Extending Alon-Tarsi Orientations

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Joint with Hal Kierstead
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a motivating problem

coloring a graph with around Δ colors

by greed: every graph is $(\Delta + 1)$-colorable

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hard conjecture: can $(\Delta - 1)$-color when no K_Δ and $\Delta \geq 9$
(Borodin-Kostochka 1977)

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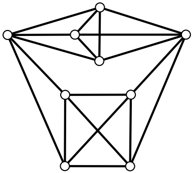
- can $\left\lfloor \frac{\theta}{2} \right\rfloor$-color when $\theta \geq 8$ and no $K_{\left\lfloor \frac{\theta}{2} \right\rfloor + 1}$ nor O_5 (Kostochka-Stiebtiz-R. 2011)



the graph O_5

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for (online) list coloring

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known bounds: combining with best known lower bound on edges in list-critical graphs (Kostochka-Stiebitz 2003) can $\left\lfloor\frac{\theta}{2}\right\rfloor$-list-color when $\theta \geq 56$ and no $K_{\left\lfloor\frac{\theta}{2}\right\rfloor+1}$

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- new lower bound on edges in online-list-critical graphs proved via extending Alon-Tarsi orientations

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many edges or nicely orientable subgraph

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corollary: since we can complete a $\delta(G)$-coloring of $G - H$ to such an H, this implies that online-list-critical graphs have "lots" of edges

Lemma

Let G be a multigraph without loops and $f : V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:

1. *any multiple edges in G are contained in $G[Y]$; and*
2. *$f(v) \geq d_G(v)$ for all $v \in V(G - Y)$; and*
3. *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and*
4. *For each component T of $G - Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:*
    1. *there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or*
    2. *$|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,*

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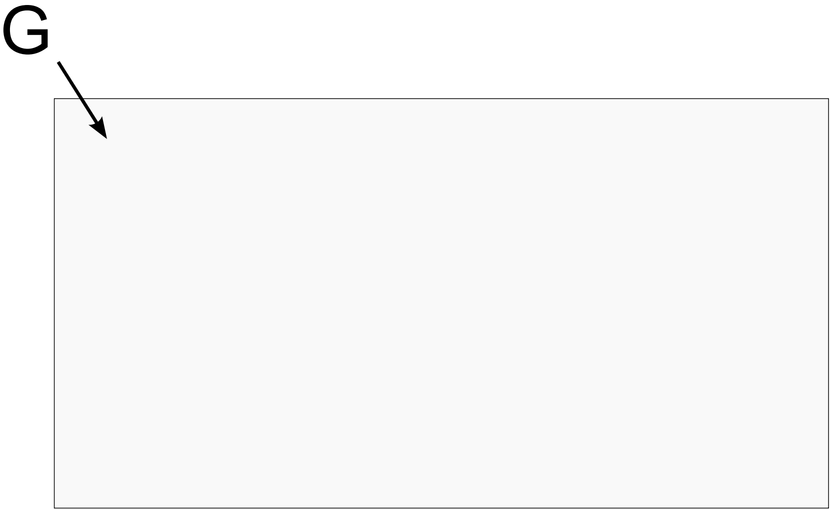
we need a picture

extending Alon-Tarsi orientations

key lemma in pictures



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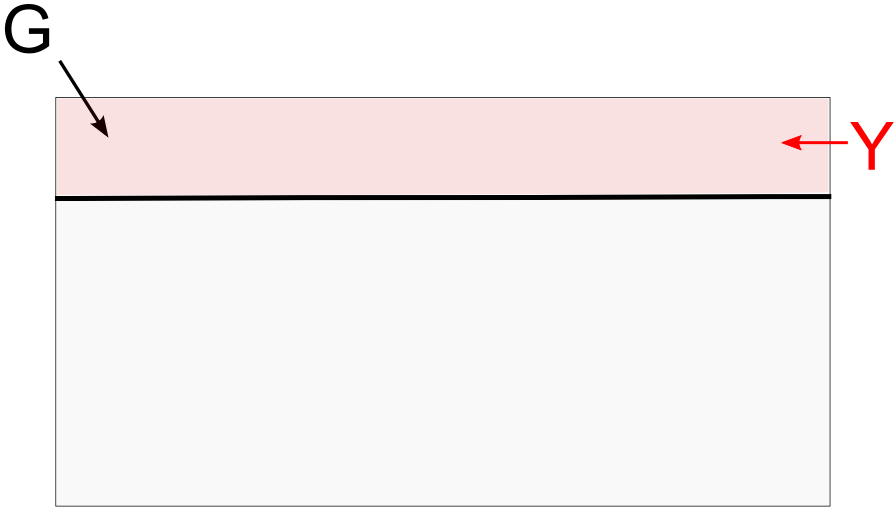

