Destroying non-complete regular components in graph partitions

Landon Rabern
landon.rabern@gmail.com
June 27, 2011

Abstract

We prove that if \( G \) is a graph and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0} \) such that \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \) then \( V(G) \) can be partitioned into sets \( V_1, \ldots, V_k \) such that \( \Delta(G[V_i]) \leq r_i \) and \( G[V_i] \) contains no non-complete \( r_i \)-regular components for each \( 1 \leq i \leq k \). In particular, the vertex set of any graph \( G \) can be partitioned into \( \lceil \Delta(G) + 2 \rceil / 3 \) sets, each of which induces a disjoint union of triangles and paths.

1 Introduction

In [5] Kostochka modified an algorithm of Catlin to show that every triangle-free graph \( G \) can be colored with at most \( 2 \Delta(G) + 3 \) colors. In fact, his modification proves that the vertex set of any triangle-free graph \( G \) can be partitioned into \( \lceil \Delta(G) + 2 \rceil / 3 \) sets, each of which induces a disjoint union of paths. We generalize this as follows.

Main Lemma. Let \( G \) be a graph and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0} \) such that \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \). Then \( V(G) \) can be partitioned into sets \( V_1, \ldots, V_k \) such that \( \Delta(G[V_i]) \leq r_i \) and \( G[V_i] \) contains no non-complete \( r_i \)-regular components for each \( 1 \leq i \leq k \).

Setting \( k = \lceil \Delta(G) + 2 \rceil / 3 \) and \( r_i = 2 \) for each \( i \) gives a slightly more general form of Kostochka’s theorem.
**Corollary 1.** The vertex set of any graph $G$ can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.

For coloring, this actually gives the bound $\chi(G) \leq 2 \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}(\Delta(G) + 3)$, just use $r_k = 0$ when $\Delta \equiv 2(\text{mod } 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each $i$ gives the following.

**Corollary 2.** Fix $r \geq 2$. The vertex set of any $K_{r+1}$-free graph $G$ can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most $r$.

For the purposes of coloring it is more economical to split off $\Delta + 2 - (r + 1) \left\lfloor \frac{\Delta+2}{r+1} \right\rfloor$ parts with $r_j = 0$.

**Corollary 3.** Fix $r \geq 2$. The vertex set of any $K_{r+1}$-free graph $G$ can be partitioned into $\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most $r$ and $\Delta(G)+2 - (r+1) \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ independent sets. In particular, $\chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$.

For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks’ Theorem. In [7] Lovász proved a decomposition lemma of the same form as the Main Lemma. The Main Lemma gives a more restrictive partition at the cost of replacing $\Delta(G) + 1$ with $\Delta(G) + 2$.

**Lovász’s Decomposition Lemma.** Let $G$ be a graph and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $1 \leq i \leq k$.

For $r \geq 3$, combining this with Brooks’ Theorem gives the following better bound for a $K_{r+1}$-free graph $G$ (first proved in [1], [3] and [6]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G)+1}{r+1} \right\rfloor.$$
2 The proofs

Instead of proving directly that we can destroy all non-complete \(r\)-regular components in the partition, we prove the theorem for the more general class of what we call \(r\)-permissible graphs and show that non-complete \(r\)-regular graphs are \(r\)-permissible.

**Definition 1.** For a graph \(G\) and \(r \geq 0\), let \(G^r\) be the subgraph of \(G\) induced on the vertices of degree \(r\) in \(G\).

**Definition 2.** Fix \(r \geq 2\). A collection \(T\) of graphs is \(r\)-permissible if it satisfies all of the following conditions.

1. Every \(G \in T\) is connected.
2. \(\Delta(G) = r\) for each \(G \in T\).
3. \(\delta(G^r) > 0\) for each \(G \in T\).
4. If \(G \in T\) and \(x \in V(G^r)\), then \(G - x \not\in T\).
5. If \(G \in T\) and \(x \in V(G^r)\), then there exists \(y \in V(G^r) - \{x\} \cup N_G(x)\) such that \(G - y\) is connected.
6. Let \(G \in T\) and \(x \in V(G^r)\). Put \(H := G - x\). Let \(A \subseteq V(H)\) with \(|A| = r\). Let \(y\) be some new vertex and form \(H_A\) by joining \(y\) to \(A\) in \(H\); that is, \(V(H_A) := V(H) \cup \{y\}\) and \(E(H_A) := E(H) \cup \{xy \mid x \in A\}\). If \(H_A \in T\), then \(A \cap N_G(x) \cap V(G^r) \neq \emptyset\).

