\textbf{\Large \textsc{\textit{$\Delta$-CRITICAL GRAPHS WITH SMALL HIGH VERTEX CLIQUES}}}

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\textsc{Abstract.} We prove that $K_{\chi(G)}$ is the only vertex critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(H(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$. Here $H(G)$ is the subgraph of $G$ induced on the vertices of degree at least $\chi(G)$. Setting $\omega(H(G)) = 1$ proves a conjecture of Kierstead and Kostochka.

1. \textbf{Introduction}

For a graph $G$ let $H(G)$ be the subgraph of $G$ induced on the vertices of degree at least $\chi(G)$. Recently, Kierstead and Kostochka [1] proved the following theorem and conjectured that the 7 could be improved to 6.

\textbf{Theorem 1} (Kierstead and Kostochka). $K_{\chi(G)}$ is the only vertex critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 7$ such that $H(G)$ is edgeless.

We prove this conjecture by establishing the following generalization.

\textbf{Theorem 2.} $K_{\chi(G)}$ is the only vertex critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(H(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

Setting $\omega(H(G)) = 1$ proves the conjecture.

\textbf{Corollary 1.} $K_{\chi(G)}$ is the only vertex critical graph $G$ with $\chi(G) \geq \Delta(G) \geq 6$ such that $H(G)$ is edgeless.

We can restate this in terms of Ore-degree as in [1] to get a generalization of Brooks’ theorem.

\textbf{Definition 1.} The \textit{Ore-degree} of an edge $xy$ in a graph $G$ is $\theta(xy) = d(x) + d(y)$. The \textit{Ore-degree} of a graph $G$ is $\theta(G) = \max_{xy \in E(G)} \theta(xy)$.

\textbf{Corollary 2.} If $6 \leq \chi(G) = \left\lfloor \frac{\theta(G)}{2} \right\rfloor + 1$, then $G$ contains the complete graph $K_{\chi(G)}$.

This is best possible as shown by the following example from [1].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{counterexample.png}
\caption{A counterexample to Corollary 2 with $\chi = 5$.}
\end{figure}
2. The Proof

We will use part of an algorithm of Mozhan [2]. The following is a generalization of his main lemma.

Definition 2. Let $G$ be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-partitioned coloring of $G$ we mean a proper coloring of $G$ of the form:

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each $L_{ij}$ is a color class.

Lemma 3. Let $G$ be a graph containing at least one critical vertex. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of $G$ pick one (call it $\pi$) minimizing

$$\sum_{i=1}^a \left| E \left( G \left[ \bigcup_{j=1}^{r_i} L_{ij} \right] \right) \right|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of $x$ in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

Proof. Let $1 \leq i \leq a$ such that $d_{Z_i(x)}(x) = r_i$. Put $Z_i = Z_i(x)$.

First suppose $\Delta(Z_i) > r_i$. Take $y \in V(Z_i)$ with $d_{Z_i}(y) > r_i$ closest to $x$ and let $x_1x_2\cdots x_t$ be a shortest $x - y$ path in $Z_i$. Clearly, for $k < t$, each $x_k$ is adjacent to exactly one vertex in each color class besides its own. Thus we may recolor $x_k$ with $\pi(x_{k+1})$ for $k < t$ and $x_t$ with $\pi(x_1)$ to produce a new $\chi(G)$-coloring of $G$ (this can be seen as a generalized Kempe chain). But we’ve moved a vertex $(x_t)$ of degree $r_i + 1$ out of $U_i$ while moving in a vertex $(x_1)$ of degree $r_i$ and it violates the minimality condition on $\pi$. This is a contradiction.

Thus $\Delta(Z_i) \leq r_i$. But $\chi(Z_i) = r_i + 1$, so Brooks’ theorem implies that $Z_i$ is complete if $r_i \geq 3$ and $Z_i$ is an odd cycle if $r_i = 2$. □

Now to prove Theorem 2, we suppose it is false and derive a contradiction from properties of a minimal counterexample. Let $G \neq K_{\chi(G)}$ be a vertex critical graph with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(H(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$ having the minimum number of vertices.

Definition 3. We call $v \in V(G)$ low if $d(v) = \chi(G) - 1$ and high otherwise.

Lemma 4. If $\Delta(G) = 6$, then $G$ contains no $K_6 - e$.

