Δ-CRITICAL GRAPHS WITH SMALL HIGH VERTEX CLIQUES

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Abstract. We prove that $K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$. Here $\mathcal{H}(G)$ is the subgraph of G induced on the vertices of degree at least $\chi(G)$. Setting $\omega(\mathcal{H}(G)) = 1$ proves a conjecture of Kierstead and Kostochka.

1. Introduction

For a graph G let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree at least $\chi(G)$. Recently, Kierstead and Kostochka [1] proved the following theorem and conjectured that the 7 could be improved to 6.

Theorem 1 (Kierstead and Kostochka). *$K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 7$ such that $\mathcal{H}(G)$ is edgeless.*

We prove this conjecture by establishing the following generalization.

Theorem 2. *$K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.*

Setting $\omega(\mathcal{H}(G)) = 1$ proves the conjecture.

Corollary 1. *$K_{\chi(G)}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.*

We can restate this in terms of Ore-degree as in [1] to get a generalization of Brooks' theorem.

Definition 1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) = d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) = \max_{xy \in E(G)} \theta(xy)$.

Corollary 2. *If $6 \leq \chi(G) = \left\lfloor \frac{\theta(G)}{2} \right\rfloor + 1$, then G contains the complete graph $K_{\chi(G)}$.*

This is best possible as shown by the following example from [1].

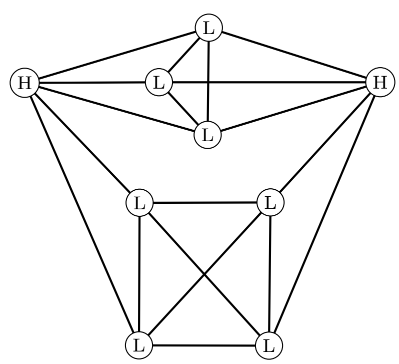

Figure 1. A counterexample to Corollary 2 with $\chi = 5$.

2. The Proof

We will use part of an algorithm of Mozhan [2]. The following is a generalization of his main lemma.

Definition 2. Let G be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$*-partitioned coloring* of G we mean a proper coloring of G of the form:

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

Lemma 3. *Let G be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of G pick one (call it π) minimizing*

$$\sum_{i=1}^{a} \left| E\left(G\left[\bigcup_{j=1}^{r_i} L_{ij}\right]\right)\right|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

Proof. Let $1 \leq i \leq a$ such that $d_{Z_i(x)}(x) = r_i$. Put $Z_i = Z_i(x)$.

First suppose $\Delta(Z_i) > r_i$. Take $y \in V(Z_i)$ with $d_{Z_i}(y) > r_i$ closest to x and let $x_1 x_2 \cdots x_t$ be a shortest $x-y$ path in Z_i. Plainly, for $k < t$, each x_k is adjacent to exactly one vertex in each color class besides its own. Thus we may recolor x_k with $\pi(x_{k+1})$ for $k < t$ and x_t with $\pi(x_1)$ to produce a new $\chi(G)$-coloring of G (this can be seen as a generalized Kempe chain). But we've moved a vertex (x_t) of degree $r_i + 1$ out of U_i while moving in a vertex (x_1) of degree r_i violating the minimality condition on π. This is a contradiction.

Thus $\Delta(Z_i) \leq r_i$. But $\chi(Z_i) = r_i + 1$, so Brooks' theorem implies that Z_i is complete if $r_i \geq 3$ and Z_i is an odd cycle if $r_i = 2$. □

Now to prove Theorem 2, we suppose it is false and derive a contradiction from properties of a minimal counterexample. Let $G \neq K_{\chi(G)}$ be a vertex critical graph with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$ having the minimum number of vertices.

Definition 3. We call $v \in V(G)$ *low* if $d(v) = \chi(G) - 1$ and *high* otherwise.