For \(r = 0, 1\) the empty set is the only \(r\)-permissible collection.

**Lemma 4.** Fix \(r \geq 2\) and let \(T\) be the collection of all non-complete connected \(r\)-regular graphs. Then \(T\) is \(r\)-permissible.

**Proof.** Let \(G \in T\). We have \(G^r = G\) and (1), (2), (3) and (4) are clearly satisfied. That (6) holds is immediate from regularity. It remains to check (5). Let \(x \in V(G)\). First, suppose \(G\) is 2-connected. If (5) did not hold, then \(x\) would need to be adjacent to every other vertex in \(G\). But then \(|G| \leq \Delta(G) + 1 = r + 1\) and hence \(G = K_r\) violating our assumption. Otherwise \(G\) has at least two end blocks and so we can pick some \(y\) in an end block not containing \(x\) such that \(G - y\) is connected. Hence (5) holds. Therefore \(T\) is \(r\)-permissible. \(\Box\)
Now to prove the Main Lemma we just need to prove the following result. For a graph \( G, x \in V(G) \) and \( D \subseteq V(G) \) we use the notation \( N_D(x) := N(x) \cap D \) and \( d_D(x) := |N_D(x)| \).

**Lemma 5.** Let \( G \) be a graph and \( r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0} \) such that \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \). If \( T_i \) is an \( r_i \)-permissible collection for each \( 1 \leq i \leq k \), then \( V(G) \) can be partitioned into sets \( V_1, \ldots, V_k \) such that \( \Delta(G[V_i]) \leq r_i \) and \( G[V_i] \) contains no element of \( T_i \) as a component for each \( 1 \leq i \leq k \).

**Proof.** For a graph \( H \), let \( c(H) \) be the number of components in \( H \) and let \( p_i(H) \) be the number of components of \( H \) that are members of \( T_i \). For a partition \( P := (V_1, \ldots, V_k) \) of \( V(G) \) let

\[
f(P) := \sum_{i=1}^{k} (|E(G[V_i])| - r_i|V_i|),
\]

\[
c(P) := \sum_{i=1}^{k} c(G[V_i]),
\]

\[
p(P) := \sum_{i=1}^{k} p_i(G[V_i]).
\]

Let \( P := (V_1, \ldots, V_k) \) be a partition of \( V(G) \) minimizing \( f(P) \), and subject to that \( c(P) \), and subject to that \( p(P) \).

Let \( 1 \leq i \leq k \) and \( x \in V_i \) with \( d_{V_i}(x) \geq r_i \). Since \( \sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k \) there is some \( j \neq i \) such that \( d_{V_j}(x) \leq r_j \). Moving \( x \) from \( V_i \) to \( V_j \) gives a new partition \( P^* \) with \( f(P^*) \leq f(P) \). Note that if \( d_{V_i}(x) > r_i \) we would have \( f(P^*) < f(P) \) contradicting the minimality of \( P \). This proves that \( \Delta(G[V_i]) \leq r_i \) for each \( 1 \leq i \leq k \).

