The list-chromatic index of $K_8$ and $K_{10}$

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July 31, 2014

Abstract

In [4], Cariolaro et al. demonstrated how colorability problems can be approached numerically by the use of computer algebra systems and the Combinatorial Nullstellensatz. In particular, they verified a case of the List Coloring Conjecture by proving that the list-chromatic index of $K_6$ is 5. In this short note, we show that using the coefficient formula of Schauz [16] is much more efficient than using partial derivatives. As a consequence we are able to show that list-chromatic index of $K_8$ is 7 and the list-chromatic index of $K_{10}$ is 9.

1 Introduction

List coloring was introduced by Vizing [17] and independently Erdős, Rubin and Taylor [9]. Let $G$ be a graph. A list assignment on $G$ is a function $L$ from $V(G)$ to the subsets of $\mathbb{N}$. A graph $G$ is $L$-colorable if there is $\pi: V(G) \rightarrow \mathbb{N}$ such that $\pi(v) \in L(v)$ for each $v \in V(G)$ and $\pi(x) \neq \pi(y)$ for each $xy \in E(G)$. For $k \in \mathbb{N}$, a list assignment $L$ is a $k$-assignment if $|L(v)| = k$ for each $v \in V(G)$. We say that $G$ is $k$-choosable if $G$ is $L$-colorable for every $k$-assignment $L$. The least $k$ for which $G$ is $k$-choosable is the choice number of $G$, written $\text{ch}'(G)$. The choice number of the line graph of $G$ is the list-chromatic index of $G$, written $\chi'(G)$. We write $\chi'(G)$ for the chromatic index of $G$; that is, the chromatic number of the line graph of $G$.

Since any $k$-choosable graph is $L$-colorable from the $k$-assignment given by $L(v) = [k]$, we have $\text{ch}'(G) \geq \chi'(G)$. That this inequality is always an equality has been conjectured independently by multiple researchers (see [13], Section 12.20). This is the List Coloring Conjecture.

Conjecture 1.1 (List Coloring Conjecture). $\text{ch}'(G) = \chi'(G)$ for every multigraph $G$.

This conjecture is open even for complete graphs. Häggkvist and Janssen [11] settled the conjecture for $K_n$ when $n$ is odd by showing that $\chi'(K_n) = \chi'(K_n) = n$. When $n$ is even, we have $\chi'(K_n) = n - 1$, so the conjecture reduces to the following.

Conjecture 1.2. $\text{ch}'(K_{2k}) = 2k - 1$ for all $k \geq 1$.

Conjecture 1.2 holds trivially for $k = 1$. Since $K_4$ is planar, then $k = 2$ case follows from the result of Ellingham and Goddyn [8] that the List Coloring Conjecture holds for...
every 1-factorable planar graph. Moreover, Ellingham and Goddyn \cite{EG} state that they have verified the $k \leq 5$ cases by computer. Cariolaro et al. verified the $k = 3$ case using the Combinatorial Nullstellensatz and a computer algebra system. We verify the $k = 4, 5$ cases using Schauz’s coefficient formula for the Combinatorial Nullstellensatz \cite{S}. The ability to quickly determine coefficients of the graph polynomial has many other uses. With Cranston, in \cite{CR} we proved that the choice number of the square of a graph is at most its degree squared minus one unless the graph is one of a few exceptions. This proof involved showing that many small induced subgraphs could be excluded by the Combinatorial Nullstellensatz. There is a web version of the author’s graph software at \url{http://bit.ly/webraphs} which may be useful to others.

## 2 Combinatorial Nullstellensatz

In \cite{AT}, Alon and Tarsi introduced a beautiful algebraic technique for proving the existence of list colorings. Alon \cite{A} further developed this technique into the Combinatorial Nullstellensatz. Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1, \ldots, k_n}$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$.

**Lemma 2.1** (Alon \cite{A}). Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$. If $f_{k_1, \ldots, k_n} \neq 0$, then for any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$, there exists $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ with $f(a_1, \ldots, a_n) \neq 0$.

Michalek \cite{M} gave a very short proof of Lemma 2.1 just using long division. Schauz \cite{S} sharpened the Combinatorial Nullstellensatz by proving the following coefficient formula. Versions of this result were also proved by Hefetz \cite{H} and Lasoń \cite{L}. Our presentation follows Lasoń.

