpoints of the path are unlinked with different vertices in $\mathcal{C}$. Since the number of odd degree vertices in any graph is even, here the number of $B_1$-vertices is even.

**Case 2.1: $G$ has a $B_1$-vertex $w_1$ and a $B_2$-vertex $w_2$.**

Let $H = G^2[\mathcal{C} \cup \{w_1, w_2\}]$. Suppose the four vertices of $\mathcal{C}$ linked to $w_1$ include the three vertices of $\mathcal{C}$ linked to $w_2$. If $w_1$ and $w_2$ are linked, then $H = K_3 \vee P_4$, and if $w_1$ and $w_2$ are unlinked, then $H = K_3 \vee (K_1 + P_3)$. In each case, $H$ is $d_1$-paintable, by Lemmas 11 and 12, respectively.

Suppose instead that the four vertices of $C$ linked to $w_1$ do not include all three vertices of $C$ linked to $w_2$. If $w_1$ is linked with $w_2$, then $H \supset K_2 \vee C_4$, which is $d_1$-paintable by Lemma 10. If $w_1$ is unlinked with $w_2$, then $H$ is again $d_1$-paintable, by Lemma 15. Thus, $G^2$ cannot contain both $B_1$-vertices and $B_2$-vertices.

**Case 2.2: $G$ has no $B_1$-vertices, but only some $B_2$-vertices, and possibly also $B_0$-vertices.**

Now $L$ consists of disjoint cycles, each with length a multiple of 5. This implies that each $V_i$ contains the same number of $B_2$-vertices; by assumption this number is at least 1. We call a pair of $B_2$ vertices with distinct cycle neighbors *near* if their cycle neighbors are adjacent and *far* if their cycle neighbors are nonadjacent. If any pair of far $B_2$-vertices are linked, then $G$ has a $d_1$-paintable subgraph, by Lemma 13. If any pair of near $B_2$-vertices are linked, then, together with their adjacent cycle vertices, they induce $K_2 \vee C_4$, which is $d_1$-paintable by Lemma 10. Thus, we consider the subgraph induced by $\mathcal{C}$ and 3 non-successive $B_2$-vertices, say with cycle neighbors $v_1, v_2, v_4$. Each such subgraph is $d_1$-paintable, by Lemma 14. Combining this with Case 2.1, we conclude that $G$ contains no $B_2$-vertices.

**Case 2.3: $G$ has $B_1$-vertices and possibly $B_0$-vertices.**

Recall that $G$ has an even number of $B_1$-vertices and they are all unlinked with the same vertex. By symmetry, assume that $G$ has $B_1$-vertices $w_2 \in V_2$ and $w_3 \in V_3$ and they are both unlinked with $v_5$. We will find two disjoint pairs of nonadjacent vertices, such that all four are linked with $\mathcal{C} - v_5$.

Since $w_3$ is a $B_1$-vertex, it is the endpoint of some path in $L$; let $w_1 \in V_1$ be the neighbor of $w_3$ on this path. We will show that $w_1$ is unlinked with some vertex in $\mathcal{D}$.

Recall that $|\mathcal{D}| = 5k$. Suppose that $w_1$ is linked to each vertex of $\mathcal{D}$. Since $d_L(w_1) = 2$ and $d_L(w_3) = 1$, at most 3 of these $5k - 1$ vertices linked with $w_1$ can be reached from $w_1$ by following edges in $L$. Clearly $w_1$ is linked to the other $k-1$ vertices of $V_1$. Now for each vertex $w$ of the remaining $(5k-1)-3-(k-1) = 4k-3$ vertices in $\mathcal{D}$, $w_1$ must have a common neighbor $x$ with $w$ and $x \notin \mathcal{D} \cup \mathcal{C}$. Furthermore, each such common neighbor $x$ can link $u$ to at most 4 of these vertices (at most one in each other $V_i$, since the girth is 5). However, this requires at least $\lceil \frac{4k-3}{4} \rceil = k$ additional neighbors of $w_1$, but we have already accounted for 3 neighbors of $w_1$. Thus, $w_1$ is unlinked with some vertex $y \in \mathcal{D}$.

Let $z$ be a $B_1$ vertex distinct from $y$. Now $z$ and $v_5$ are unlinked and $w_1$ and $y$ are unlinked. But every vertex of $\{w_1, v_5, y, z\}$ is linked to $\mathcal{C} - v_5$. Thus $G^2[(\mathcal{C} - v_5) \cup \{w_1, v_5, y, z\}] = K_4 \vee H$, where $H$ contains disjoint pairs of nonadjacent vertices. So $K_4 \vee H$ is $d_1$-paintable, by Lemma 7.

**Case 2.4: $\mathcal{D}$ has only $B_0$-vertices.**

Let $H = G^2[\mathcal{C} \cup \mathcal{D}]$. We will show that if $H$ is not a clique, then we can choose a different 5-cycle and be in an earlier case. Suppose that $H$ is not a clique. Since $\mathcal{D}$ is linked to $\mathcal{C}$ and $G^2[\mathcal{C}] = K_5$, we must have $w_1, w_2 \in \mathcal{D}$ with $w_1$ and $w_2$ unlinked. By symmetry, we have only two cases.

First suppose that $w_1 \in V_1$ and $w_2 \in V_2$ and $w_1$ and $w_2$ are unlinked. Since $w_1$ is a $B_0$-vertex, we have $w_3 \in V_3$ with $w_1 \leftrightarrow w_3$. Consider the 5-cycle $w_1v_1v_2v_3w_3$. Now $w_2$ is not linked to $w_1$, which makes $w_2$ not a $B_0$-vertex for that 5-cycle. So we are in Case 2.1, 2.2, or 2.3 above. Now suppose instead that $w_1 \in V_1$ and $w_3 \in V_3$ and $w_1$ and $w_3$ are unlinked. Now we pick some $w_3' \in V_3$ with $w_1 \leftrightarrow w_3'$ and consider the 5-cycle $w_1v_1v_2v_3w_3'$. Since $w_3$ and $w_1$ are unlinked, $w_3$ is not a $B_0$-vertex for this 5-cycle, so we are in Case 2.1, 2.2, or 2.3 above. Hence $G^2[\mathcal{C} \cup \mathcal{D}]$ must be a clique.

To link all vertices in $\mathcal{D}$, we must have $k(k-1)$ additional vertices in $G$, at distance 2 from $\mathcal{C}$; call the set of them $\mathcal{F}$. We see that $|\mathcal{F}| \geq k(k-1)$ as follows. All $\binom{5k}{2}$ pairs of vertices in $\mathcal{D}$ are linked. The $5\binom{k}{2}$ pairs contained within a common $V_i$ are linked via vertices of $\mathcal{C}$. Each of the $5k$ vertices is linked with exactly 4 vertices via edges of $L$. The remaining links all must be due to vertices of $\mathcal{F}$, and each vertex of $\mathcal{F}$ can link at most $\binom{5}{2} = 10$ pairs of vertices in $\mathcal{D}$ (at most one vertex in each $V_i$, since $G$ has girth 5). Thus $|\mathcal{F}| \geq (\binom{5k}{2} - 5\binom{k}{2} - 5k(4)/2)/\binom{5}{2} = k(k-1)$. If any vertex $x \in \mathcal{F}$ has fewer than exactly one neighbor in each $V_i$, then some pair of vertices in $\mathcal{D}$ will be unlinked. Thus, each $x \in \mathcal{F}$ has exactly one neighbor in each $V_i$. This implies that $\mathcal{F}$ is linked to $\mathcal{C}$, and hence that $|\mathcal{F}| = k(k-1)$. We will show that every pair of vertices in $\mathcal{C} \cup \mathcal{D} \cup \mathcal{F}$ is linked.

Suppose there exists $w \in \mathcal{D}$ and $x \in \mathcal{F}$ with $w$ and $x$ unlinked. By symmetry, we assume $w \in V_1$. There exist $w_1 \in V_1$ and $w_2 \in V_2$ with $x \leftrightarrow w_1$ and $x \leftrightarrow w_2$. Now consider the 5-cycle $xw_1v_1v_2w_2$. Since $w$ and $x$ are unlinked, $w$ is not a $B_0$-vertex for that 5-cycle. This puts us in Case 2.1, 2.2., or 2.3 above. So $\mathcal{F}$ must be linked to $\mathcal{D}$.

Finally suppose there exist $x_1, x_2 \in \mathcal{F}$ with $x_1$ and $x_2$ unlinked. Now there exist $w_1, w_2 \in V_1$ with $x_1 \leftrightarrow w_1$ and $x_2 \leftrightarrow w_2$. Since $G$ has girth 5, we have $x_1 \not\leftrightarrow w_2$. And since $x_1$ is linked with $w_2$, they have some common neighbor $y \in \mathcal{D} \cup \mathcal{F}$. Now consider the 5-cycle $x_1w_1v_1w_2y$. Since $x_1$ and $x_2$ are unlinked, $x_2$ is not a $B_0$-vertex for this 5-cycle. Hence, we are in Case 2.1, 2.2, or 2.3.

Thus, all vertices of $\mathcal{C} \cup \mathcal{D} \cup \mathcal{F}$ are pairwise linked. Now $|\mathcal{C} \cup \mathcal{D} \cup \mathcal{F}| = 5 + 5k + k(k-1) = k^2 + 4k + 5 = (k+2)^2 + 1 = \Delta^2 + 1$. This contradicts that $\omega(G^2) \leq \Delta^2 - 1$ and completes the proof. □

We note that many of the cases of the above proof actually prove that $G^2$ is $d_1$-paintable, and hence has paint number at most $\Delta(G^2) - 1$. In particular, this is true when $G$ has girth 6, 7, or at least 9. Probably with more work, we could also adapt the proof to the case when $G$ has girth 8. The Conjecture that $G^2$ is $(\Delta(G^2) - 1)$-paintable unless $\omega(G^2) \geq \Delta(G^2)$ is a special case of Conjecture 5. The main obstacle to proving this stronger result is the case when $G$ has girth at most 5, particularly girth 3 or girth 4.

