YET ANOTHER PROOF OF BROOKS' THEOREM

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Theorem 1 (Brooks 1941)**.** *Every graph G with $\chi(G) = \Delta(G) + 1 \geq 4$ contains $K_{\Delta(G)+1}$.*

Proof. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$. Using minimality of $|G|$, we see that $\chi(G - v) \leq \Delta$ for all $v \in V(G)$. In particular, G is Δ-regular.

Let M be a maximal independent set in G. Since $\Delta(G - M) < \Delta$ and $\chi(G - M) \geq \Delta$, minimality of $|G|$ shows that $G - M$ has an induced subgraph T where $T = K_\Delta$ or T is an odd cycle if $\Delta = 3$. Suppose G contains $K_{\Delta+1}$ less an edge, say $K_{\Delta+1} - xy = D \subseteq G$. Then we may Δ-color $G - D$ and extend the coloring to D by first coloring x and y the same and then finishing greedily on the rest.

Since $K_{\Delta+1} \not\subseteq G$ we have $|N(T)| \geq 2$. So, we may take different $x, y \in N(T)$ and put $H := G - T$ if x is adjacent to y and $H := (G - T) + xy$ otherwise. Then, H doesn't contain $K_{\Delta+1}$ as G doesn't contain $K_{\Delta+1}$ less an edge. By minimality of $|G|$, H is Δ-colorable. That is, we have a Δ-coloring of $G - T$ where x and y receive different colors. We can easily extend this partial coloring to all of G since each vertex of T has a set of $\Delta - 1$ available colors and some pair of vertices in T get different sets. $\square$