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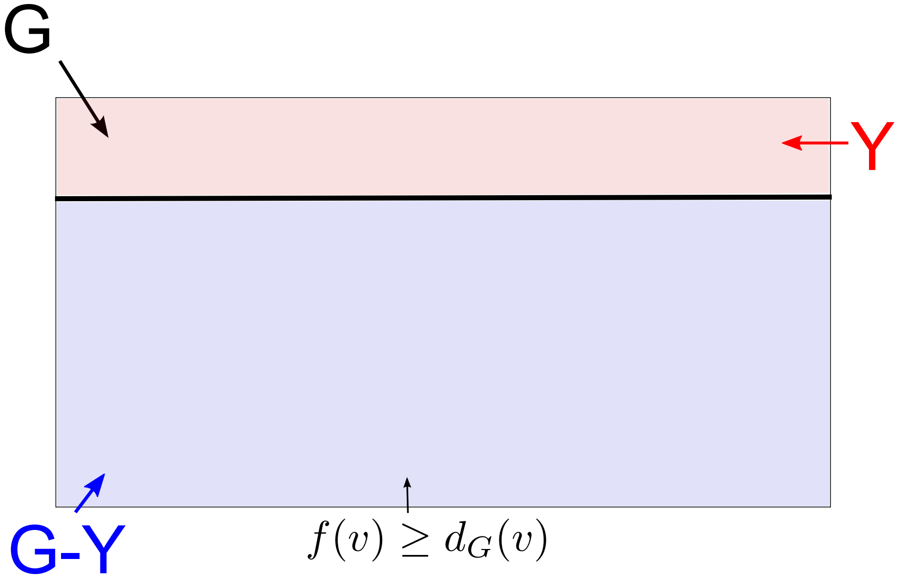

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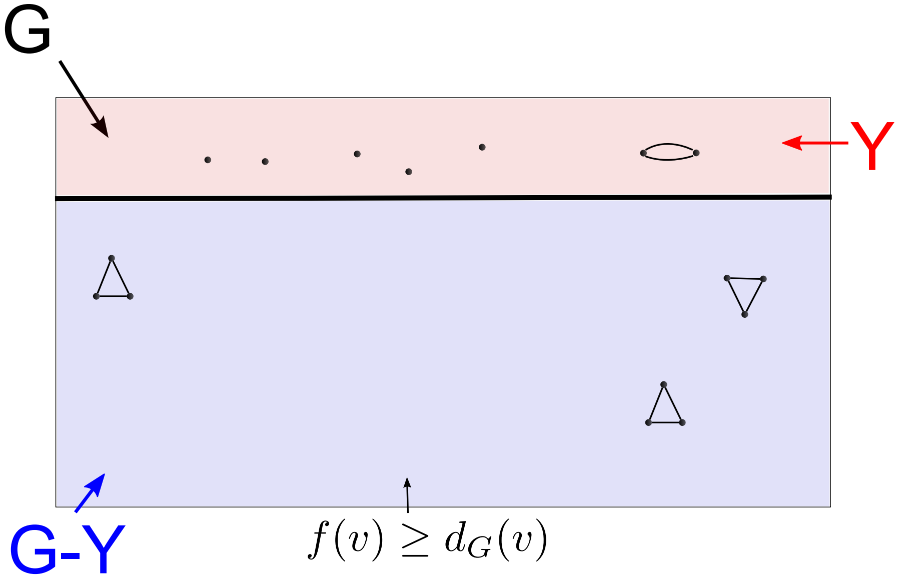