## 4 Proofs of forbidden subgraph lemmas

In what follows, we slightly abuse the terminology of high and low vertices defined earlier. Now a vertex is *high* if its list size is one less than its degree and *low* if its list size equals its degree. Note that if a vertex $v$ is high (resp. low) in $G$ by our old definition, then it will be high (resp. low) in each induced subgraph $H$ by our new definition. A vertex is *very low* if its list size is greater than its degree. When a vertex $v$ in a graph $G$ is very low, we may say that we *delete* $v$. If $G - v$ is paintable from its lists, then so is $G$. On each round, we play the game on $G - v$ and consider $v$ after all other vertices, coloring it only if its list contained the color for that round and we have colored none of its neighbors on that round. Recall that $S_k$ denotes the vertices with lists containing color $k$. We write $E_k$ for the empty graph on $k$ vertices, i.e., $E_k = \overline{K_k}$. In what follows, all vertices not specified to be low are assumed to be high.

### 4.1 Direct proofs

For pictures of the graphs in Lemmas 4 through 12, see Figures 9 and 10 in Section 5.

**Lemma 4.** *If $G$ is $K_4 - e$ with one degree 3 vertex high and the other vertices low, then $G$ is $f$-paintable.*

*Proof.* Let $v_1, v_2$ denote the degree 3 vertices, with $v_1$ low, and let $w_1, w_2$ denote the degree 2 vertices. If $w_1, w_2 \in S_1$, then color them both with 1. Now the remaining vertices are low and very low, so we can finish. Otherwise, color some $v_i$ with 1, choosing $v_2$ if possible. Now at least one $w_j$ becomes very low and the uncolored $v_k$ is low, so we can finish. $\square$

**Lemma 5.** *If $G$ is $K_3 \vee E_2$ with a low vertex in the $K_3$ and a low vertex in the $E_2$, then $G$ is $f$-paintable.*

*Proof.* Denote the vertices of the $K_3$ by $v_1, v_2, v_3$, with $v_1$ low, and the vertices of $E_2$ by $w_1, w_2$, with $w_1$ low. If $w_1, w_2 \in S_1$, then color them both 1. Now $v_1$ becomes very low and $v_2$ and $v_3$ each become low, so we finish greedily, ending with $v_2$ and $v_1$. Suppose $w_2 \in S_1$. If $v_2 \in S_1$ (or $v_3 \in S_1$, by symmetry), then color $v_2$ with 1. Now $w_1$ becomes very low (since $S_1 \not\supseteq \{w_1, w_2\}$), and $v_1$ remains low, so we can finish greedily. If instead $v_1 \in S_1$ and $v_2, v_3 \notin S_1$, then color $v_1$ with 1. Again $w_1$ becomes very low and $v_2$ and $v_3$ become low, so we can finish greedily. The situation is similar if $S_1$ contains only a single $w_i$. Thus, $w_2 \notin S_1$. Since $S_1 \neq \{w_1\}$, some $v_i$ is in $S_1$. Use color 1 on $v_i$, choosing $v_2$ or $v_3$ if possible. What remains is $K_4 - e$ with one degree 3 vertex high and all others low (or very low). So we finish by Lemma 4. $\square$

**Lemma 6.** *If $G$ is $K_4 \vee E_2$ with a low vertex in the $K_4$, then $G$ is $f$-paintable.*

*Proof.* Denote the vertices of the $K_4$ by $v_1, \ldots, v_4$, with $v_1$ low and the vertices of $E_2$ by $w_1, w_2$. If $w_1, w_2 \in S_1$, then color them both 1. Now $v_1$ becomes very low and the other $v_i$ become low, so we can finish by coloring greedily, with $v_1$ last. So $S_1$ contains at most one $w_i$, say $w_2$. Suppose $S_1$ contains a $v_j$ other than $v_1$. Color $v_j$ with 1. Now $w_1$ becomes low, $v_1$ remains low, and the other vertices remain high. So we can finish the coloring by Lemma 5. If the only $v_i$ in $S_1$ is $v_1$, then color it 1. Now the other $v_j$ become low, so again we finish by Lemma 5. Finally, if the only vertex in $S_1$ is $w_2$, then color it 1. Now $v_1$ becomes very low, and the other $v_i$ become low, so again we can finish by coloring greedily, ending with a low vertex and a very low vertex. $\square$

**Lemma 7.** *If $G$ is $K_4 \vee H$ with $H$ containing two disjoint nonadjacent pairs, then $G$ is $d_1$-paintable.*

*Proof.* We may assume $|H| = 4$. Denote the vertices of $K_4$ by $v_1, \ldots, v_4$ and the vertices of $H$ by $w_1, \ldots, w_4$ with $w_1 \not\leftrightarrow w_2$ and $w_3 \not\leftrightarrow w_4$. If $w_1, w_2 \in S_1$, then color $w_1$ and $w_2$ with 1. Now every $v_i$ becomes low, so we can finish by Lemma 6. Similarly, if $w_3, w_4 \in S_1$.

If some $v_i$ is missing from $S_1$, then use 1 to color either some $v_j$ or some $w_k$. In the first case, we finish by Lemma 5 and in the second by Lemma 6. So color $v_4$ with 1. Now, by symmetry, $w_2, w_4 \notin S_1$, so they each become low. If $w_1, w_2 \in S_2$, then color them both with 2. Now every $v_i$ becomes low, so we can finish by Lemma 5. Similarly if $w_3, w_4 \in S_2$. So $S_2$ contains at most one of $w_1, w_2$ and at most one of $w_3, w_4$. If $S_2$ contains no $v_i$, then we color some $w_j$ with 2. This makes every $v_i$ low. Now we can finish by Lemma 5. So $S_2$ contains some $v_i$, say $v_3$.

Color $v_3$ with 1. Recall that $S_1$ was missing at least one of $w_1, w_2$ and at least one of $w_3, w_4$. (i) If $w_2, w_4 \notin S_2$, then they both become very low, so we can delete them. This in turn makes $v_1$ and $v_2$ both very low, so we can finish greedily. (ii) If $w_2, w_3 \notin S_2$, then $w_2$ becomes very low, so we delete it. Now $v_1$ and $v_2$ become low; also $w_3$ and $w_4$ are low. Since $v_1, v_2, w_3, w_4$ induce $K_4 - e$ with all vertices low, we can finish by Lemma 4. By symmetry, this handles the case $w_1, w_4 \notin S_2$. (iii) If $w_1, w_3 \notin S_2$, then the uncolored vertices induce $K_2 \vee H$, with all vertices of $H$ low. Now consider $S_3$. If $S_3$ contains a nonadjacent pair in $H$, then color them both 3. This makes $v_1$ and $v_2$ low, so what remains is $K_4 - e$ with all vertices low. We now finish by Lemma 4. Similarly, if $S_3$ contains no $v_i$, then color some $w_j$ with 3, and we can finish by Lemma 4. So $S_3$ contains some $v_i$, say $v_2$, and we color $v_2$ with 3. Now one of $w_1, w_2$ becomes very low and one of $w_3, w_4$ becomes very low. We can delete the very low vertices, which in turn makes $v_1$ very low. We can now finish greedily, since what remains is a 3-vertex path with two low vertices and a very low vertex. □

We won't use Lemma 8 in the proof, but it is generally useful so we record it here.

**Lemma 8.** *If $G$ is $K_6 \vee E_3$, then $G$ is $d_1$-paintable.*

*Proof.* Denote the vertices of $K_6$ by $v_1, \ldots, v_6$ and the vertices of $E_3$ by $w_1, w_2, w_3$. If $w_1, w_2, w_3 \in S_1$, then color $w_1, w_2, w_3$ all with 1. Now all $v_i$ are very low, so we finish greedily. If no $v_i$ appears in $S_1$, then color some $w_j$ with 1. Now all the $v_i$ are low, so we can finish by Lemma 6. So some $v_i$ is in $S_1$, say $v_6$. Color $v_6$ with 1. This makes some $w_i$ low, say $w_3$. Repeating this argument, we get by symmetry that $v_5 \in S_2$ and $S_2$ is missing some $w_j$. If $S_2$ is missing $w_3$, then color $v_5$ with 2. Now $w_3$ becomes very low, so we delete it. This in turn makes all uncolored $v_k$ low. Now we can finish by Lemma 6. So instead $S_2$ is missing (by symmetry) $w_2$. Again repeating the argument, we must have $v_4 \in S_3$ and $w_1 \notin S_3$; otherwise we finish by Lemma 5 or Lemma 6. Now we color $v_4$ with 3. What remains is $K_3 \vee E_3$ with every $w_i$ low.

Now consider $S_4$. If $w_1, w_2, w_3 \in S_4$, then color them all with 3. Now all remaining vertices become very low, so we finish greedily. Suppose instead that $w_1 \in S_4$ and $v_1 \notin S_4$. Color $w_1$ with 4. What remains is $K_3 \vee E_2$ with both $w_i$ low and some $v_j$ low. So we can finish by Lemma 4. A similar approach works for any $w_i \in S_4$ and $v_j \notin S_4$. So instead, assume by symmetry that $v_1 \in S_4$ and $w_1 \notin S_4$. Color $v_1$ with 4. Now $w_1$ becomes very low, so we delete it. This in turn makes $v_2$ and $v_3$ low. Now we can finish by Lemma 4. □

**Lemma 9.** *If $G$ is $C_6^2$, then $G$ is $d_1$-paintable.*

*Proof.* Denote the vertices of the 6-cycle by $v_1, \ldots, v_6$ in order. So $v_i$ is adjacent to all but $v_{(i+3) \bmod 6}$. Consider $S_1$. If $S_1$ contains some nonadjacent pair, then color them with 1. What remains is $C_4$ with all vertices low, so we can complete the coloring since $C_4$ is 2-paintable. So assume that $S_1$ contains no nonadjacent pairs. Now without loss of generality, we assume $S_1 = \{v_1, v_2, v_3\}$, since adding vertices to $S_1$ only makes things harder to color, as long as $S_1$ induces a clique; we may also need to permute a nonadjacent pair. Color $v_1$ with 1.