Lemma 4. *If $\Delta(G) = 6$, then G contains no $K_6 - e$.*

Proof. Suppose $\Delta(G) = 6$ and that G contains a $K_6 - e$, call it H. Let $x_1, x_2 \in V(H)$ with $d_H(x_i) = 4$. Color $G - H$ with 5 colors and let J be the resulting list assignment on H. Then $|J(x_1)| + |J(x_2)| \geq d_H(x_1) + d_H(x_2) - 2 \geq 2 * 6 - 6 \geq 6$. Hence we have $c \in J(x_1) \cap J(x_2)$. Color both x_1 and x_2 with c to get a list assignment J' on $F = H - \{x_1, x_2\}$. Since $\Delta(G) = 6$, $\mathcal{H}(G)$ is edgeless. Thus at most one vertex $y \in V(F)$ is high. Hence $|J'(y)| \geq 3$ and $|J'(z)| \geq 4$ for all $z \in V(F) - \{y\}$. Since F has 4 vertices we can complete the 5-coloring using Hall's theorem. This contradiction completes the proof. □

Lemma 5. *Suppose $\Delta(G) = 6$. Let C be a K_5 in G with at most one high vertex. Then each vertex in $G - C$ is adjacent to at most one low vertex in C.*

Proof. Suppose otherwise that some $x \in V(G - C)$ is adjacent to all of $S \subseteq C$ where each vertex in S is low and $|S| \geq 2$. Put $F = G - C$. Then F is 5-colorable. Since each vertex in C is adjacent to at least one vertex in F and G contains no $K_6 - e$, we have $y \in V(F)$ with $y \neq x$ such that $N(y) \cap C$ contains low vertices. Consider the graph $T = F + xy$. Note that $d_T(x) \leq 5$ and $d_T(y) \leq 6$. By minimality of G,

T is either 5-colorable or contains a $K_{\Delta(G)}$. In the former case we get a 5-coloring of F where x and y receive different colors, but this is easily completable to a coloring of G. Thus T contains K_6 and hence G contains a $K_6 - e$ giving a contradiction. □

Note that in Lemma 3, if $d_{Z_i(x)}(x) = r_i$ then we can *swap* x with any other $y \in Z_i(x)$ by changing π so that x is colored with $\pi(y)$ and y is colored with $\pi(x)$ to get another minimal $\chi(G)$-coloring of G.

Proof of Theorem 2. First, if $\chi(G) > \Delta(G)$ the theorem follows from Brooks' theorem.

Hence we may assume that $\chi(G) = \Delta(G)$. Put $\Delta = \Delta(G)$, $r_1 = \left\lfloor \frac{\Delta-1}{2} \right\rfloor$ and $r_2 = \left\lceil \frac{\Delta-1}{2} \right\rceil$. Of all (r_1, r_2)-partitioned colorings of G, pick one minimizing

$$\sum_{i=1}^{2} \left| E\left(G\left[\bigcup_{j=1}^{r_i} L_{ij} \right] \right) \right|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Throughout the proof we refer to a coloring that minimizes the above function as a *minimal* coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $C_i = \pi(U_i)$ (the colors used on U_i). For a minimal coloring γ of G, let $Z_{\gamma,i}(x)$ be the component of x in $G[\{x\} \cup \gamma^{-1}(C_i)]$. Put $Z_i(x) = Z_{\pi,i}(x)$.

Note that $r_1 \geq 2$ and $r_2 \geq 3$ and if $r_1 = 2$ then $r_2 = 3$, $\Delta = 6$ and $\omega(\mathcal{H}(G)) \leq 1$. First suppose x is high. Then $d(x) = r_1 + r_2 + 1$ and hence $d_{Z_i(x)}(x) = r_i$ for some $i \in \{1, 2\}$. Hence, by Lemma 3, either $Z_i(x)$ is complete or is an odd cycle with at least 5 vertices. In the first case, $Z_i(x)$ contains at least $r_i - \left\lfloor \frac{\Delta(G}{2} \right\rfloor + 2 \geq i \geq 1$. In the second case, $r_i = r_1 = 2$, so $\mathcal{H}(G)$ is independent. Thus $Z_i(x)$ contains at least 3 low vertices. Hence we can swap x with a low vertex in U_i to get another minimal $\chi(G)$ coloring.

Thus we may assume that x is low. For $i \geq 0$, let $p_i = 1$ if i is odd and $p_i = 2$ if i is even. Consider the following algorithm.