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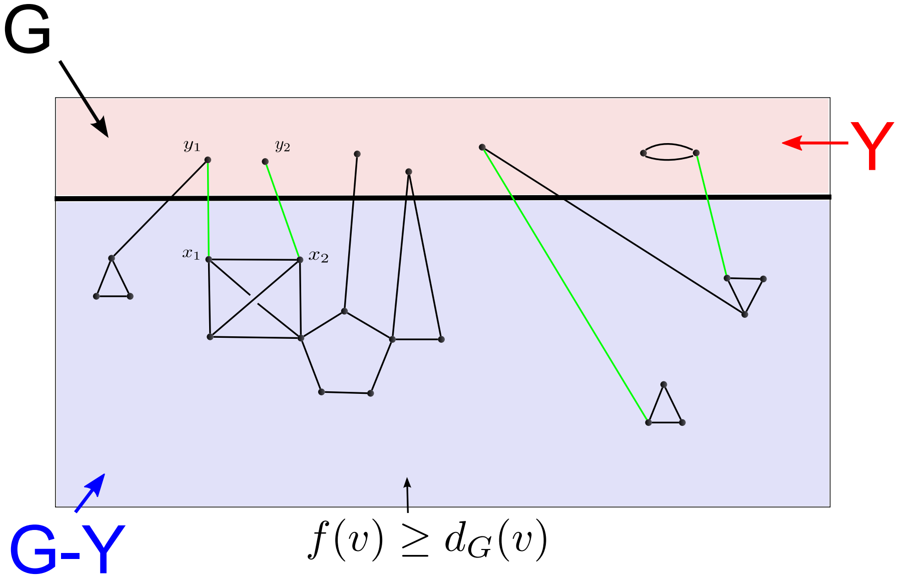

