On Graph Associations

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January 20, 2006

Abstract

We introduce a notion of vertex association and consider sequences of these associations. This allows for slick proofs of a few known theorems as well as showing that for any induced subgraph H of G, $\chi(G) \leq \chi(H) + \frac{1}{2}(\omega(G) + |G| - |H| - 1)$. As a special case of this, we have $\chi(G) \leq \left\lceil \frac{\omega(G)+\tau(G)}{2} \right\rceil$ (here $\chi(G)$ denotes the chromatic number, $\omega(G)$ the clique number and $\tau(G)$ the vertex cover number), which is a generalization of the Nordhaus-Gaddum upper bound. In addition, this settles a conjecture of Reed that $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta(G)+1}{2} \right\rceil$ in the case when $\delta(\overline{G}) \leq \omega(\overline{G})$.

1 Definitions and Basic Properties

All graphs will be assumed finite and simple. We let $|G|$ denote the order of G, $s(G)$ the size of G, $\chi(G)$ the chromatic number, $\omega(G)$ the clique number, $\tau(G)$ the vertex cover number, $\Delta(G)$ the maximum degree, $\delta(G)$ the minimum degree, $d_G(x)$ the degree of x in G, and $N_G(x)$ the set of neighbors of x in G.

Definition 1.1. Given a graph G and non-adjacent vertices a and b, we write $G/[a,b]$ for the graph obtained from G by associating (i.e., identifying) a and b into a single vertex $[a,b]$ and discarding multiple edges.

Proposition 1.2. *Let G be a graph and $a, b, x \in V(G)$ with $a \notin N_G(b)$. Then*

$$d_{G/[a,b]}(x) = \begin{cases} d_G(x) - 1 & \text{if } x \in N_G(a) \cap N_G(b), \\ d_G(a) + d_G(b) - |N_G(a) \cap N_G(b)| & \text{if } x \in \{a,b\}, \\ d_G(x) & \text{otherwise.} \end{cases}$$

Proof. Immediate from the definitions. □

The content of the following proposition is that the operations of vertex removal and association commute.

Proposition 1.3. *Let G be a graph. If $a, b \in V(G)$ with $a \notin N_G(b)$ and $S \subseteq V(G) \setminus \{a, b\}$, then*

$$(G \setminus S)/[a, b] = G/[a, b] \setminus S.$$

Proof. Again, this is immediate from the definitions. □

Lemma 1.4. *Let a and b be non-adjacent vertices in a graph G. Then*

(i) $\chi(G) \leq \chi(G/[a, b]) \leq \chi(G) + 1$,

(ii) $\chi(G/[a, b]) = \chi(G)$ *if and only if there exists a coloring of G with $\chi(G)$ colors in which a and b receive the same color.*

Proof.

(i) Since a and b are non-adjacent, any k-coloring of $G/[a, b]$ lifts to a k-coloring of G. This gives the first inequality. The second follows by noting that any k-coloring of G induces a k-coloring of $G/[a, b] \setminus \{[a, b]\}$ and hence a $(k + 1)$-coloring of $G/[a, b]$ by introducing a new color.

(ii) Assume $\chi(G/[a, b]) = \chi(G)$. Then we have a $\chi(G)$-coloring of $G/[a, b]$ and lifting this to G gives a $\chi(G)$-coloring of G in which a and b receive the same color.
For the converse, assume we have a $\chi(G)$-coloring of G in which a and b receive the same color. Then the induced $\chi(G)$-coloring of $G/[a, b] \setminus \{[a, b]\}$ extends to a $\chi(G)$-coloring of $G/[a, b]$ by coloring $[a, b]$ the color that a and b share.

□

Proposition 1.5. *Let a and b be non-adjacent vertices in a graph G. Then*

$$\chi(G) = \min\{\chi(G/[a, b]), \chi(G + ab)\}.$$

Proof. If $\chi(G) = \chi(G/[a, b])$, then we are done since $\chi(G + ab) \geq \chi(G)$. Otherwise, by Lemma 1.4(ii), a and b must receive different colors in every $\chi(G)$-coloring of G. Hence, any $\chi(G)$-coloring of G extends to a $\chi(G)$-coloring of $G + ab$. Thus $\chi(G) = \chi(G + ab)$, completing the proof. □

2 Sequences of Associations

We consider sequences of the form

$$G = H_0 \to H_1 \to \ldots \to H_r = K_t,$$

where each term is obtained from the previous one by associating two non-adjacent vertices. The process clearly terminates at some complete graph K_t.