Proof. Suppose $\Delta(G) = 6$ and that $G$ contains a $K_6 - e$, call it $H$. Let $x_1, x_2 \in V(H)$ with $d_H(x_1) = 4$. Color $G - H$ with 5 colors and let $J$ be the resulting list assignment on $H$. Then $|J(x_1)| + |J(x_2)| \geq d_H(x_1) + d_H(x_2) - 2 \geq 2*6 - 6 \geq 6$. Hence we have $c \in J(x_1) \cap J(x_2)$. Color both $x_1$ and $x_2$ with $c$ to get a list assignment $J'$ on $F = H - \{x_1, x_2\}$. Since $\Delta(G) = 6$, $H(G)$ is edgeless. Thus at most one vertex $y \in V(F)$ is high. Hence $|J'(y)| \geq 3$ and $|J'(z)| \geq 4$ for all $z \in V(F) - \{y\}$. Since $F$ has 4 vertices we can complete the 5-coloring using Hall’s theorem. This contradiction completes the proof. □

Lemma 5. Suppose $\Delta(G) = 6$. Let $C$ be a $K_5$ in $G$ with at most one high vertex. Then each vertex in $G - C$ is adjacent to at most one low vertex in $C$.

Proof. Suppose otherwise that some $x \in V(G - C)$ is adjacent to all of $S \subseteq C$ where each vertex in $S$ is low and $|S| \geq 2$. Put $F = G - C$. Then $F$ is 5-colorable. Since each vertex in $C$ is adjacent to at least one vertex in $F$ and $G$ contains no $K_6 - e$, we have $y \in V(F)$ with $y \neq x$ such that $N(y) \cap C$ contains low vertices. Consider the graph $T = F + xy$. Note that $d_F(x) \leq 5$ and $d_T(y) \leq 6$. By minimality of $G$,
T is either 5-colorable or contains a $K_{\Delta(G)}$. In the former case we get a 5-coloring of $F$ where $x$ and $y$ receive different colors, but this is easily completable to a coloring of $G$. Thus $T$ contains $K_6$ and hence $G$ contains a $K_6 - e$ giving a contradiction.

Note that in Lemma 3, if $d_{Z_i(x)}(x) = r_i$ then we can swap $x$ with any other $y \in Z_i(x)$ by changing $\pi$ so that $x$ is colored with $\pi(y)$ and $y$ is colored with $\pi(x)$ to get another minimal $\chi(G)$-coloring of $G$.

**Proof of Theorem 2.** First, if $\chi(G) > \Delta(G)$ the theorem follows from Brooks’ theorem.

Hence we may assume that $\chi(G) = \Delta(G)$. Put $\Delta = \Delta(G)$, $r_1 = \lceil \frac{\Delta-1}{2} \rceil$ and $r_2 = \lceil \frac{\Delta-1}{2} \rceil$. Of all $(r_1, r_2)$-partitioned colorings of $G$, pick one minimizing

$$\sum_{i=1}^{2} \left| E \left( G \left[ \bigcup_{j=1}^{r_i} L_{ij} \right] \right) \right|.$$ 

Remember that $\{x\}$ is a singleton color class in the coloring. Throughout the proof we refer to a coloring that minimizes the above function as a *minimal* coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $C_i = \pi(U_i)$ (the colors used on $U_i$). For a minimal coloring $\gamma$ of $G$, let $Z_{\gamma,i}(x)$ be the component of $x$ in $G[\{x\} \cup \gamma^{-1}(C_i)]$.

Put $Z_i(x) = Z_{\pi,i}(x)$.

Note that $r_1 \geq 2$ and $r_2 \geq 3$ and if $r_1 = 2$ then $r_2 = 3$, $\Delta = 6$ and $\omega(\mathcal{H}(G)) \leq 1$. First suppose $x$ is high. Then $d(x) = r_1 + r_2 + 1$ and hence $d_{Z_i(x)}(x) = r_i$ for some $i \in \{1, 2\}$. Hence, by Lemma 3, either $Z_i(x)$ is complete or is an odd cycle with at least 5 vertices. In the first case, $Z_i(x)$ contains at least $r_i - \lceil \frac{\Delta(G)}{2} \rceil + 2 \geq i \geq 1$. In the second case, $r_i = r_1 = 2$, so $\mathcal{H}(G)$ is independent. Thus $Z_i(x)$ contains at least 3 low vertices. Hence we can swap $x$ with a low vertex in $U_i$ to get another minimal $\chi(G)$ coloring.

Thus we may assume that $x$ is low. For $i \geq 0$, let $p_i = 1$ if $i$ is odd and $p_i = 2$ if $i$ is even. Consider the following algorithm.

