The list-chromatic index of K_8 and K_{10}

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Abstract

In [4], Cariolaro et al. demonstrated how colorability problems can be approached numerically by the use of computer algebra systems and the Combinatorial Nullstellensatz. In particular, they verified a case of the List Coloring Conjecture by proving that the list-chromatic index of K_6 is 5. In this short note, we show that using the coefficient formula of Schauz [16] is much more efficient than using partial derivatives. As a consequence we are able to show that list-chromatic index of K_8 is 7 and the list-chromatic index of K_{10} is 9.

1 Introduction

List coloring was introduced by Vizing [17] and independently Erdős, Rubin and Taylor [9]. Let G be a graph. A list assignment on G is a function L from $V(G)$ to the subsets of $\mathbb{N}$. A graph G is *L-colorable* if there is $\pi\colon V(G) \to \mathbb{N}$ such that $\pi(v) \in L(v)$ for each $v \in V(G)$ and $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. For $k \in \mathbb{N}$, a list assignment L is a *k-assignment* if $|L(v)| = k$ for each $v \in V(G)$. We say that G is *k-choosable* if G is L-colorable for every k-assignment L. The least k for which G is k-choosable is the *choice number* of G, written $\text{ch}(G)$. The choice number of the line graph of G is the *list-chromatic index* of G, written $\text{ch}'(G)$. We write $\chi'(G)$ for the chromatic index of G; that is, the chromatic number of the line graph of G.

Since any k-choosable graph is L-colorable from the k-assignment given by $L(v) = [k]$, we have $\text{ch}'(G) \geq \chi'(G)$. That this inequality is always an equality has been conjectured independently by multiple researchers (see [13], Section 12.20). This is the *List Coloring Conjecture*.

Conjecture 1.1 (List Coloring Conjecture)**.** *$ch'(G) = \chi'(G)$ for every multigraph G.*

This conjecture is open even for complete graphs. Häggkvist and Janssen [11] settled the conjecture for K_n when n is odd by showing that $\text{ch}'(K_n) = \chi'(K_n) = n$. When n is even, we have $\chi'(K_n) = n - 1$, so the conjecture reduces to the following.

Conjecture 1.2. $\text{ch}'(K_{2k}) = 2k - 1$ *for all* $k \geq 1$.

Conjecture 1.2 holds trivially for $k = 1$. Since K_4 is planar, then $k = 2$ case follows from the result of Ellingham and Goddyn [8] that the List Coloring Conjecture holds for every 1-factorable planar graph. Moreover, Ellingham and Goddyn [8] state that they have verified the $k \leq 5$ cases by computer. Cariolaro et al. verified the $k = 3$ case using the Combinatorial Nullstellensatz and a computer algebra system. We verify the $k = 4, 5$ cases using Schauz's coefficient formula for the Combinatorial Nullstellensatz [16]. The ability to quickly determine coefficients of the graph polynomial has many other uses. With Cranston, in [5] we proved that the choice number of the square of a graph is at most its degree squared minus one unless the graph is one of a few exceptions. This proof involved showing that many small induced subgraphs could be excluded by the Combinatorial Nullstellensatz. There is a web version of the author's graph software at `http://bit.ly/webraphs` which may be useful to others.

2 Combinatorial Nullstellensatz

In [2], Alon and Tarsi introduced a beautiful algebraic technique for proving the existence of list colorings. Alon [1] further developed this technique into the *Combinatorial Nullstellensatz*. Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1,\ldots,k_n}$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$.

Lemma 2.1 (Alon [1])**.** *Suppose* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *and* $k_1, \ldots, k_n \in \mathbb{N}$ *with* $\sum_{i\in[n]} k_i = \deg(f)$*. If* $f_{k_1,\ldots,k_n} \neq 0$*, then for any* $A_1, \ldots, A_n \subseteq \mathbb{F}$ *with* $|A_i| \geq k_i + 1$*, there exists* $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ *with* $f(a_1, \ldots, a_n) \neq 0$*.*

Michałek [15] gave a very short proof of Lemma 2.1 just using long division. Schauz [16] sharpened the Combinatorial Nullstellensatz by proving the following coefficient formula. Versions of this result were also proved by Hefetz [12] and Lasoń [14]. Our presentation follows Lasoń.

Lemma 2.2 (Schauz [16])**.** *Suppose* $f \in \mathbb{F}[x_1, \ldots, x_n]$ *and* $k_1, \ldots, k_n \in \mathbb{N}$ *with* $\sum_{i\in[n]} k_i = \deg(f)$*. For any* $A_1, \ldots, A_n \subseteq \mathbb{F}$ *with* $|A_i| = k_i + 1$*, we have*

$$f_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)},$$

where

$$N(a_1, \ldots, a_n) := \prod_{i\in[n]} \prod_{b\in A_i - a_i} (a_i - b).$$

3 List edge-coloring complete graphs

To apply the Combinatorial Nullstellensatz to graph coloring, we need one further definition. The *graph polynomial* of a graph G with vertex set $\{1, \ldots, n\}$ is

$$P_G(x_1, \ldots, x_n) := \prod_{\substack{ij\in E(G)\\ i<j}} (x_i - x_j).$$

For a graph G, let $L(G)$ be the line graph of G. Applying Lemma 2.1 to the graph polynomial of $L(G)$ gives the following.