Lemma 2.1. *Let G be a graph. If G is not complete, then there exist non-adjacent vertices a and b which receive the same color in some $\chi(G)$-coloring of G.*

Proof. If not, then any given vertex must be colored differently from every other vertex in any $\chi(G)$-coloring of G. Hence, $\chi(G) = |G|$ and thus G is complete. □

Proposition 2.2. *The smallest t for which there exists a sequence*

$$G = H_0 \to H_1 \to \ldots \to H_r = K_t$$

is $t = \chi(G)$.

Proof. The first inequality of Lemma 1.4(i) and the fact that $\chi(K_t) = t$ yield $t \geq \chi(G)$. We just need to show that $K_{\chi(G)}$ can be attained. If G is complete, then we are done. Otherwise, by Lemma 2.1, we have two vertices a and b which receive the same color in some $\chi(G)$-coloring of G. By Lemma 1.4(ii), $\chi(G/[a,b]) = \chi(G)$. Since $|G/[a,b]| < |G|$, the result follows by induction. □

Definition 2.3. We denote by $\psi(G)$ the largest t for which there exists a sequence

$$G = H_0 \to H_1 \to \ldots \to H_r = K_t.$$

With a little thought, one can see that this is the same thing as the achromatic number of G.

Loose upper bounds on $\psi(G)$ can be easily obtained.

Proposition 2.4. *Let G be a graph. Then*

(i) $\psi(G) \leq |G|$,

(ii) $\psi(G) \leq \frac{1+\sqrt{1+8s(G)}}{2}$.

Proof. Consider the sequence

$$G = H_0 \to H_1 \to \ldots \to H_r = K_{\psi(G)}.$$

As we move from left to right, both the order and size of the graphs do not increase; hence, $|G| \geq \psi(G)$ and $s(G) \geq \binom{\psi(G)}{2}$. The results follow. □

3 Some Slick Proofs

Lemma 3.1. *If a and b are non-adjacent vertices in a graph G, then*

$$\chi(\overline{G}) - 1 \leq \chi(\overline{G/[a,b]}) \leq \chi(\overline{G}).$$

Proof. Note that the chromatic number of $\overline{G}$ is the clique cover number of G. Assume we have a partition of $V(G)$ into n disjoint sets $\{K_1, \ldots, K_n\}$, each of which induces a clique. Since a and b are non-adjacent, they are in distinct cliques, say $a \in K_i$, $b \in K_j$ with $i \neq j$. We see that replacing K_i with $K_i \setminus \{a\}$ and K_j with $(K_j \setminus \{b\}) \cup \{[a,b]\}$ yields a covering of $G/[a,b]$ with n cliques. This gives the second inequality. To get the first, assume we have a partition of $V(G/[a,b])$ into n disjoint sets $\{K_1, \ldots, K_n\}$, each of which induces a clique. Then $[a,b]$ is in one of the sets, say $[a,b] \in K_i$. Let $K_i^{'} = ((K_i \setminus \{[a,b]\}) \cap N_G(a)) \cup \{a\}$ and $K_{n+1}^{'} = ((K_i \setminus \{[a,b]\}) \setminus K_i^{'}) \cup \{b\}$. Then $\{K_1, \ldots, K_{i-1}, K_i^{'}, K_{i+1}, \ldots, K_n, K_{n+1}^{'}\}$ is a partition of $V(G)$ into $n+1$ disjoint sets, each of which induces a clique. □