**Lemma 2.2** (Schauz \cite{S}). Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$. For any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$, we have

$$f_{k_1, \ldots, k_n} = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)},$$

where

$$N(a_1, \ldots, a_n) := \prod_{i \in [n]} \prod_{b \in A_i - a_i} (a_i - b).$$

## 3 List edge-coloring complete graphs

To apply the Combinatorial Nullstellensatz to graph coloring, we need one further definition. The graph polynomial of a graph $G$ with vertex set $\{1, \ldots, n\}$ is

$$P_G(x_1, \ldots, x_n) := \prod_{i,j \in E(G)} (x_i - x_j).$$

For a graph $G$, let $L(G)$ be the line graph of $G$. Applying Lemma 2.1 to the graph polynomial of $L(G)$ gives the following.
Corollary 3.1. Let $G$ be a $d$-regular graph with $\chi'(G) = d$. Let $f$ be the graph polynomial of $L(G)$. If $f_{d-1,\ldots,d-1} \neq 0$, then $\chi'(G) = d$.

Cariolaro et al. used partial derivatives and the $k = 3$ case of Corollary 3.1 to show that $\chi'(K_6) = 5$. In particular, they showed that $f_{4,\ldots,4} = -720$. Using Lemma 2.2, this coefficient can be computed in under a second on a basic laptop. To try this out, go to http://bit.ly/webgraphs_LK_6 in the “Orientations” menu, select “compute coefficient” and then “use current orientation”. For $k = 4$, the same method gives $f_{6,\ldots,6} = 21772800$, but the computation takes about six hours to complete. To do $K_{10}$, we need a more efficient method.

To get a more efficient method, we use the fact that Lemma 2.2 simplifies greatly in the case of $d$-regular graphs with chromatic index $d$. In fact each term of the sum is either 1 or $-1$. Alon [3] proved this in slightly different form; we follow the presentation in Hefetz [12]. Put $\text{sign}(x) := \frac{x}{|x|}$ for $x \neq 0$ and $\text{sign}(0) := 0$.

Lemma 3.2. Let $G$ be a $d$-regular graph with $\chi'(G) = d$. Let $n = |E(G)|$ and let $f$ be the graph polynomial of $L(G)$. Then

$$f_{d-1,\ldots,d-1} = \sum_{(a_1,\ldots,a_n) \in [d]^n} \text{sign}(f(a_1,\ldots,a_n)).$$

Proof. Since every permutation can be written as a product of adjacent transpositions, it will suffice to prove the lemma for $\pi = (c \ c+1)$. Also, we may assume $\text{sign}(f(a_1,\ldots,a_n)) \neq 0$. For a factor $(a_i - a_j)$ of $f(a_1,\ldots,a_n)$ we have $\text{sign}(\pi(a_i) - \pi(a_j)) = \text{sign}(a_i - a_j)$ unless $\{a_i, a_j\} = \{c, c+1\}$ in which case $\text{sign}(\pi(a_i) - \pi(a_j)) = -\text{sign}(a_i - a_j)$. Consider the edge-coloring of $L(G)$ given by $(a_1,\ldots,a_n)$. In this edge-coloring, each vertex is incident to an edge colored $c$ and an edge colored $c+1$. So there is a factor $(a_i - a_j)$ with $\{a_i, a_j\} = \{c, c+1\}$ for each vertex of $G$. Since $|G|$ is even, we have $\text{sign}(f(\pi(a_1),\ldots,\pi(a_n))) = \text{sign}(f(a_1,\ldots,a_n))$. \hfill $\square$

So, by Lemma 3.3 to compute $f_{d-1,\ldots,d-1}$, we only need to sum $\text{sign}(f(a_1,\ldots,a_n))$ over all one-factorizations up to color permutation and then multiply by $d!$. For $K_6$ there are 6 distinct one-factorizations, plugging them into $f$ shows that they all have negative sign. So for $K_6$, we have $f_{4,\ldots,4} = 5!(6240) = -720$ which agrees with the result in [4].

Theorem 3.4. $K_8$ has 5280 positive one-factorizations and 960 negative one-factorizations. Therefore $f_{6,\ldots,6} = 7!(5280 - 960) = 21772800$. In particular, $\chi'(K_8) = 7$.

We also conclude that the total number of one-factorizations of $K_8$ is 6240 which is in agreement with [6].

Theorem 3.5. $K_{10}$ has 598993920 positive one-factorizations and 626572800 negative one-factorizations. Therefore $f_{8,\ldots,8} = 9!(598993920 - 626572800) = -10007823974400$. In particular, $\chi'(K_{10}) = 9$. 

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We also conclude that the total number of one-factorizations of $K_{10}$ is 1225566720 which is in agreement with [10]. There was a bit of confusion around the correctness of this count because it is incorrectly cited in [18] as 1255566720; this discrepancy was noted by Dinitz, Garnick and McKay [7].

References