1. Put $q_0(y) = 0$ for each $y \in V(G)$.
2. Put $x_0 = x$, $\pi_0 = \pi$ and $i = 0$.
3. Pick a low vertex $x_{i+1} \in Z_{\pi_i, p_i}(x_i) - x_i$ first minimizing $q_i(x_{i+1})$ and then minimizing $d(x_i, x_{i+1})$.

Swap $x_{i+1}$ with $x_i$. Let $\pi_{i+1}$ be the resulting coloring.
4. Put $q_i(x_i) = q_i(x_{i+1}) + 1$.
5. Put $q_{i+1} = q_i$.
6. Put $i = i + 1$.
7. Goto (3).

Since $V(G)$ is finite, we have a smallest $k$ such that we are at step 3, $p_k = 2$, and $q_k(z) = 1$ for some low vertex $z \in Z_{\pi_k, 2}(x_k) - x_k$.

**Claim:** $q_k(y) \leq 1$ for all $y \in V(G)$.

Suppose to the contrary that we have $y \in V(G)$ with $q_k(y) > 1$, then there is a first $j < k$ for which $q_j(y) > 1$. From the first minimality condition in step 3 we see that we must have $q_j(t) = 1$ for each low vertex $t \in Z_{\pi_j, p_j}(x_j) - x_j$. In addition, $p_j = 1$ by the minimality of $k$.

For each low $t \in Z_{\pi_j, p_j}(x_j) - x_j$, let $m(t)$ be the least $a$ such that $t = x_a$. We will show that there exists low $t \in Z_{\pi_j, p_j}(x_j) - x_j$ such that $x_a(t)$ is adjacent to $x_{m(t)+1}$. Plainly, this is the case if $r_1 \geq 3$ since then $Z_{\pi_j, p_j}(x_j)$ is complete for all $j$ and $x_a(t)$ is always adjacent to $x_{m(t)+1}$. Thus we may assume that $r_1 = 2$, $r_2 = 3$, $\Delta = 6$ and $\mathcal{H}(G)$ is independent. Let $t_1, t_2, \ldots, t_k$ be the low vertices of $Z_{\pi_j, p_j}(x_j)$ ordered by $m(t_i)$. Since $Z_{\pi_{m(t_1)+1}}(t_1)$ is an odd cycle and $\mathcal{H}(G)$ is independent, $Z_{\pi_{m(t_1)+1}}(t_1)$ contains a pair of adjacent low vertices, say $u$ and $v$. If $N(t_1) \cap Z_{\pi_{m(t_1)+1}}(t_1)$ contains a low vertex, then $t_1$ is our desired $t$ by the second minimality condition in step 3. Thus $t_1 \notin \{u, v\}$. Take $l$ minimal such that $u = x_{m(t_l)+1}$...
or \( v = x_{m(t_i) + 1} \). Without loss of generality, say \( u = x_{m(t_i) + 1} \). Then \( t_{l_1 + 1} \) must be adjacent to \( v \) and thus \( t_{l_1 + 1} \) is our desired \( t \) by the second minimality condition in step 3.

Now, put \( a = m(t), H_a = N(x_a) \cap \pi_a^{-1}(C_2) \) and \( H_j = N(x_a) \cap \pi_j^{-1}(C_2) \). Since \( x_{a_1} \in H_a \) and \( q_{a_1}(x_{a_1}) = 1 \), by the minimality of \( k \), \( N(x_m) \cap H_a = \emptyset \) for \( a \leq m < k \). Thus \( H_a \subseteq H_j \). Since \( x_{a+1} \) is adjacent to \( x_a \) we have \( x_{a+1} \in H_j - H_a \) and thus \( |H_j| \geq |H_a| + 1 = r_2 + 1 \). But then \( d(x_a) \geq r_1 + r_2 + 1 \geq \Delta \) contradicting the fact that \( x_a \) is low. This proves the claim.

Now, remember our low vertex \( z \in Z_{\pi_k,2}(x_k) - x_k \) with \( q_k(z) = 1 \). Let \( w \in Z_{\pi_k,2}(x_k) - \{x_k, z\} \) be a low vertex and let \( e \) be minimal such that \( x_e = z \). Consider the change of \( \pi_k \) given by swapping \( x_k \) with \( z \) to get a minimal coloring \( \pi' \). Also consider the change of \( \pi_k \) given by swapping \( x_k \) with \( w \) to get a minimal coloring \( \pi'' \). Since \( q_k(x_{e+1}) \leq 1 \), it must be that \( x_{e+1} \in Z_{\pi''}(z) \cap Z_{\pi''}(w) \) and hence \( Z_{\pi',1}(z) = Z_{\pi'',1}(w) \). Let \( T = V(Z_{\pi',1}(z)) - z, D = V(Z_{\pi_k,2}(x_k)) \), and \( F = G[T \cup D] \).