Now suppose that for some \( i_1 \) there is \( A_1 \in T_{i_1} \) which is a component of \( G[V_{i_1}] \). Plainly, we may assume that \( r_{i_1} \geq 2 \). Put \( P_1 := P \) and \( V_{1,i} := V_i \) for \( 1 \leq i \leq k \). Take \( x_1 \in V(A_{i_1}^{i_1}) \) such that \( A_1 - x_1 \) is connected (this exists by condition (5) of \( r \)-permissibility). By the above we have \( i_2 \neq i_1 \) such that moving \( x_1 \) from \( V_{1,i_1} \) to \( V_{1,i_2} \) gives a new partition \( P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k}) \) such that \( f(P_2) = f(P_1) \). By the minimality of \( c(P_1) \), \( x_1 \) is adjacent to only one component \( C_2 \) in \( G[V_{2,i_2}] \). Let \( A_2 := G[V(C_2) \cup \{x_1\}] \). Since (by condition (4)) we destroyed a \( T_{i_1} \) component when we moved \( x_1 \) out of \( V_{1,i_1} \), by the minimality of \( p(P_1) \), it must be that \( A_2 \in T_{i_2} \). Now pick \( x_2 \in A_{i_2}^{i_2} \) not adjacent to \( x_1 \) such that \( A_2 - x_2 \) is connected (again by condition (5)). Continue
on this way to construct sequences \(i_1, i_2, \ldots, A_1, A_2, \ldots, P_1, P_2, P_3, \ldots\) and \(x_1, x_2, \ldots\). Since \(G\) is finite, this process cannot continue forever. At some point we will need to reuse a destroyed component; that is, there is a smallest \(t\) such that \(A_{t+1} - x_t = A_s - x_s\) for some \(s < t\). Put \(B := V(A_s - x_s)\). Notice that \(A_{t+1}\) is constructed from \(A_s - x_s\) by joining the vertex \(x_t\) to \(N_B(x_t)\). By condition (6) of \(r_{is}\)-permissibility, we have \(z \in N_B(x_t) \cap N_B(x_s) \cap A_{rs}\).

We now modify \(P_s\) to contradict the minimality of \(f(P)\). At step \(t + 1\), \(x_t\) was adjacent to exactly \(r_{is}\) vertices in \(V_{t+1, is}\). This is what allowed us to move \(x_t\) into \(V_{t+1, is}\). Our goal is to modify \(P_s\) so that we can move \(x_t\) into the \(is\) part without moving \(x_s\) out. Since \(z\) is adjacent to both \(x_s\) and \(x_t\), moving \(z\) out of the \(is\) part will then give us our desired contradiction.

So, consider the set \(X\) of vertices that could have been moved out of \(V_{s,is}\) between step \(s\) and step \(t + 1\); that is, \(X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,is}\).

For \(x_j \in X\), since \(x_j \in A_{rs}\) and \(x_j\) is not adjacent to \(x_{j-1}\) we see that \(d_{V_{s,is}}(x_j) \geq r_{is}\). Similarly, \(d_{V_{s,ii}}(x_t) \geq r_{ii}\). Also, by the minimality of \(t\), \(X\) is an independent set in \(G\). Thus we may move all elements of \(X\) out of \(V_{s,is}\) to get a new partition \(P^* := (V_{s,1}, \ldots, V_{s,k})\) with \(f(P^*) = f(P)\).

Since \(x_t\) is adjacent to exactly \(r_{is}\) vertices in \(V_{t+1, is}\) and the only possible neighbors of \(x_t\) that were moved out of \(V_{s,is}\) between steps \(s\) and \(t + 1\) are the elements of \(X\), we see that \(d_{V_{s,is}}(x_t) = r_{is}\). Since \(d_{V_{s,ii}}(x_t) \geq r_{ii}\) we can move \(x_t\) from \(V_{s,ii}\) to \(V_{s,is}\) to get a new partition \(P^{**} := (V_{s,1}, \ldots, V_{s,k})\) with \(f(P^{**}) = f(P^*)\). Now, recall that \(z \in V_{s,is}\). Since \(z\) is adjacent to \(x_t\) we have \(d_{V_{s,is}}(z) \geq r_{is} + 1\). Thus we may move \(z\) out of \(V_{s,is}\) to get a new partition \(P^{***}\) with \(f(P^{***}) < f(P^{**}) = f(P)\). This contradicts the minimality of \(f(P)\).

\(\square\)

**Question.** Are there any other interesting \(r\)-permissible collections?

**Question.** The definition of \(r\)-permissibility can be weakened in various ways and the proof will still go through. Does this yield anything interesting?

**Acknowledgments**

Thanks to Dieter Gernert for finding and sending me a copy of Kostochka’s paper.
References