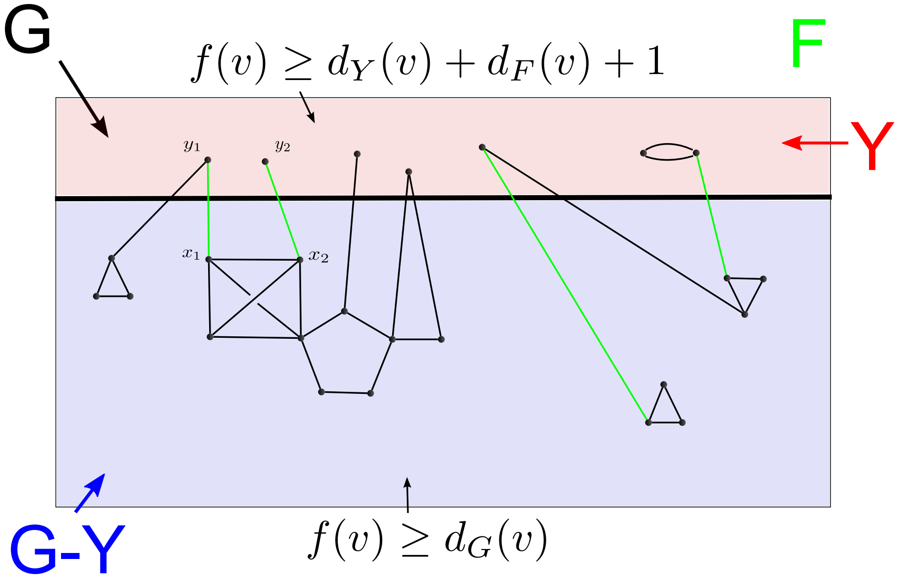

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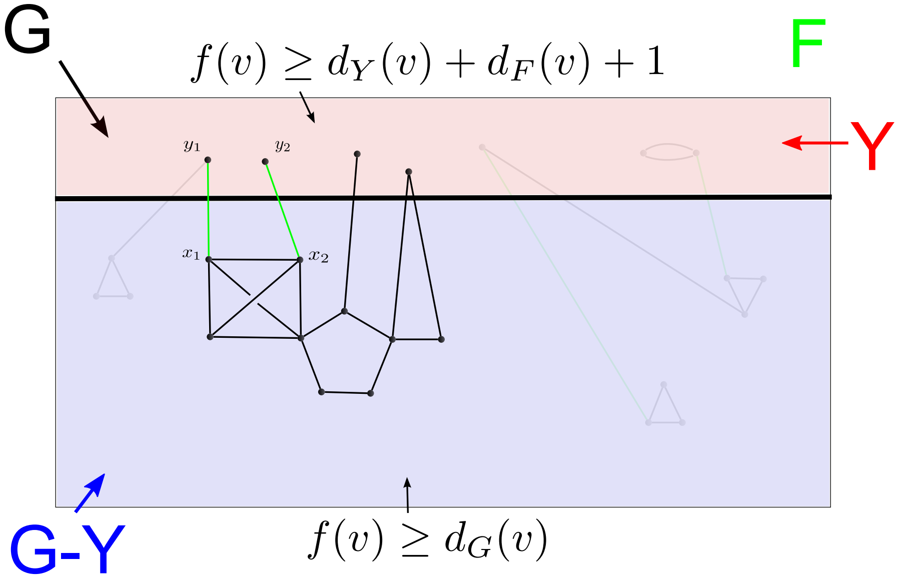
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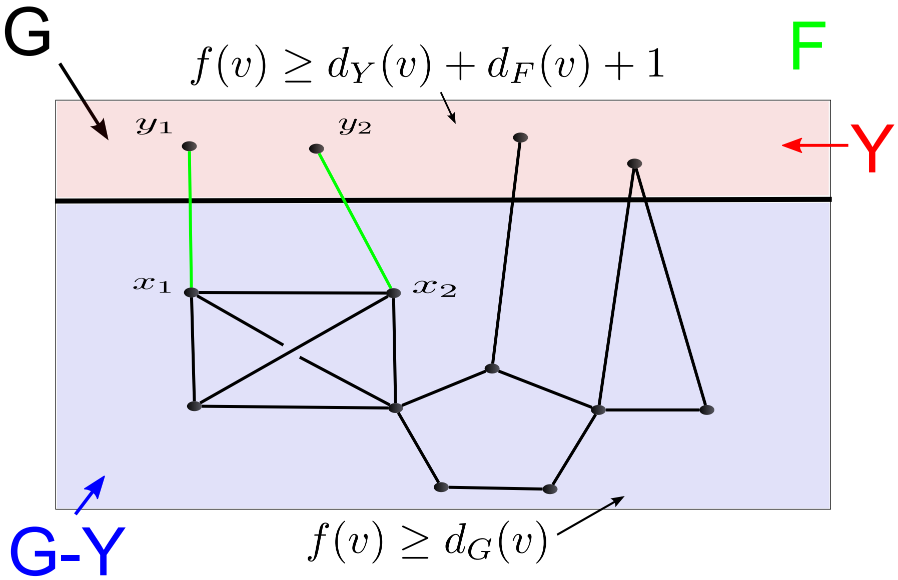
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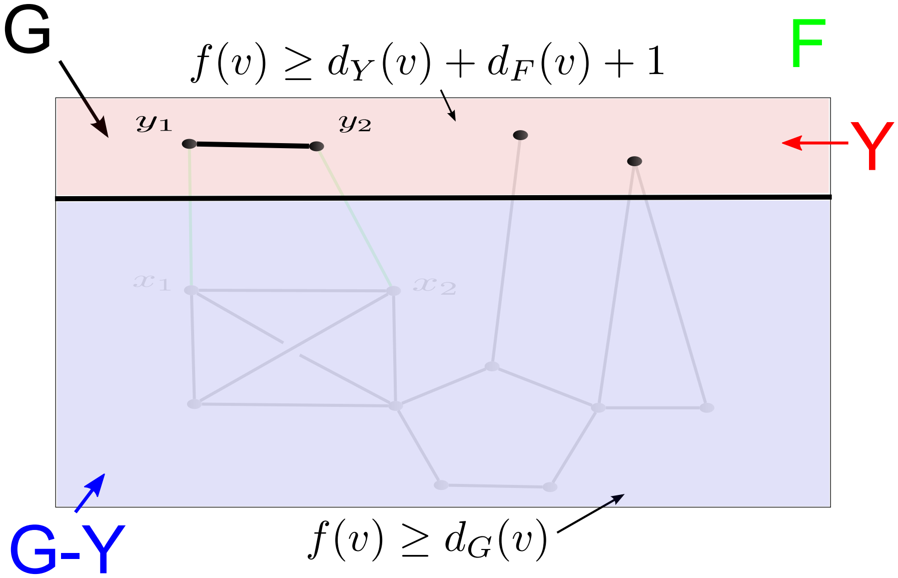