Now $v_5$ and $v_6$ become low. Consider $S_2$. Again, if $S_2$ contains a nonadjacent pair, then we color both vertices with 2 and can finish greedily since all remaining vertices are low, except for one that is very low. If $v_2, v_3 \in S_2$, then color $v_2$ with 2. Now $v_6$ becomes very low and $v_5$ remains low, so we can finish greedily. So $S_2$ misses at least one of $v_2, v_3$. Suppose $v_4 \in S_2$. Color $v_4$ with 3. What remains is $C_4$. If $v_2, v_3 \notin S_2$, then all vertices are low, and we can finish since $C_4$ is 2-paintable. Otherwise, $v_5$ or $v_6$ becomes very low and the other remains low. Now we can finish greedily. So $v_4 \notin S_2$. If $v_2 \in S_2$, then color $v_2$ with 2. Now $v_3$ and $v_4$ become low, so we can finish by Lemma 4. An analogous argument works if $v_3 \in S_2$. So assume $v_2, v_3, v_4 \notin S_2$. Now color $v_5$ or $v_6$ with 2. Again we can finish by Lemma 4. $\square$

**Lemma 10.** *If $G$ is $K_2 \vee C_4$, then $G$ is $d_1$-paintable.*

*Proof.* Denote the vertices of $K_2$ by $v_1, v_2$ and the vertices of $C_4$ by $w_1, \ldots w_4$ in order. If $S_1$ contains a pair of nonadjacent vertices, then color them both 1. What remains is $K_4 - e$, with all vertices low. So we can finish by Lemma 4. So $S_1$ misses at least one of $w_1, w_3$ and at least one of $w_2, w_4$. By symmetry, say it misses $w_1$ and $w_2$. Suppose $v_1, v_2 \notin S_1$. Now by symmetry $w_3 \in S_1$, so color $w_3$ with 1. This makes each of $w_2, v_1, v_2$ low. So what remains is $K_3 \vee E_2$ with two low vertices in the $K_3$ and a low vertex in the $E_2$. Hence, we can finish by Lemma 5.

So instead (by symmetry) $v_2 \in S_1$. Color $v_2$ with 1. What remains is $K_1 \vee C_4$ with $w_1$ and $w_2$ low. Consider $S_2$. Again if $S_2$ contains a nonadjacent pair, then we color them both 2, and we can finish greedily. Suppose that $w_3 \in S_2$. If $w_4 \notin S_2$, then we color $w_3$ with 4; now $w_4$ becomes low, so we can finish by Lemma 4. If instead $w_4 \in S_2$, then $w_2 \notin S_2$. Now when we color $w_3$ with 2, $w_2$ becomes very low, so we can finish greedily. So assume $w_3, w_4 \notin S_2$. If $v_1 \in S_2$, then color $v_1$ with 1. What remains is $C_4$ with all vertices low. Now we can finish the coloring since $C_4$ is 2-paintable. The proof is similar to that for 2-choosability, so we omit it. So assume that $v_1 \notin S_2$. By symmetry, we have $w_1 \in S_2$. Color $w_1$ with 2. What remains is $K_4 - e$ with only $w_3$ high. Hence we can finish by Lemma 4. $\square$

**Lemma 11.** *If $G$ $K_3 \vee P_4$, then $G$ is $d_1$-paintable.*

*Proof.* Let $v_1, v_2, v_3$ denote the vertices of $K_3$ and $w_1, \ldots, w_4$ denote the vertices of the $P_4$ in order. If $w_1, w_3 \in S_1$, then color them both 1. Now what remains is $K_3 \vee E_2$ with all but one vertex low, so we can finish by Lemma 5. An analagous strategy works if $w_2, w_4 \in S_1$. So assume $S_1$ misses at least one of $w_1, w_3$ and at least one of $w_2, w_4$. If $S_1$ misses $v_1$, then use color 1 on some $w_j$, choosing $w_2$ or $w_3$ if possible. Again, we can finish by Lemma 5. So assume $v_1 \in S_1$. Now color $v_3$ with 1. What remains is $K_2 \vee P_4$ with at least two vertices of the $P_4$ low. Consider $S_2$. If $w_1, w_3 \in S_2$ (or $(w_2, w_4 \in S_2)$, then color them both 2, and we can finish greedily since all vertices are low except for one that is very low. If $v_2 \in S_2$, then color it with 2. Now in each case we can finish by repeatedly deleting very low vertices, possibly using Lemma 4. So $v_2 \notin S_2$ (and by symmetry $v_3 \notin S_2$). If possible use color 2 on $w_1$ or $w_4$. This leaves $K_3 \vee E_2$ with enough low vertices to finish by Lemma 5. Finally, if $w_1, w_4 \notin S_2$, then by symmetry $w_2 \in S_2$, so color $s_2$ with 2. What remains contains a $K_4 - e$ with all vertices low, so we can finish by Lemma 4. $\square$

**Lemma 12.** *If $G$ is $K_3 \vee (K_1 + P_3)$, then $G$ is $d_1$-paintable.*

*Proof.* Let $v_1, v_2, v_3$ denote the vertices of $K_3$; let $w_1$, $w_2$, $w_3$ denote the vertices of $P_3$ in order, and let $w_4$ be the $K_1$. If $w_1, w_3 \in S_1$, then color them both 1 and we can finish by Lemma 5. If instead $w_2, w_4 \in S_1$, then color them both 1, and again we can finish by Lemma 5. If $S_1 = \{w_4\}$, then color $w_4$ with 1. What remains is $K_3 \vee P_3$ with all vertices of the $K_3$ low. Since $K_3 \vee P_3 \cong K_4 \vee E_2$, we can finish by Lemma 6. If $w_1 \in S_1$ (or $w_2 \in S_1$ or $w_3 \in S_1$) and $v_3 \notin S_1$, then color $w_1$ with 1. Again we can finish by Lemma 5. This implies that $v_3 \in S_1$.

Since $v_3 \in S_1$, color $v_3$ with 1. Now at least one of $w_1, w_3$ becomes low and at least one of $w_2, w_4$ becomes low. What remains is $K_2 \vee (K_1 + P_3)$, and by symmetry either (i) $w_1$ and $w_2$ are low or (ii) $w_1$ and $w_4$ are low. Consider (i). If we ignore $w_4$, then what remains is

$K_2 \vee P_3 \cong K_3 \vee E_2$. Since $w_1$ and $w_2$ are low, we can finish by Lemma 5. Instead consider (ii). If $w_1, w_3 \in S_2$, then color them both with 2. What remains is $K_4 - e$ and all vertices are low, so we finish by Lemma 4. Suppose instead that $w_2, w_4 \in S_2$. Color them both with 2, which makes $v_1$ and $v_2$ low. If $w_1$ became very low, then we finish greedily. Otherwise $w_3$ became low, so we finish by Lemma 4. Now suppose $v_1 \in S_2$, and color $v_1$ with 2. We have four possibilities. If $w_2$ and $w_3$ become low, then we can finish by Lemma 4. Similarly, if $w_4$ becomes very low, we delete it; now $v_2$ becomes low, so we can finish by Lemma 4. In the two remaining cases, we can finish greedily by repeatedly deleting very low vertices. □

## 4.2 Proofs via the Alon-Tarsi Theorem

Our goal in each of the next lemmas is to prove that a certain graph is $d_1$-paintable. For a digraph $\overrightarrow{D}$, we write $\text{diff}(\overrightarrow{D})$ to denote $|EE(\overrightarrow{D})| - |EO(\overrightarrow{D})|$. In each case we find an orientation $\overrightarrow{D}$ such that each vertex has indegree at least 2 and $\text{diff}(\overrightarrow{D}) \neq 0$. Now the Alon-Tarsi Theorem, specifically the generalization in Theorem B, proves the graph is $d_1$-paintable. To compute $\text{diff}(\overrightarrow{D})$, we typically want to avoid calculating $|EE(\overrightarrow{D})|$ and $|EO(\overrightarrow{D})|$ explicitly. Rather, we look for a parity-reversing bijection that pairs elements of $EE(\overrightarrow{D})$ with elements of $EO(\overrightarrow{D})$. In computing $\text{diff}(\overrightarrow{D})$, we can ignore all circulations paired by such a bijection. We also use the following trick to reduce our work. We explain it via an example, but it holds more generally.

Let $\overrightarrow{D}$ contain a 5-clique and two other vertices $w_1$ and $w_2$ such that for each $v$ either $d^+(v) \leq 3$ or $d^+(v) = 4$ and $w_1, w_2 \in N^+(v)$. In computing $\text{diff}(\overrightarrow{D})$, we want to restrict the difference to the set of circulations in which $d^+(w_1) \geq 1$ and $d^+(w_2) \geq 1$; call this $\text{diff}'(\overrightarrow{D})$. By inclusion-exclusion, we have $\text{diff}'(\overrightarrow{D}) = \text{diff}(\overrightarrow{D}) - \text{diff}(\overrightarrow{D} - w_1) - \text{diff}(\overrightarrow{D} - w_2) + \text{diff}(\overrightarrow{D} - w_1 - w_2)$. So it suffices to show that the final three terms on the right side are 0. If any term were nonzero, then, by the Alon-Tarsi Theorem, we would be able to color the corresponding subgraph from lists of size at most 4. However, the subgraph contains a 5-clique, making this impossible. Thus, each term is 0, and we have the desired equality. (In some cases we use a slight variation of this approach, instead concluding that the induced subgraph $H$ with $\text{diff}(H) \neq 0$ is $d_1$-paintable.) Finally, we combine this technique with the parity-reversing bijection mentioned above, by restricting the bijection only to the set of circulations where $d^+(w_1) \geq 1$ and $d^+(w_2) \geq 1$.

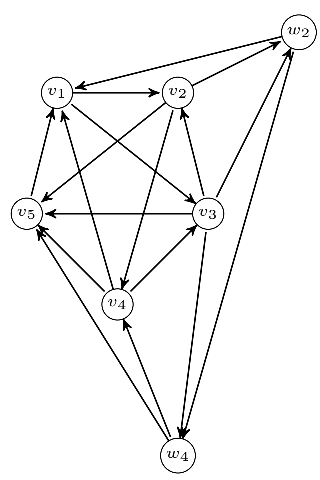



Figure 3: The orientation for Lemma 13.

**Lemma 13.** *Let $H$ be a 5-cycle $v_1, \dots, v_5$ with pendant edges at $v_2$ and $v_4$, leading to vertices $w_2$ and $w_4$, respectively, and let $w_2$ and $w_4$ have a common neighbor $x$ (off the cycle). Let $G = H^2 - x$; now $G$ is $d_1$-paintable.*

*Proof.* We orient $G$ to form $\overrightarrow{D}$ with the following out-neighborhoods: $N^+(v_1) = \{v_2, v_3\}$, $N^+(v_2) = \{w_2, v_4, v_5\}$, $N^+(w_2) = \{v_1, w_4\}$, $N^+(v_3) = \{v_2, w_2, w_4, v_5\}$, $N^+(v_4) = \{v_1, v_3, v_5\}$, $N^+(w_4) = \{v_4, v_5\}$, $N^+(v_5) = \{v_1\}$. See Figure 3.