(1) Put $q_0(y) = 0$ for each $y \in V(G)$.
(2) Put $x_0 = x$, $\pi_0 = \pi$ and $i = 0$.
(3) Pick a low vertex $x_{i+1} \in Z_{\pi_i, p_i}(x_i) - x_i$ first minimizing $q_i(x_{i+1})$ and then minimizing $d(x_i, x_{i+1})$. Swap x_{i+1} with x_i. Let π_{i+1} be the resulting coloring.
(4) Put $q_i(x_i) = q_i(x_{i+1}) + 1$.
(5) Put $q_{i+1} = q_i$.
(6) Put $i = i + 1$.
(7) Goto (3).

Since $V(G)$ is finite, we have a smallest k such that we are at step 3, $p_k = 2$, and $q_k(z) = 1$ for some low vertex $z \in Z_{\pi_k,2}(x_k) - x_k$.

Claim: $q_k(y) \leq 1$ **for all** $y \in V(G)$**.**

Suppose to the contrary that we have $y \in V(G)$ with $q_k(y) > 1$, then there is a first $j < k$ for which $q_j(y) > 1$. From the first minimality condition in step 3 we see that we must have $q_j(t) = 1$ for each low vertex $t \in Z_{\pi_j, p_j}(x_j) - x_j$. In addition, $p_j = 1$ by the minimality of k.

For each low $t \in Z_{\pi_j, p_j}(x_j) - x_j$, let $m(t)$ be the least a such that $t = x_a$. We will show that there exists low $t \in Z_{\pi_j, p_j}(x_j) - x_j$ such that $x_{m(t)}$ is adjacent to $x_{m(t)+1}$. Plainly, this is the case if $r_1 \geq 3$ since then $Z_{\pi_j, p_j}(x_j)$ is complete for all j and $x_{m(t)}$ is always adjacent to $x_{m(t)+1}$. Thus we may assume that $r_1 = 2$, $r_2 = 3$, $\Delta = 6$ and $\mathcal{H}(G)$ is independent. Let $t_1, t_2, \ldots, t_b$ be the low vertices of $Z_{\pi_j, p_j}(x_j)$ ordered by $m(t_l)$. Since $Z_{\pi_{m(t_1)},1}(t_1)$ is an odd cycle and $\mathcal{H}(G)$ is independent, $Z_{\pi_{m(t_1)},1}(t_1)$ contains a pair of adjacent low vertices, say u and v. If $N(t_1) \cap Z_{\pi_{m(t_1)},1}(t_1)$ contains a low vertex, then t_1 is our desired t by the second minimality condition in step 3. Thus $t_1 \notin \{u, v\}$. Take l minimal such that $u = x_{m(t_l)+1}$

or $v = x_{m(t_l)+1}$. Without loss of generality, say $u = x_{m(t_l)+1}$. Then t_{l+1} must be adjacent to v and thus t_{l+1} is our desired t by the second minimality condition in step 3.

Now, put $a = m(t)$, $H_a = N(x_a) \cap \pi_a^{-1}(C_2)$ and $H_j = N(x_a) \cap \pi_j^{-1}(C_2)$. Since $x_{a-1} \in H_a$ and $q_{a-1}(x_{a-1}) = 1$, by the minimality of k, $N(x_m) \cap H_a = \emptyset$ for $a \leq m < k$. Thus $H_a \subseteq H_j$. Since x_{a+1} is adjacent to x_a we have $x_{a+1} \in H_j - H_a$ and thus $|H_j| \geq |H_a| + 1 = r_2 + 1$. But then $d(x_a) \geq r_1 + r_2 + 1 \geq \Delta$ contradicting the fact that x_a is low. This proves the claim.

Now, remember our low vertex $z \in Z_{\pi_k,2}(x_k) - x_k$ with $q_k(z) = 1$. Let $w \in Z_{\pi_k,2}(x_k) - \{x_k, z\}$ be a low vertex and let e be minimal such that $x_e = z$. Consider the change of π_k given by swapping x_k with z to get a minimal coloring π'. Also consider the change of π_k given by swapping x_k with w to get a minimal coloring π''. Since $q_k(x_{e+1}) \leq 1$, it must be that $x_{e+1} \in Z_{\pi',1}(z) \cap Z_{\pi'',1}(w)$ and hence $Z_{\pi',1}(z) - z = Z_{\pi'',1}(w) - w$. Let $T = V(Z_{\pi',1}(z)) - z$, $D = V(Z_{\pi_k,2}(x_k))$, and $F = G[T \cup D]$.