Corollary 3.1. *Let G be a d-regular graph with $\chi'(G) = d$. Let f be the graph polynomial of $L(G)$. If $f_{d-1,\ldots,d-1} \neq 0$, then $\mathrm{ch}'(G) = d$.*

Cariolaro et al. used partial derivatives and the $k = 3$ case of Corollary 3.1 to show that $\mathrm{ch}'(K_6) = 5$. In particular, they showed that $f_{4,\ldots,4} = -720$. Using Lemma 2.2, this coefficient can be computed in under a second on a basic laptop. To try this out, go to `http://bit.ly/webgraphs_LK_6`, in the "Orientations" menu, select "compute coefficient" and then "use current orientation". For $k = 4$, the same method gives $f_{6,\ldots,6} = 21772800$, but the computation takes about six hours to complete. To do K_{10}, we need a more efficient method.

To get a more efficient method, we use the fact that Lemma 2.2 simplifies greatly in the case of d-regular graphs with chromatic index d. In fact each term of the sum is either 1 or -1. Alon [3] proved this in slightly different form; we follow the presentation in Hefetz [12]. Put $\mathrm{sign}(x) := \frac{x}{|x|}$ for $x \neq 0$ and $\mathrm{sign}(0) := 0$.

Lemma 3.2. *Let G be a d-regular graph with $\chi'(G) = d$. Let $n = |E(G)|$ and let f be the graph polynomial of $L(G)$. Then*

$$f_{d-1,\ldots,d-1} = \sum_{(a_1,\ldots,a_n)\in[d]^n} \mathrm{sign}(f(a_1,\ldots,a_n)).$$

Lemma 3.3. *Let G be a d-regular graph with $\chi'(G) = d$. Let $n = |E(G)|$ and let f be the graph polynomial of $L(G)$. For any permutation π of $[d]$ and any $(a_1,\ldots,a_n) \in [d]^n$, we have $\mathrm{sign}(f(\pi(a_1),\ldots,\pi(a_n)) = \mathrm{sign}(f(a_1,\ldots,a_n))$.*

Proof. Since every permutation can be written as a product of adjacent transpositions, it will suffice to prove the lemma for $\pi = (c\;\; c{+}1)$. Also, we may assume $\mathrm{sign}(f(a_1,\ldots,a_n)) \neq 0$. For a factor $(a_i - a_j)$ of $f(a_1,\ldots,a_n)$ we have $\mathrm{sign}(\pi(a_i) - \pi(a_j)) = \mathrm{sign}(a_i - a_j)$ unless $\{a_i, a_j\} = \{c, c+1\}$ in which case $\mathrm{sign}(\pi(a_i) - \pi(a_j)) = -\mathrm{sign}(a_i - a_j)$. Consider the edge-coloring of $L(G)$ given by $(a_1,\ldots,a_n)$. In this edge-coloring, each vertex is incident to an edge colored c and an edge colored $c+1$. So there is a factor $(a_i - a_j)$ with $\{a_i, a_j\} = \{c, c+1\}$ for each vertex of G. Since $|G|$ is even, we have $\mathrm{sign}(f(\pi(a_1),\ldots,\pi(a_n)) = \mathrm{sign}(f(a_1,\ldots,a_n))$. $\square$

So, by Lemma 3.3, to compute $f_{d-1,\ldots,d-1}$, we only need to sum $\mathrm{sign}(f(a_1,\ldots,a_n))$ over all one-factorizations up to color permutation and then multiply by $d!$. For K_6 there are 6 distinct one-factorizations, plugging them into f shows that they all have negative sign. So for K_6, we have $f_{4,\ldots,4} = 5!(-6) = -720$ which agrees with the result in [4].

Theorem 3.4. *K_8 has 5280 positive one-factorizations and 960 negative one-factorizations. Therefore $f_{6,\ldots,6} = 7!(5280 - 960) = 21772800$. In particular, $ch'(K_8) = 7$.*

We also conclude that the total number of one-factorizations of K_8 is 6240 which is in agreement with [6].

Theorem 3.5. *K_{10} has 598993920 positive one-factorizations and 626572800 negative one-factorizations. Therefore $f_{8,\ldots,8} = 9!(598993920 - 626572800) = -10007823974400$. In particular, $ch'(K_{10}) = 9$.*

We also conclude that the total number of one-factorizations of K_{10} is 1225566720 which is in agreement with [10]. There was a bit of confusion around the correctness of this count because it is incorrectly cited in [18] as 1255566720; this discrepancy was noted by Dinitz, Garnick and McKay [7].

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