We will show that $\mathrm{diff}(\overrightarrow{D}) \neq 0$. Since each vertex has at least two in-edges, this proves that $G$ is $d_1$-paintable. Let $R = \{v_3w_2, v_3w_4\}$. For any nonempty subset $S$ of $R$, we must have $\mathrm{diff}(\overrightarrow{D} \setminus R) = 0$. This is because each vertex on the 5-cycle has outdegree at most 3, so will get a list of size at most 4. And clearly, we cannot always color $K_5$ from lists of size at most 4. Thus, it suffices to count the difference, when restricted to the set $A$ of circulations $\overrightarrow{T}$ such that $v_3w_2, v_3w_4 \in \overrightarrow{T}$.

Let $\overrightarrow{T}$ be such a circulation. Note that $v_3v_2, v_3v_5 \notin \overrightarrow{T}$, and thus $v_1v_3, v_4v_3 \in \overrightarrow{T}$. Now we consider the 8 possible subsets of $\{w_4v_4, w_4v_5, v_4v_5\}$ in $\overrightarrow{T}$. Clearly $d^+(w_4) \geq 1$ and $d^-(v_5) \leq 1$. Also, we can pair the case $w_4v_4, v_4v_5 \in \overrightarrow{T}$ and $w_4v_5 \notin \overrightarrow{T}$ with the case coming from its complement. Thus, we can restrict to the case when $w_4v_4 \in \overrightarrow{T}$ and $v_4v_5 \notin \overrightarrow{T}$ (and we're not specifying whether $w_4v_5$ is in or out). Now consider the directed triangle $v_1v_2, v_2v_4, v_4v_1$. We can pair the cases when all or none of these edges are in $\overrightarrow{T}$. Thus we may assume that either exactly 1 or exactly 2 of these edges are in. Considering indegree and outdegree of $v_2$ shows that we must have $v_1v_2 \in \overrightarrow{T}$ and $v_2v_4, v_4v_1 \notin \overrightarrow{T}$. This implies $w_2v_1, v_5v_1 \in \overrightarrow{T}$. Now we have two ways to complete $\overrightarrow{T}$. We can have $v_2w_2, w_2w_4, w_4v_5 \in \overrightarrow{T}$ and $v_2v_4 \notin \overrightarrow{T}$ or vice versa. Each of these gives $|E(\overrightarrow{T})|$ odd; thus, we get $|\mathrm{diff}(D)| = 2$. □

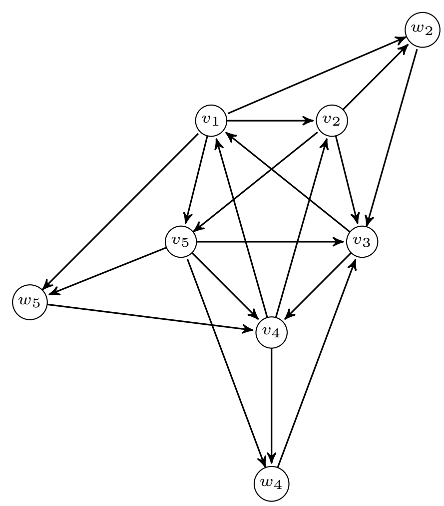



Figure 4: The orientation for Lemma 14.

**Lemma 14.** *Let $H$ be a 5-cycle $v_1, \dots, v_5$ with pendant edges at $v_2$, $v_4$, and $v_5$, leading to vertices $w_2$, $w_4$, and $w_5$, respectively. Let $G = H^2$; now $G$ is $d_1$-paintable.*

*Proof.* We orient $G$ to form $\overrightarrow{D}$ with the following out-neighborhoods: $N^+(v_1) = \{v_2, w_2, v_5, w_5\}$, $N^+(v_2) = \{w_2, v_3, v_5\}$, $N^+(w_2) = \{v_3\}$, $N^+(v_3) = \{v_1, v_4\}$, $N^+(v_4) = \{v_1, v_2, w_4\}$, $N^+(w_4) = \{v_3\}$, $N^+(v_5) = \{v_3, v_4, w_4, w_5\}$, $N^+(w_5) = \{v_4\}$. See Figure 4.

We will show that $\text{diff}(\overrightarrow{D}) \neq 0$. Since each vertex has at least two in-edges, this proves that $G$ is $d_1$-paintable. If $\text{diff}(\overrightarrow{D} - w_2) \neq 0$, then we are done, since $\overrightarrow{D} - w_2$ is $d_1$-paintable. Thus, we can assume that $\text{diff}(\overrightarrow{D} - w_2) = 0$. Similarly, we can assume that $\text{diff}(\overrightarrow{D} \setminus S) = 0$ for every $S \subseteq \{w_2, w_4, w_5\}$. Thus, it suffices to count the difference, when restricted to the set $A$ of circulations such that $d^+(w_2) = 1$, $d^+(w_4) = 1$, and $d^+(w_5) = 1$. Let $\overrightarrow{T}$ be such a circulation. So $w_2v_3, w_4v_3, w_5v_4 \in \overrightarrow{T}$. Now $d^+(v_3) = 2$, so $v_3v_1, v_3v_4 \in \overrightarrow{T}$ and $v_2v_3, v_5v_3 \notin \overrightarrow{T}$. In particular, $d^-(v_1) \geq 1$, so $d^+(v_1) \geq 1$.

Now we will pair some circulations in $A$ via a parity-reversing bijection. Consider the paths $v_1w_2$ and $v_1v_2, v_2w_2$. If a circulation contains all edges in one path and none in the other, then we can pair it via a bijection. The same is true for the paths $v_1w_5$ and $v_1v_5, v_5w_5$. Since $1 \leq d^+(v_1) \leq 2$, and also $d^-(w_2) = d^-(w_5) = 1$, the only way that $\overrightarrow{T}$ can avoid these cases is if either (i) $v_1v_2, v_1w_2 \in \overrightarrow{T}$ or (ii) $v_1v_5, v_1w_5 \in \overrightarrow{T}$. Before we consider these cases, note that in each case $v_4v_1 \in \overrightarrow{T}$.

Case (i): Now we must have $v_1w_5, v_1v_5 \notin \overrightarrow{T}$. Note that $v_2w_2 \notin \overrightarrow{T}$, which implies $v_4v_2 \notin \overrightarrow{T}$. Also $v_2v_5 \in \overrightarrow{T}$. Further, $d^-(w_5) = 1$ implies $v_5w_5 \in \overrightarrow{T}$, which in turn yields $v_5v_4, v_5w_4 \notin \overrightarrow{T}$. Finally, $v_4w_4 \in \overrightarrow{T}$. Thus, we have a unique $\overrightarrow{T}$ (with an odd number of edges).

Case (ii): Now we must have $v_1w_2, v_1v_2 \notin \overrightarrow{T}$ and also $v_5w_5 \notin \overrightarrow{T}$. Note that $v_2w_2 \in \overrightarrow{T}$, which implies that $v_4v_2 \in \overrightarrow{T}$ and also that $v_2v_5 \notin \overrightarrow{T}$. Now we get that either (a) $v_5v_4 \in \overrightarrow{T}$, and thus $v_4w_4 \in \overrightarrow{T}$ and $v_5w_4 \notin \overrightarrow{T}$ or else (b) $v_5w_4 \in \overrightarrow{T}$ and $v_5v_4, v_4w_4 \notin \overrightarrow{T}$. Again, by a parity-reversing bijection, we see that together these circulations contribute nothing to $\text{diff}(A)$ (in fact there is only one of each). Now combining Cases (i) and (ii), we get that $|\text{diff}(A)| = 1$, and in fact $|\text{diff}(\overrightarrow{D})| = 1$. Thus, $G$ is $d_1$-paintable. □

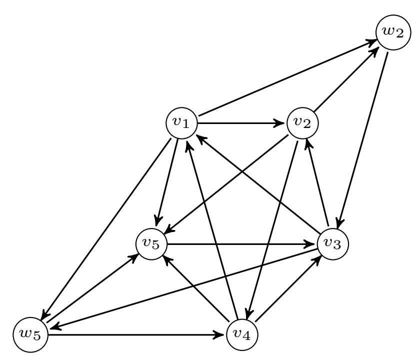



Figure 5: The orientation for Lemma 15.

**Lemma 15.** *Let $H$ be a 5-cycle $v_1, \ldots, v_5$ with pendant edges at $v_2$ and $v_5$, leading to vertices $w_2$ and $w_5$, respectively, and let $w_5$ and $v_3$ have a common neighbor $x$ (off the cycle). Let $G = H^2 - x$; now $G$ is $d_1$-paintable.*

*Proof.* We orient $G$ to form $\overrightarrow{D}$ with the following out-neighborhoods: $N^+(v_1) = \{v_2, w_2, v_5, w_5\}$, $N^+(v_2) = \{w_2, v_4, v_5\}$, $N^+(w_2) = \{v_3\}$, $N^+(v_3) = \{v_1, v_2, w_5\}$, $N^+(v_4) = \{v_1, v_3, v_5\}$, $N^+(v_5) = \{v_3\}$, $N^+(w_5) = \{v_4, v_5\}$. See Figure 5.

We will show that $\text{diff}(\overrightarrow{D}) \neq 0$. Since each vertex has at least two in-edges, this proves that $G$ is $d_1$-paintable. Note that for each nonempty subset $S \subseteq \{w_2, w_5\}$, we have $\text{diff}(\overrightarrow{D} \setminus S) = 0$, since otherwise we can color the corresponding subgraph from lists of size 4, even though it contains a 5-clique. So by inclusion-exclusion, we can restrict our count of diff to the set of circulations

$A$ where $w_2$ and $w_5$ each have positive indegree. Consider the paths $v_1w_2$ and $v_1v_2, v_2w_2$. Let $\overrightarrow{T}$ be a circulation in $A$. If $T$ contains all edges of one path and none of the other, then we can pair it via a parity-reversing bijection. So we assume we are not in these situations. Since $w_2$ has positive indegree, and hence indegree 1, we either have (i) $v_1w_2, v_1v_2 \in \overrightarrow{T}$ and $v_2w_2 \notin \overrightarrow{T}$ or (ii) $v_2w_2 \in \overrightarrow{T}$ and $v_1w_2, v_1v_2 \notin \overrightarrow{T}$.