Since G is vertex critical, we may $(\Delta - 1)$-color $G - F$. Doing so leaves a list assignment J on F where $|J(v)| = d_F(v)$ if $v \in V(F)$ is low and $|J(v)| = d_F(v) - 1$ if $v \in V(F)$ is high. Suppose x_k is not adjacent to x_{e+1}. Since both are low vertices we have $|J(x_k)| + |J(x_{e+1})| \geq d_F(x_k) + d_F(x_{e+1})$. Clearly, $d_F(x_k) \geq r_2$. Also, since x_{e+1} is adjacent to all of D we have $d_F(x_{e+1}) \geq r_2 + r_1 - 1$ if $r_1 \geq 3$ and $d_F(x_{e+1}) \geq r_2$ if $r_1 = 2$. Note that in both cases, $d_F(x_k) + d_F(x_{e+1}) \geq r_1 + r_2 + 1$. Since the lists together contain at most $\Delta - 1 = r_1 + r_2$ colors, we have $c \in J(x_k) \cap J(x_{e+1})$. If we color both x_k and x_{e+1} with c it is easy to complete the coloring to the rest of F by first coloring $F - \{z, w, x_k, x_{e+1}\}$ and then coloring z and w. This is a contradiction, hence x_k is adjacent to x_{e+1}.

First suppose $\Delta = 6$. Then $|T| = 2$, say $T = \{z', x_{e+1}\}$. Now $D \cup \{x_{e+1}\}$ induces a K_5 with at most one high vertex and z' is adjacent to the low vertices $w, z \in D$. Thus Lemma 5 gives a contradiction.

Hence we may assume that $\Delta \geq 7$. Put $C = \{z, w\}$, $A = T - \{x_{e+1}\}$ and $B = D - \{z, w\} \cup \{x_{e+1}\}$ and $F' = F - \{z, w\}$. Then A and B are cliques that cover F' and x_{e+1} is joined to A. As above we may $(\Delta - 1)$-color $G - F$. Doing so leaves a list assignment J on F where $|J(v)| = d_F(v)$ if $v \in V(F)$ is low and $|J(v)| = d_F(v) - 1$ if $v \in V(F)$ is high. If we can find non-adjacent $y_1, y_2 \in V(F')$ such that $J(y_1) \cap J(y_2) \neq \emptyset$, then after coloring y_1 and y_2 the same we can easily complete the coloring to the rest F' and then to F. Since G contains no K_Δ we have non-adjacent vertices $y_1 \in A$ and $y_2 \in B$. Let $l(y_1, y_2) = |\{i \mid y_i \text{ is low}\}|$ and $n(y_1) = |N(y_1) \cap V(B)|$. Since x_{e+1} is joined to A, $n(y_1) \geq 1$. We have

$$\begin{aligned}
|L(y_1)| + |L(y_2)| &\geq d_F(y_1) + d_F(y_2) - 2 + l(y_1, y_2) \\
&\geq d_{F'}(y_1) + d_{F'}(y_2) + 2 + l(y_1, y_2) \\
&\geq |A| - 1 + n(y_1) + |B| - 1 + 2 + l(y_1, y_2) \\
&= |A| + |B| + n(y_1) + l(y_1, y_2) \\
&= \Delta - 2 + n(y_1) + l(y_1, y_2).
\end{aligned}$$

Since there are at most $\Delta - 1$ colors in both lists, if $n(y_1) + l(y_1, y_2) \geq 2$ we have $L(y_1) \cap L(y_2) \neq \emptyset$ giving a contradiction. Whence $n(y_1) + l(y_1, y_2) \leq 1$, giving $l(y_1, y_2) = 0$ and $n(y_1) = 1$. But $x_k \in B$ is low, so using $y_2 = x_k$ shows that x_k is joined to A. But then $n(y_1) \geq 2$ for any $y_1 \in A$. This final contradiction completes the proof. □

References

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