Since \( G \) is vertex critical, we may \((\Delta - 1)\)-color \( G - F \). Doing so leaves a list assignment \( J \) on \( F \) where \( |J(v)| = d_F(v) \) if \( v \in V(F) \) is low and \( |J(v)| = d_F(v) - 1 \) if \( v \in V(F) \) is high. Suppose \( x_k \) is not adjacent to \( x_{e+1} \). Since both are low vertices we have \( |J(x_k)| + |J(x_{e+1})| \geq d_F(x_k) + d_F(x_{e+1}) \). Clearly, \( d_F(x_k) \geq r_2 \). Also, since \( x_{e+1} \) is adjacent to all of \( D \) we have \( d_F(x_{e+1}) \geq r_2 + r_1 - 1 \) if \( r_1 \geq 3 \) and \( d_F(x_{e+1}) \geq r_2 \) if \( r_1 = 2 \). Note that in both cases, \( d_F(x_k) + d_F(x_{e+1}) \geq r_1 + r_2 + 1 \). Since the lists together contain at most \( \Delta - 1 = r_1 + r_2 \) colors, we have \( c \in J(x_k) \cap J(x_{e+1}) \). If we color both \( x_k \) and \( x_{e+1} \) with \( c \) it is easy to complete the coloring to the rest of \( F \) by first coloring \( F - \{z, w, x_k, x_{e+1}\} \) and then coloring \( z \) and \( w \). This is a contradiction, hence \( x_k \) is adjacent to \( x_{e+1} \).

First suppose \( \Delta = 6 \). Then \( |T| = 2 \), say \( T = \{z', x_{e+1}\} \). Now \( D \cup \{x_{e+1}\} \) induces a \( K_5 \) with at most one high vertex and \( z' \) adjacent to the low vertices \( w, z \in D \). Thus Lemma 5 gives a contradiction.

Hence we may assume that \( \Delta \geq 7 \). Put \( C = \{z, w\}, A = T - \{x_{e+1}\} \) and \( B = D - \{z, w\} \cup \{x_{e+1}\} \) and \( F' = F - \{z, w\} \). Then \( A \) and \( B \) are cliques that cover \( F' \) and \( x_{e+1} \) is joined to \( A \). As above we may \((\Delta - 1)\)-color \( G - F \). Doing so leaves a list assignment \( J \) on \( F \) where \( |J(v)| = d_F(v) \) if \( v \in V(F) \) is low and \( |J(v)| = d_F(v) - 1 \) if \( v \in V(F) \) is high. If we can find non-adjacent \( y_1, y_2 \in V(F') \) such that \( J(y_1) \cap J(y_2) \neq \emptyset \), then after coloring \( y_1 \) and \( y_2 \) the same we can easily complete the coloring to the rest \( F' \) and then \( F \). Since \( G \) contains no \( K_\Delta \) we have non-adjacent vertices \( y_1 \in A \) and \( y_2 \in B \). Let \( l(y_1, y_2) = |\{i \mid y_i \text{ is low}\}| \) and \( n(y_1) = |N(y_1) \cap V(B)| \). Since \( x_{e+1} \) is joined to \( A \), \( n(y_1) \geq 1 \). We have

\[
\begin{align*}
|L(y_1)| + |L(y_2)| &\geq d_F(y_1) + d_F(y_2) - 2 + l(y_1, y_2) \\
&\geq d_{F'}(y_1) + d_{F'}(y_2) + 2 + l(y_1, y_2) \\
&\geq |A| - 1 + n(y_1) + |B| - 1 + 2 + l(y_1, y_2) \\
&= |A| + |B| + n(y_1) + l(y_1, y_2) \\
&= \Delta - 2 + n(y_1) + l(y_1, y_2).
\end{align*}
\]

Since there are at most \( \Delta - 1 \) colors in both lists, if \( n(y_1) + l(y_1, y_2) \geq 2 \) we have \( L(y_1) \cap L(y_2) \neq \emptyset \) giving a contradiction. Whence \( n(y_1) + l(y_1, y_2) \leq 1 \), giving \( l(y_1, y_2) = 0 \) and \( n(y_1) = 1 \). But \( x_k \in B \) is low, so using \( y_2 = x_k \) shows that \( x_k \) is joined to \( A \). But then \( n(y_1) \geq 2 \) for any \( y_1 \in A \). This final contradiction completes the proof.

\[ \square \]

References