Case (i): $v_1w_2, v_1v_2 \in \overrightarrow{T}$ and $v_2w_2 \notin \overrightarrow{T}$. Clearly $w_2v_3 \in \overrightarrow{T}$. Since $d^+(v_1) = 2$, we have $v_3v_1, v_4v_1 \in \overrightarrow{T}$ and $v_1v_5, v_1w_5 \notin \overrightarrow{T}$. Suppose $v_3v_2 \in \overrightarrow{T}$. Now also $v_2v_4, v_2v_5, v_5v_3 \in \overrightarrow{T}$. Finally, since $w_5$ has positive indegree, $v_3w_5, w_5v_4, v_4v_3 \in \overrightarrow{T}$. The resulting circulation is even. Suppose instead that $v_3v_2 \notin \overrightarrow{T}$. If $v_2v_5 \in \overrightarrow{T}$, then we get $v_5v_3, v_3w_5, w_5v_4 \in \overrightarrow{T}$. The resulting circulation is odd. If instead $v_2v_5 \notin \overrightarrow{T}$ and $v_2v_4 \in \overrightarrow{T}$, then we have three possibilities to ensure $d^+(w_5) > 0$. Either $v_3w_5, w_5v_4, v_4v_5, v_5v_3 \in \overrightarrow{T}$ or $v_3w_5, w_5v_4, v_4v_3 \in \overrightarrow{T}$ or $v_3w_5, w_5v_5, v_5v_3 \in \overrightarrow{T}$. Two of the resulting circulations are odd and one is even. Thus in total for Case (i), we have one more odd circulation than even.

Case (ii): $v_2w_2 \in \overrightarrow{T}$ and $v_1w_2, v_1v_2 \notin \overrightarrow{T}$. We have $v_2w_2 \in \overrightarrow{T}$, which implies $w_2v_3 \in \overrightarrow{T}$ and $v_3v_2 \in \overrightarrow{T}$. This further yields $v_2v_4, v_2v_5 \notin \overrightarrow{T}$. Again we will pair some of the circulations in $A$ via a parity-reversing bijection. Consider the paths $v_3w_5$ and $v_3v_1, v_1w_5$. If a circulation contains all edges in one path and none in the other, then we can pair it via a bijection. Since $1 \le d^-(w_5)$, the only way that $\overrightarrow{T}$ can avoid these cases is if either (a) $v_1w_5 \in \overrightarrow{T}$ and $v_3v_1 \notin \overrightarrow{T}$ or (b) $v_3v_1 \in \overrightarrow{T}$ and $v_1w_5 \notin \overrightarrow{T}$ (and thus $v_3w_5 \in \overrightarrow{T}$ or (c) $v_3v_1, v_1w_5, v_3w_5 \in \overrightarrow{T}$. Consider (a). $v_1w_5 \in \overrightarrow{T}$ implies $v_4v_1 \in \overrightarrow{T}$, and thus $w_5v_4 \in \overrightarrow{T}$. We also have the option of all or none of $v_3w_5, w_5v_5, v_5v_3$ in $\overrightarrow{T}$. One of the resulting circulations is odd and the other is even. Consider (b). Now $v_3v_1 \in \overrightarrow{T}$ and $v_1w_5 \notin \overrightarrow{T}$ imply $v_1v_5 \in \overrightarrow{T}$, and thus $v_5v_3 \in \overrightarrow{T}$. Now $d^+(w_5) > 0$ implies $v_3w_5, w_5v_4, v_4v_3 \in \overrightarrow{T}$. The resulting circulation is odd. Consider (c). Now we get $w_5v_5 \in \overrightarrow{T}$, which implies $v_5v_3 \in \overrightarrow{T}$. We also get $w_5v_4 \in \overrightarrow{T}$, which implies $\overrightarrow{v_4v_3} \in \overrightarrow{T}$. The resulting circulation is even. Thus in total for Case (ii), we have the same number of even and odd circulations.

So combining Cases (i) and (ii), we have one more odd circulation than even. Thus $\mathrm{diff}(\overrightarrow{D}) \neq 0$, so $G$ is $d_1$-paintable. ☐

Form $\overrightarrow{P_n}$ from $(P_n)^2$ by orienting all edges from left to right. Number the vertices as $v_1, \ldots, v_n$ from left to right. A subgraph $\overrightarrow{T} \subseteq \overrightarrow{P_n}$ is *weakly eulerian* if each vertex $w \notin \{v_1, v_n\}$ satisfies $d^+(w) = d^-(w)$ and $d^+(v_1) = d^-(v_n) = i$ for some $i \in \{1, 2\}$. Let $EE_i(\overrightarrow{P_n})$ (resp. $EO_i(\overrightarrow{P_n})$) denote the set of even (resp. odd) weakly eulerian subgraphs where $d^+(v_1) = d^-(v_n) = i$. Finally, let $f_i(n) = |EE_i(\overrightarrow{P_n})| - |EO_i(\overrightarrow{P_n})|$. We will not apply the following lemma directly to find $d_1$-paintable subgraphs. However, it will be helpful in the proof for the remaining $d_1$-paintable graph, which includes cycles of arbitrary length.

**Lemma 16.** *If $n = 3k + j$ for some positive integer $k$ and $j \in \{-1, 0, 1\}$, then $f_1(n) = j$ and for $n \ge 4$ also $f_2(n) = -f_1(n-2)$, with $f_i(n)$ as defined above.*

*Proof.* Rather than directly counting weakly eulerian subgraphs, we again use a parity-reversing bijection. We first prove that $f_2(n) = -f_1(n-2)$. The complement of each $\overrightarrow{D} \in EE_2(\overrightarrow{P_n}) \cup EO_2(\overrightarrow{P_n})$ has $d^+(v_2) = d^-(v_{n-1}) = 1$ and $d^+(w) = d^-(w)$ for each $w \notin \{v_1, v_2, v_{n-1}, v_n\}$ (and $d^+(v_1) = d^-(v_n) = d^-(v_2) = d^+(v_{n-1}) = 0$). Since $\overrightarrow{P_n}$ has $2n-3$ edges, each digraph has parity opposite its complement; so $f_2(n) = -f_1(n-2)$.

Now we determine $f_1(n)$. Let $\overrightarrow{T}$ be a weakly eulerian subgraph with $d^+(v_1) = 1$. Consider the directed paths $v_1v_3$ and $v_1v_2, v_2v_3$. If $\overrightarrow{T}$ contains all of one path and none of the other, then we can pair $\overrightarrow{T}$ with its complement, which has opposite parity. If neither of these cases holds, then we must have $v_1v_2, v_2v_4 \in \overrightarrow{T}$ and $v_1v_3, v_2v_3 \notin \overrightarrow{T}$. This yields $f_1(n) = f_1(n-3)$. It remains only to check that $f_1(2) = -1$, $f_1(3) = 0$, and $f_1(4) = 1$. □

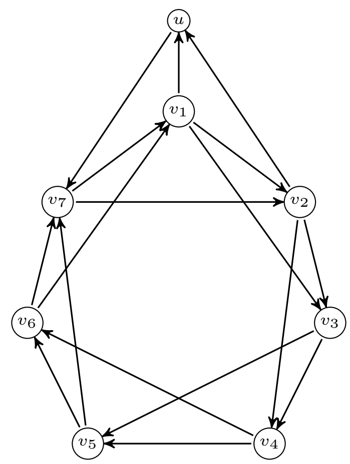



Figure 6: The orientation for Lemma 17 with $n = 7$.

**Lemma 17.** *Cycle + one pendant edge: Let $J_n$ consist of an $n$-cycle on vertices $v_1, \ldots, v_n$ (in clockwise order) with a pendant edge at $v_1$ leading to vertex $u$. Form $\overrightarrow{D_n}$ by squaring $J_n$ and orienting the edges as follows. Orient edges $v_iv_{i+1}$ and $v_iv_{i+2}$ away from $v_i$ (with subscripts modulo $n$). Orient $uv_n$ away from $u$ and $v_1u$ and $v_2u$ toward $u$. We will show that $\mathrm{diff}(\overrightarrow{D_n}) \neq 0$ when $n \not\equiv 2 \bmod 3$ (or else $f(\overrightarrow{D_n} - u) \not\equiv 0$).*

*Proof.* Form $\overrightarrow{D_n}$ as in the lemma. We will show that $\mathrm{diff}(\overrightarrow{D_n}) \neq 0$, and thus $J_n^2$ is $d_1$-paintable. We may assume that $\mathrm{diff}(\overrightarrow{D_n - u}) \neq 0$, for otherwise $\overrightarrow{D_n - u}$ is $d_1$-paintable. Thus, restricting our count to the set $A$ of circulations with $d^+(u) = 1$ does not affect the difference. Let $\overrightarrow{T}$ be a circulation in $A$. Consider the directed paths $v_1u$ and $v_1v_2, v_2u$. If $\overrightarrow{T}$ contains all edges of one path and none of the other, then we can pair $\overrightarrow{T}$ via a parity-reversing bijection. So we assume we are not in one of those cases. Clearly $\overrightarrow{T}$ contains $\overrightarrow{uv_n}$ and exactly one of $v_1u$ and $v_2u$. Thus either (i) $v_2u \in \overrightarrow{T}$ and $v_1u, v_1v_2 \notin \overrightarrow{T}$ or (ii) $v_1u, v_1v_2 \in \overrightarrow{T}$ and $v_2u \notin \overrightarrow{T}$.

Case (i): $v_2u \in \overrightarrow{T}$ and $v_1u, v_1v_2 \notin \overrightarrow{T}$. Since $v_2u \in \overrightarrow{T}$ and $v_1v_2 \notin \overrightarrow{T}$, we must have $v_nv_2 \in \overrightarrow{T}$ and $v_2v_3, v_2v_4 \notin \overrightarrow{T}$. By removing edges $uv_n, v_nv_2, v_2u$, we see that these circulations are in bijection with the circulations in $\overrightarrow{D}_n - u - v_2$ (with the parity of each subgraph reversed). If we exclude the empty graph, these circulations are in bijection with those counted by $f_1(n-1)$, since $d^+(v_1) = 1$ and $d^-(v_3) = 1$. Adding 1 for the empty subgraph, this difference is $1 - f_1(n-1)$, and when we account for removing edges $uv_n, v_nv_2, v_2u$, the difference is $-1 + f_1(n-1)$.