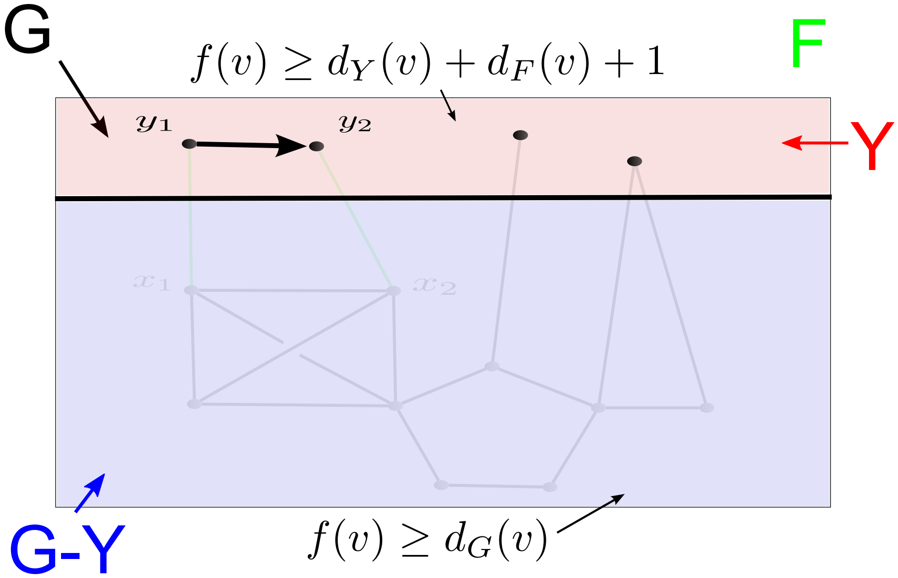
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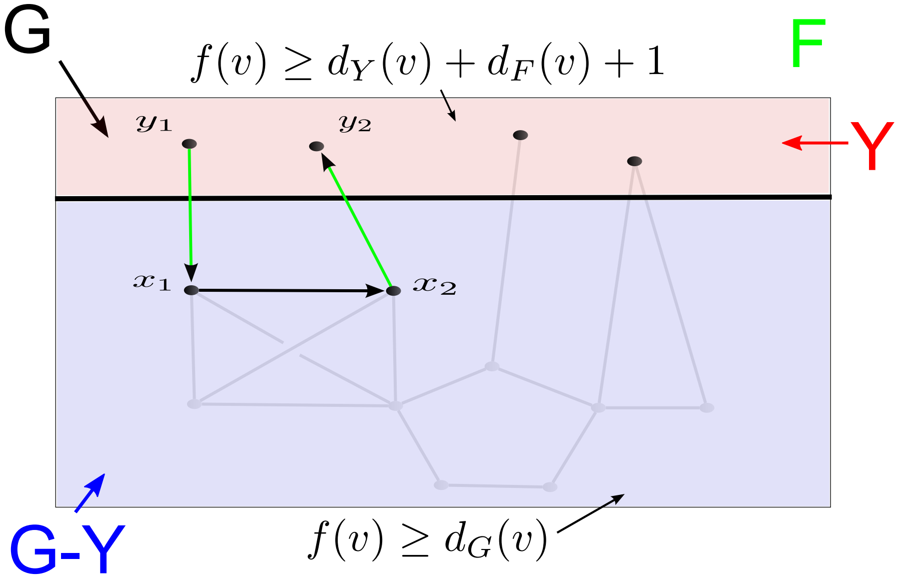