Case (ii): $v_1u, v_1v_2 \in \overrightarrow{T}$ and $v_2u \notin \overrightarrow{T}$. Since $v_1u, v_1v_2 \in \overrightarrow{T}$, we must have $v_{n-1}v_1, v_nv_1 \in \overrightarrow{T}$ and $v_1v_3 \notin \overrightarrow{T}$. After removing edges $v_nv_1, v_1u, uv_n$, we see that these circulations are in bijection with the circulations in $\overrightarrow{D}_n - u - v_nv_1 - v_1v_3$ that contain edges $v_{n-1}v_1$ and $v_1v_2$. We will count the difference of these even and odd circulations, then multiply the total by $-1$ (to account for removing edges $v_1u, uv_n, v_nv_1$) before adding to the total above.

We consider two subcases: $v_nv_2 \notin \overrightarrow{T}$ and $v_nv_2 \in \overrightarrow{T}$. In the first case, these circulations are in bijection with circulations of $\overrightarrow{D}_{n-1} - u - v_1$ (since $d^+(v_n) = 0$ and $v_1$ may be suppressed). This difference is counted by $f_1(n-2)$. In the second case, the difference is counted by $-f_2(n)$, since we may think of deleting $v_1v_2$ and replacing $v_nv_2$ with $v_nv_1$; our path now starts at $v_2$ and runs through $v_n$ to $v_1$ (and the parity is changed when accounting for $v_1v_2$).

Thus, the total difference in Case (ii) is counted by $f_1(n-2)-f_2(n)$. Thus, the total difference overall is counted by $-1+f_1(n-1)-f_1(n-2)+f_2(n) = -1+f_1(n-1)-2f_1(n-2)$. Substituting values from Lemma 16 shows that this expression is non-zero when $n \not\equiv 2 \bmod 3$. ☐

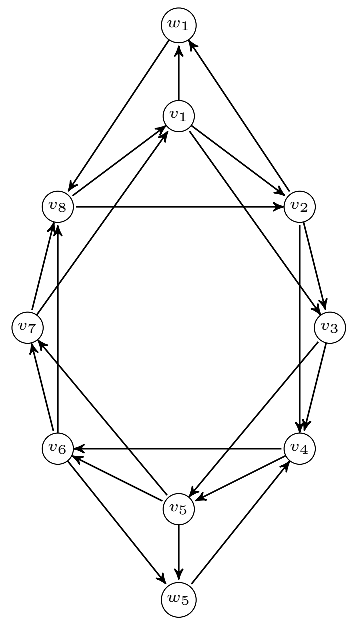



Figure 7: The orientation for Lemma 18 with $n = 8$.

**Lemma 18.** *Cycle + two pendant edges: For $n \geq 7$, let $J_n$ consist of an $n$-cycle on vertices $v_1, \ldots, v_n$ (in clockwise order) with pendant edges at $v_1$ and $v_5$ leading to vertices $w_1$ and $w_5$. Form $\overrightarrow{D}_n$ by squaring $J_n$ and orienting the edges as follows. Orient edges $v_iv_{i+1}$ and $v_iv_{i+2}$ away from $v_i$ (with subscripts modulo $n$). Orient $w_1v_n$ away from $w_1$ and $v_1w_1$ and $v_2w_1$ toward $w_1$; similarly, orient $w_5v_4$ away from $w_5$ and $v_5w_5$ and $v_6w_5$ toward $w_5$. We will show that $f(\overrightarrow{D_n}) \neq 0$ (or else $f(\overrightarrow{D_n} \setminus B) \neq 0$ for some subset $B \subseteq \{w_1, w_5\}$).*

*Proof.* Form $\overrightarrow{D_n}$ as in the lemma. We will show that $\mathrm{diff}(\overrightarrow{D_n}) \neq 0$, and thus $J_n^2$ is $d_1$-paintable. For each nonempty $B \subseteq \{w_1, w_5\}$, we may assume that $\mathrm{diff}(\overrightarrow{D_n} \setminus B) = 0$, for otherwise $\overrightarrow{D_n} \setminus B$ is $d_1$-paintable. Thus, restricting our count to the set $A$ of circulations with $d^+(w_1) = 1$ and $d^+(w_5) = 1$ does not affect the difference.

Let $\overrightarrow{T}$ be a circulation in $A$. Clearly $\overrightarrow{T}$ contains $\overrightarrow{w_1v_n}$ and exactly one of $\overrightarrow{v_1w_1}$ and $\overrightarrow{v_2w_1}$. Consider the directed paths $v_1w_1$ and $v_1v_2, v_2w_1$. If $\overrightarrow{T}$ contains all edges of one path and none of the other, then we can pair $\overrightarrow{T}$ via a parity-reversing bijection. So we assume we are not in one of those cases. Thus either (i) $v_2w_1 \in \overrightarrow{T}$ and $v_1w_1, v_1v_2 \notin \overrightarrow{T}$ or (ii) $v_1w_1, v_1v_2 \in \overrightarrow{T}$ and $v_2w_1 \notin \overrightarrow{T}$.

Now we consider the directed paths $v_5w_5$ and $v_5v_6, v_6w_5$. Among those circulations, within Cases (i) and (ii), where $\overrightarrow{T}$ contains all of one path and none of the other we again pair $\overrightarrow{T}$ via a parity-reversing bijection, by removing the edges of one path and adding the edges of the other.

Thus, we need only consider two subcases in each case: (1) $v_6w_5 \in \overrightarrow{T}$ and $v_5w_5, v_5v_6 \notin \overrightarrow{T}$ and (2) $v_5w_5, v_5v_6 \in \overrightarrow{T}$ and $v_6w_5 \notin \overrightarrow{T}$.

Case (i.1): $v_2w_1 \in \overrightarrow{T}$ and $v_1w_1, v_1v_2 \notin \overrightarrow{T}$ and also $v_6w_5 \in \overrightarrow{T}$ and $v_5w_5, v_5v_6 \notin \overrightarrow{T}$. Since $v_2w_1 \in \overrightarrow{T}$, we must have $v_nv_2 \in \overrightarrow{T}$ and also $v_2v_3, v_2v_4 \notin \overrightarrow{T}$. Similarly, since $v_6w_5 \in \overrightarrow{T}$, we must have $v_4v_6 \in \overrightarrow{T}$ and also $v_6v_7, v_6v_8 \notin \overrightarrow{T}$. Since both triangles $w_1v_nv_2$ and $v_4v_6w_5$ must be included in every circulation under consideration, we may remove $w_1, v_2, w_5, v_6$ without changing the total difference. Now any non-empty circulation must contain both $v_1v_3$ and $v_5v_7$. But we have a parity reversing bijection between those circulations containing $v_3v_5$ and those containing $v_3v_4, v_4v_5$, so for non-empty circulations the difference is zero. Thus after adding in the empty circulation, we see that the total difference is 1 for this case.

Case (i.2): $v_2w_1 \in \overrightarrow{T}$ and $v_1w_1, v_1v_2 \notin \overrightarrow{T}$ and also $v_5w_5, v_5v_6 \in \overrightarrow{T}$ and $v_6w_5 \notin \overrightarrow{T}$. Since $v_2w_1 \in \overrightarrow{T}$, we must have $v_nv_2 \in \overrightarrow{T}$ and hence $v_2v_3, v_2v_4 \notin \overrightarrow{T}$. Since the triangle $w_1v_nv_2$ must be included in every circulation under consideration, we may remove $w_1, v_2$ at the cost of negating the difference. Since $v_5w_5, v_5v_6 \in \overrightarrow{T}$, we must have $w_5v_4, v_3v_5, v_4v_5 \in \overrightarrow{T}$ and $v_5v_7 \notin \overrightarrow{T}$. But then $v_3v_4 \notin \overrightarrow{T}$ and hence $v_4v_6 \notin \overrightarrow{T}$. Now we may remove $w_5$ and $v_4$ at the cost of negating the difference again. Now removing $v_3$ and $v_5$ we lose three edges that must be in every circulation and the resulting difference is counted by $f_1(n-4)$; the paths run from $v_6$ through $v_n$ to $v_1$. Hence this case contributes $-f_1(n-4)$ to the difference.

Case (ii.1): $v_1w_1, v_1v_2 \in \overrightarrow{T}$ and $v_2w_1 \notin \overrightarrow{T}$ and also $v_6w_5 \in \overrightarrow{T}$ and $v_5w_5, v_5v_6 \notin \overrightarrow{T}$. Since $v_1w_1, v_1v_2 \in \overrightarrow{T}$, we get $v_nv_1, v_{n-1}v_1 \in \overrightarrow{T}$. Since $v_6w_5 \in \overrightarrow{T}$ and $v_5v_6 \notin \overrightarrow{T}$, we get $v_4v_5 \in \overrightarrow{T}$ and $v_6v_7, v_6v_8 \notin \overrightarrow{T}$. Since we have $v_{n-1}v_1 \in \overrightarrow{T}$, we must also have $v_5v_7 \in \overrightarrow{T}$. Since $v_6v_7, v_6v_8 \notin \overrightarrow{T}$ and $v_5v_7 \in \overrightarrow{T}$, we get $d^+(v_2) = 1$. This also implies $d^+(v_{n-1}) = 1$. Now when $n \geq 9$ our difference is counted by $-f_1(3)f_1(n-7)$. Here $f_1(3)$ accounts for the edges of the path from $v_2$ to $v_5$ and $f_1(n-7)$ accounts for the edges of the path from $v_7$ to $v_{n-1}$ (and the $-1$ accounts for the 9 edges that are present but not on either of these paths). Since $f_1(3) = 1$, the total for this case is $-f_1(n-7)$. When $n = 8$ the total is $-f_1(3) = -1$ and when $n = 7$ the total is 0, since $v_{n-1} = v_6$. Now by Lemma 16, together with checking the cases $n = 7$ and $n = 8$, we get that this case is counted by $-f_1(n-4)$.

Case (ii.2): $v_1w_1, v_1v_2 \in \overrightarrow{T}$ and $v_2w_1 \notin \overrightarrow{T}$ and also $v_5w_5, v_5v_6 \in \overrightarrow{T}$ and $v_6w_5 \notin \overrightarrow{T}$. Since $v_1w_1, v_1v_2 \in \overrightarrow{T}$, we must have $w_1v_n, v_nv_1, v_{n-1}v_1 \in \overrightarrow{T}$ and $v_1v_3 \notin \overrightarrow{T}$. Since $v_5w_5, v_5v_6 \in \overrightarrow{T}$, we must have $w_5v_4, v_3v_5, v_4v_5 \in \overrightarrow{T}$ and $v_5v_7 \notin \overrightarrow{T}$. Suppose $v_nv_2 \notin \overrightarrow{T}$. Now $v_2v_4 \notin \overrightarrow{T}$, so $d^+(v_4) = 1$. Now our problem reduces to computing $-f_1(n-6)$; the $f(n-6)$ accounts for the edges on the path from $v_6$ to $v_{n-1}$ and the $-1$ accounts for the 11 other edges that are present. Suppose instead that $v_nv_2 \in \overrightarrow{T}$. Now our problem reduces to computing $f_2(n-4)$, accounting for the edges on the two paths from to $v_1$ (after replacing $v_nv_2$ by $v_vv_1$) and the 12 edges present but not on these paths.

So, combining the contributions from all cases we get that the difference is $1 - f_1(n-4) - f_1(n-4) - f_1(n-6) + f_2(n-4)$. By Lemma 16 this is $1 - 2(f_1(n-4) + f_1(n-6)) \neq 0$ when $n \geq 8$. When $n = 7$ the difference is $1 - 2f_1(3) - 1 + f_2(3) = -1$. $\square$

For $n \geq 4$, a subgraph $\overrightarrow{T} \subseteq \overrightarrow{P_n}$ is *extra weakly eulerian* if each vertex $w \notin \{v_1, v_2, v_{n-1,}v_n\}$ satisfies $d^+(w) = d^-(w)$, $d^+(v_1) = d^-(v_n) = 1$, $d^+(v_2) = d^-(v_2)+1$ and $d^-(v_{n-1}) = d^+(v_{n-1})+1$ Let $EE^*(\overrightarrow{P_n})$ (resp. $EO^*(\overrightarrow{P_n})$) denote the set of even (resp. odd) extra weakly eulerian subgraphs. Finally, let $g(n) = |EE^*(\overrightarrow{P_n})| - |EO^*(\overrightarrow{P_n})|$. Lemma 19 is analogous to Lemma 16, but for extra weakly eulerian subgraphs.

**Lemma 19.** *If $n = 3k + j \geq 4$ for a positive integer $k$ and $j \in \{-1, 0, 1\}$, then $g(n) = -j$.*

*Proof.* Let $\overrightarrow{T} \subseteq \overrightarrow{P_n}$ be extra weakly eulerian. Consider the directed paths $v_1v_3$ and $v_1v_2, v_2v_3$. If

$\overrightarrow{T}$ contains all of one path but none of the other, then we can pair $\overrightarrow{T}$ with its complement which has opposite parity. If neither of these cases holds, then we must have either $v_1v_3, v_2v_3 \in \overrightarrow{T}$ and $v_1v_2 \notin \overrightarrow{T}$ or $v_1v_2 \in \overrightarrow{T}$ and $v_1v_3, v_2v_3 \notin \overrightarrow{T}$. The latter case is impossible, so suppose we have $v_1v_3, v_2v_3 \in \overrightarrow{T}$ and $v_1v_2 \notin \overrightarrow{T}$. Then $v_3v_4, v_3v_5 \in \overrightarrow{T}$ and $v_2v_4 \notin \overrightarrow{T}$. Hence the difference is counted by $g(n-3)$. It remains only to check that $g(4) = -1$, $g(5) = 1$ and $g(6) = 0$. $\square$

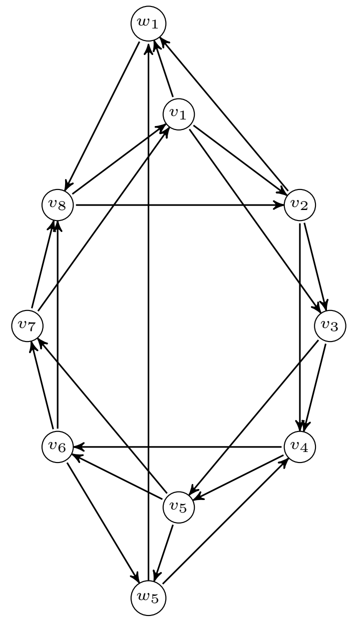



Figure 8: The orientation for Lemma 20.

**Lemma 20.** *8-cycle + two pendant edges + extra edge: Let $J_8$ consist of an 8-cycle on vertices $v_1, \ldots, v_8$ (in clockwise order) with pendant edges at $v_1$ and $v_5$ leading to vertices $w_1$ and $w_5$. Form $\overrightarrow{D}_8$ by squaring $J_8$, adding the edge $w_1w_5$ and orienting the edges as follows. Orient edges $v_iv_{i+1}$ and $v_iv_{i+2}$ away from $v_i$ (with subscripts modulo 8). Orient $w_1v_8$ away from $w_1$ and $v_1w_1$ and $v_2w_1$ toward $w_1$; similarly, orient $w_5v_4$ away from $w_5$ and $v_5w_5$ and $v_6w_5$ toward $w_5$. Finally, orient $w_5w_1$ toward $w_1$. We will show that $f(\overrightarrow{D_8}) \neq 0$ (or else $f(\overrightarrow{D_8} \setminus B) \neq 0$ for some subset $B \subseteq \{w_1, w_5\}$).*

*Proof.* Form $\overrightarrow{D_8}$ as in the lemma. Suppose $f(\overrightarrow{D_8} \setminus B) = 0$ for each subset $\emptyset \neq B \subseteq \{w_1, w_5\}$. Then by Lemma 18, we have $\mathrm{diff}(\overrightarrow{D_8} - w_5w_1) \neq 0$. Hence it will suffice to show that the circulations of $\overrightarrow{D_8}$ containing $w_5w_1$ are half odd and half even.

Let $\overrightarrow{T}$ be a circulation of $\overrightarrow{D_8}$ containing $w_5w_1$. Then $w_1v_8 \in \overrightarrow{T}$ and $v_1w_1, v_2w_1 \notin \overrightarrow{T}$. After suppressing $w_1$, we are looking at all circulations containing $w_5v_8$.

Consider the directed paths $v_5w_5$ and $v_5v_6, v_6w_5$. If $\overrightarrow{T}$ contains all edges of one path and none of the other, then we can pair $\overrightarrow{T}$ via a parity-reversing bijection. So we assume we are not in one of those cases. Thus either (i) $v_6w_5 \in \overrightarrow{T}$ and $v_5w_5, v_5v_6 \notin \overrightarrow{T}$, (ii) $v_5w_5, v_5v_6 \in \overrightarrow{T}$ and $v_6w_5 \notin \overrightarrow{T}$, (iii) $v_5w_5, v_5v_6, v_6w_5 \in \overrightarrow{T}$ or (iv) $v_6w_5, v_5w_5 \in \overrightarrow{T}$ and $v_5v_6 \notin \overrightarrow{T}$.

Case (i): $v_6w_5 \in \overrightarrow{T}$ and $v_5w_5, v_5v_6 \notin \overrightarrow{T}$. Then $v_4v_6 \in \overrightarrow{T}$ and $w_5v_4, v_6v_7, v_6v_8 \notin \overrightarrow{T}$. Now we can suppress $v_6$ and $w_5$. First suppose $v_5v_7 \notin \overrightarrow{T}$. Now $v_7, v_5 \notin \overrightarrow{T}$ and what remains is counted by $-f_1(5)$. Instead suppose $v_5v_7 \in \overrightarrow{T}$. Then the difference is counted by $g(7)$; the path is from $v_7$ to $v_5$. Hence the total difference is $g(7) - f_1(5) = -1 - (-1) = 0$.

Case (ii): $v_5w_5, v_5v_6 \in \overrightarrow{T}$ and $v_6w_5 \notin \overrightarrow{T}$. Then $v_3v_5, v_4v_5 \in \overrightarrow{T}$ and $w_5v_4, v_5v_7 \notin \overrightarrow{T}$. Now we can suppress $w_5$. First suppose $v_4v_6 \in \overrightarrow{T}$. There is only one possible circulation and it contains all edges except $v_7v_8$; this circulation is odd, hence the difference is $-1$. Now suppose $v_4v_6 \notin \overrightarrow{T}$. If $v_6v_7 \in \overrightarrow{T}$, then $v_6v_8 \notin \overrightarrow{T}$ and the difference is counted by $-g(6)$; the path is from $v_7$ to $v_4$. If $v_6v_7 \notin \overrightarrow{T}$, then $v_6v_8, v_8v_1, v_8v_2 \in \overrightarrow{T}$ and $v_7 \notin \overrightarrow{T}$. Now the difference is counted by $-g(4)$; the path is from $v_1$ to $v_4$. Hence the total difference is $-1 - g(6) - g(4) = 0$.

Case (iii): $v_5w_5, v_5v_6, v_6w_5 \in \overrightarrow{T}$. Then $w_5v_4, v_3v_5, v_4v_5 \in \overrightarrow{T}$ and $v_5v_7 \notin \overrightarrow{T}$. If $v_4v_6, v_6v_7 \in \overrightarrow{T}$, then the difference is counted by $g(6)$; the path is from $v_7$ to $v_4$. Since $v_6v_7 \in \overrightarrow{T}$ and $v_4v_6 \notin \overrightarrow{T}$ is impossible, we may assume either $v_4v_6 \in \overrightarrow{T}$ and $v_6v_7 \notin \overrightarrow{T}$ or $v_4v_6, v_6v_7 \notin \overrightarrow{T}$. Suppose we are in the former case. Then $v_6v_8, v_8v_1, v_8v_2 \in \overrightarrow{T}$ and $v_7 \notin \overrightarrow{T}$. This difference is counted by $g(4)$; the path is from $v_1$ to $v_4$. Now suppose $v_4v_6, v_6v_7 \notin \overrightarrow{T}$. Then $v_7 \notin \overrightarrow{T}$ and $v_6v_8 \notin \overrightarrow{T}$. This difference is counted by $f_1(4)$; the path is from $v_8$ to $v_3$. Hence the total difference is $g(6) + g(4) + f_1(4) = 0$.