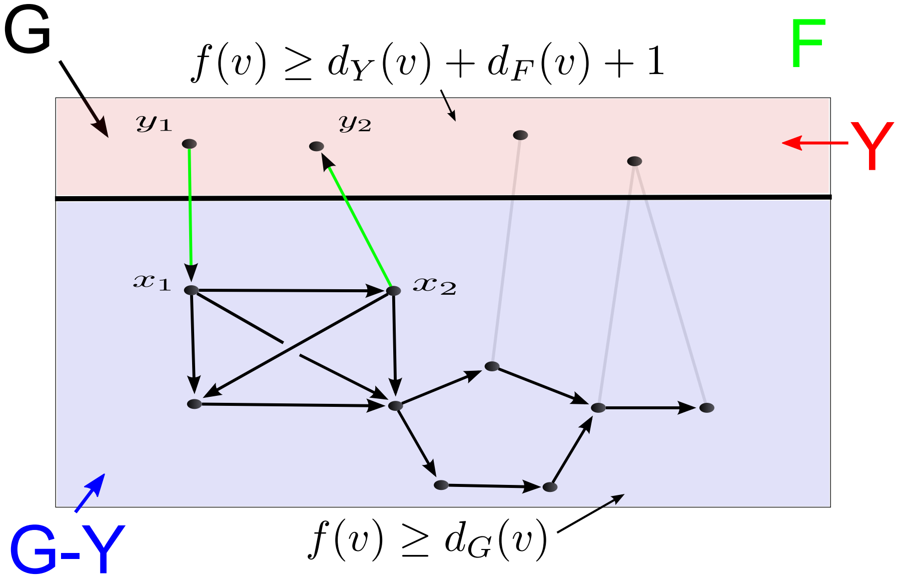
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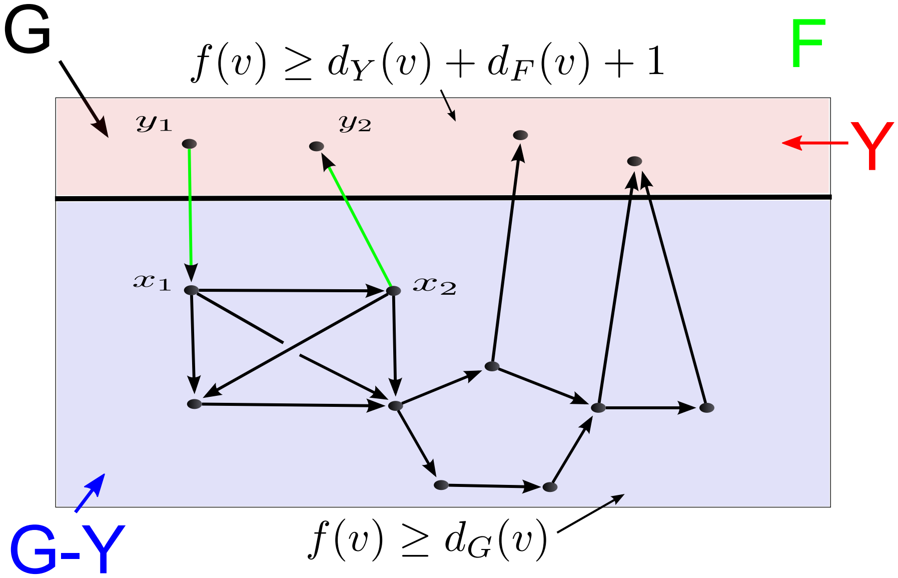

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