Case (iv): $v_6w_5, v_5w_5 \in \overrightarrow{T}$ and $v_5v_6 \notin \overrightarrow{T}$. Then $w_5v_4, v_4v_6 \in \overrightarrow{T}$ and $v_6v_7, v_6v_8 \notin \overrightarrow{T}$. If $v_5v_7 \notin \overrightarrow{T}$, then $v_7 \notin \overrightarrow{T}$ and the difference is counted by $f_1(6) = 0$; the path is from $v_8$ to $v_5$. Hence we may assume $v_5v_7 \in \overrightarrow{T}$. Then $v_3v_5, v_4v_5 \in \overrightarrow{T}$ and the difference is counted by $g(6) = 0$; the path is from $v_7$ to $v_4$.

So in each of the four cases, half the circulations are even and half are odd. Thus, the difference is not affected by the circulations that use edge $w_5w_1$. Now by Lemma 18, $f(\overrightarrow{D}) \neq 0$, so $\overrightarrow{D}$ is $d_1$-paintable. □

# 5 Generalizing to Alon-Tarsi number

Excepting the direct proofs of paintability in Section 4.1, we've actually proved that all the excluded subgraphs have a good Alon-Tarsi orientation. This suggests that the main theorem might hold more generally for the Alon-Tarsi number $\mathrm{AT}(G)$—the least $k$ for which $G$ has an orientation $\vec{D}$ with $\Delta^+(\vec{D}) \leq k-1$ and $EE(\vec{D}) \neq EO(\vec{D})$. Here we show that this is indeed the case.

**Main Theorem for** AT. *If $G$ is a connected graph with maximum degree $\Delta \geq 3$ and $G$ is not the Peterson graph, the Hoffman-Singleton graph, or a Moore graph with $\Delta = 57$, then* $\mathrm{AT}(G^2) \leq \Delta^2 - 1$.

The proof is identical to the paintability proof except we need to replace all the auxiliary lemmas with their AT counterparts. First the two subgraph lemmas; these are actually easier to prove in the AT context.

**Lemma 21.** *Let $G$ be a graph with maximum degree $\Delta$ and $H$ be an induced subgraph of $G$ that is $d_1$-AT. If $G \setminus H$ is $(\Delta-1)$-AT, then $G$ is $(\Delta-1)$-AT.*

*Proof.* Let $G$ and $H$ satisfy the hypotheses. Take an orientation of $G \setminus H$ demonstrating that it is $(\Delta-1)$-AT and an orientation of $H$ demonstrating that it is $d_1$-AT. Now orient all the edges between $H$ and $G \setminus H$ into $G \setminus H$. Call the resulting oriented graph $\vec{D}$. Then $\vec{D}$ satisfies the outdegree requirements of being $(\Delta-1)$-AT since the outdegree of the vertices in $G \setminus H$ haven't changed and the outdegree of each $v \in V(H)$ has increased by $d_G(v) - d_H(v)$. Since no directed cycle in $D$ has vertices in both $H$ and $\vec{D} \setminus H$, the circulations of $\vec{D}$ are just all pairings of circulations of $H$ and $\vec{D} \setminus H$. Therefore $EE(\vec{D}) - EO(\vec{D}) = EE(H)EE(\vec{D} \setminus H) + EO(H)EO(\vec{D} \setminus$

$H)-(EE(H)EO(\vec{D}\setminus H)+EO(H)EE(\vec{D}\setminus H)) = (EE(H)-EO(H))(EE(\vec{D}\setminus H)-EO(\vec{D}\setminus H)) \neq 0$. Hence $G$ is $(\Delta-1)$-AT. $\square$

**Lemma 22.** *Let $G$ be a graph with maximum degree $\Delta$ and let $H$ be an induced subgraph of $G^2$. If $H$ is $d_1$-AT, then $G^2$ is $d_1$-AT. If there exists $v$ with $d_{G^2}(v) < \Delta^2-1$, then $G^2$ is $(\Delta^2-1)$-AT.*

*Proof.* We prove the first statement first. Form $G'$ from $G$ by contracting $V(H)$ to a single vertex $r$. Let $T$ be a spanning tree in $G'$ rooted at $r$. Let $\sigma$ be an ordering of the vertices of $G \setminus H$ by nonincreasing distance in $T$ from $r$. Take an orientation of $H$ demonstrating that it is $d_1$-AT; direct all edges between $H$ and $G \setminus H$ towards $G \setminus H$ and direct all other edges of $G^2$ toward the vertex that comes earlier in $\sigma$. Call the resulting oriented graph $\vec{D}$. By construction, all circulations in $\vec{D}$ are contained in $H$ and hence $EE(\vec{D}) \neq EO(\vec{D})$. It is clear that every vertex in $\vec{D}$ has indegree at least two and hence $G^2$ is $d_1$-AT.

Now we prove the second statement, which has a similar proof. Suppose there exists $v$ with $d_{G^2}(v) < \Delta^2-1$. As before we order the vertices by nonincreasing distance in some spanning tree $T$ from $v$, and we put $v$ and some neighbor $u$ last in $\sigma$. Since $d_{G^2}(v) < \Delta^2-1$, either (i) $v$ lies on a 3-cycle or 4-cycle or else (ii) $d_G(v) < \Delta$ or $v$ has some neighbor $u$ with $d_G(u) < \Delta$; in Case (ii), by symmetry we assume $d_G(v) < \Delta$. In Case (i), $d_{G^2}(u) \leq \Delta^2-1$ for some neighbor $u$ of $v$ on the short cycle and by assumption $d_{G^2}(v) < \Delta^2-1$; so the two final vertices of $\sigma$ are $u$ and $v$. In Case (ii), we again have $d_{G^2}(v) < \Delta^2-1$ and $d_{G^2}(u) \leq \Delta^2-1$, so again $u$ and $v$ are last in $\sigma$. $\square$

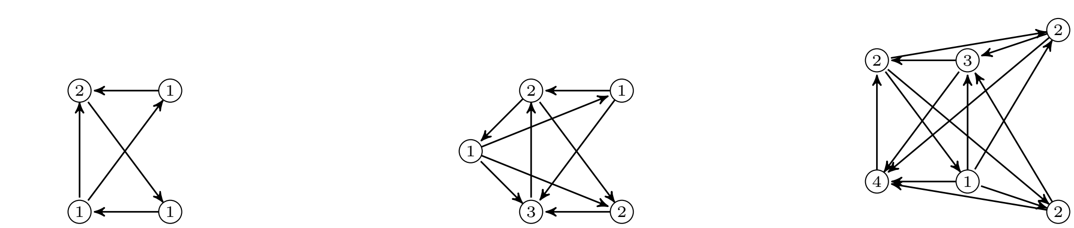
The proof of Lemma 22 proves something slightly more general, which we record in the following corollary.

**Corollary 23.** *Let $G$ be a graph with maximum degree $\Delta$ and let $H$ be an induced subgraph of $G^2$. Let $f(v) = d(v)-1$ for each high vertex of $G^2$ and $f(v) = d(v)$ for each low vertex. If $H$ is $f$-AT, then $G^2$ is $(\Delta^2-1)$-AT.*

Now each of Lemmas 13, 14, 15, 16, 17, and 18 was already proved for $AT$. It remains to prove the lemmas in Section 4.1 for $AT$. We do this by exhibiting in Figures 9 and 10 a good Alon-Tarsi orientation for each. For brevity, we will not prove here that the counts differ; instead we give the actual even/odd circulation counts for the reader to check at her leisure. Each vertex will be labeled with its indegree for easy checking. Note that three of the cases in Lemma 7 are handled by Lemmas 10, 11, and 12 (none of which depend on Lemma 7).

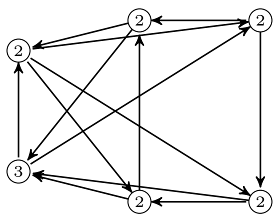
We conclude by generalizing the conjectures we mentioned in the introduction to the Alon-Tarsi number.

**Conjecture 6** (Borodin-Kostochka Conjecture (Alon-Tarsi version))**.** *If $G$ is a graph with $\Delta \geq 9$ and $\omega \leq \Delta-1$, then* $\mathrm{AT}(G) \leq \Delta-1$.

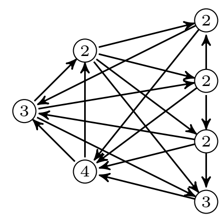


(a) Lemma 4: EE=2, EO=1 (b) Lemma 5: EE=4, EO=3 (c) Lemma 6: EE=16, EO=17

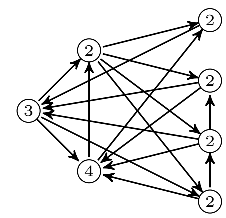
Figure 9: Good orientations for the AT versions of Lemmas 4, 5, and 6.



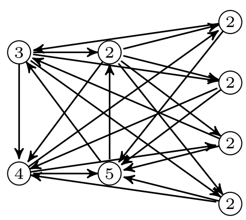

(a) Lemma 10: EE=30, EO=28



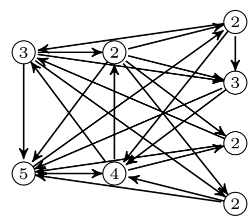

(b) Lemma 11: EE=108, EO=107



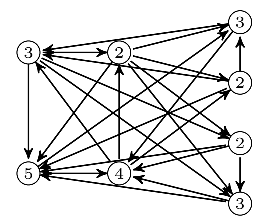

(c) Lemma 12: EE=88, EO=87



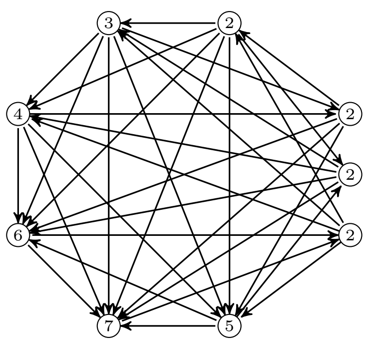

(d) Lemma 7a: EE=512, EO=515



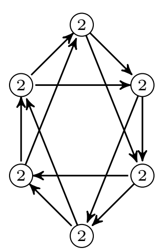

(e) Lemma 7b: EE=751, EO=750




(f) Lemma 7c: EE=1097, EO=1096




(g) Lemma 8: EE=4394, EO=4393




(h) Lemma 9: EE=22, EO=16

Figure 10: Good orientations for the AT versions of Lemmas 7, 8, 9, 10, 11, and 12.

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