Proposition 3.2 (Harary and Hedetniemi [2]). *Let G be a graph. Then*

$$\psi(G) + \chi(\overline{G}) \leq |G| + 1.$$

Proof. Consider the sequence

$$G = H_0 \to H_1 \to \ldots \to H_r = K_{\psi(G)}, \tag{1}$$

where $r = |G| - \psi(G)$. It follows from the first inequality of Lemma 3.1 that

$$\chi(\overline{G}) - (|G| - \psi(G)) = \chi(\overline{G}) - r \leq \chi(\overline{K_{\psi(G)}}) = 1,$$

so that $\psi(G) + \chi(\overline{G}) \leq |G| + 1$ as required. □

Corollary 3.3 (Nordhaus and Gaddum [3]). *Let G be a graph. Then*

$$\chi(G) + \chi(\overline{G}) \leq |G| + 1.$$

Proof. Use $\chi(G) \leq \psi(G)$ in Proposition 3.2. □

Lemma 3.4. *Let G be a graph. Then*

$$\chi(G) \geq 2\psi(G) - |G|.$$

Proof. It follows from (1) and the second inequality of Lemma 1.4(i) that

$$\psi(G) = \chi(K_{\psi(G)}) \leq \chi(G) + r = \chi(G) + |G| - \psi(G).$$

The result follows.

□

Proposition 3.5. *Let G be a graph. Then*

$$2\psi(G) + \psi(\overline{G}) \leq 2|G| + 1.$$

Proof. Lemma 3.4 applied to $\overline{G}$ yields $\chi(\overline{G}) \geq 2\psi(\overline{G}) - |G|$. Substituting this in proposition 3.2 gives $2\psi(\overline{G}) + \psi(G) \leq 2|G| + 1$. Now substiting $\overline{G}$ for G gives the result. □

Corollary 3.6 (Gupta [1]). *Let G be a graph. Then*

$$\psi(G) + \psi(\overline{G}) \leq \lceil \tfrac{4}{3}|G| \rceil.$$

Proof. Applying Proposition 3.5 to G and $\overline{G}$ yields the inequalities

$$2\psi(G) + \psi(\overline{G}) \leq 2|G| + 1$$

and

$$\psi(G) + 2\psi(\overline{G}) \leq 2|G| + 1$$

respectively. By adding these, we get

$$3(\psi(G) + \psi(\overline{G})) \leq 4|G| + 2,$$

which is

$$\psi(G) + \psi(\overline{G}) \leq \tfrac{4}{3}|G| + \tfrac{2}{3}.$$

The result follows. □

4 The Main Results

Definition 4.1. Let G be a graph and I an independent set in G. We denote by $G/[I]$ the graph obtained from G by associating I down to a single vertex $[I]$.

Lemma 4.2. *Let f be a real-valued graph function such that, for any graph G, $f(G \setminus \{v\}) \geq f(G) - 1$ for all $v \in V(G)$. Then, for any graph G and independent set I in G,*

$$f(G/[I]) \leq f(G \setminus I) + 1.$$

Proof. Observe that $G \setminus I = G/[I] \setminus \{[I]\}$. But $[I]$ is a single vertex; hence, $f(G \setminus I) = f(G/[I] \setminus \{[I]\}) \geq f(G/[I]) - 1$. The result follows. □

Definition 4.3. We say that a graph G consists of *an independent set attached to a clique* if $V(G)$ can be partitioned into two disjoint sets I and K such that I is independent and K induces a clique. We say that G consists of *an independent set strongly attached to a clique* if there is such a partition in which each vertex of K is adjacent to at least one vertex of I.

Lemma 4.4.

(a) If a graph G consists of an independent set I attached to a clique K, then $\overline{G}$ consists of an independent set $\overline{K}$ attached to a clique $\overline{I}$, and $\chi(G) = \omega(G) = |K|$ or $|K| + 1$ and $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G) = |I|$ or $|I| + 1$.

(b) If G consists of an independent set I strongly attached to a clique K, then $\chi(\overline{G}) = \omega(\overline{G}) = \alpha(G) = |I|$.

(c) If I is an independent set in a graph G, then $G/[I]$ is complete if and only if G consists of I strongly attached to a clique.

Proof.

(a) Since I is independent, $\chi(G) \leq |K| + 1$ and $\chi(G) = |K| + 1$ if and only if there exists $v \in I$ such that $N_G(v) = K$; in this case, $\omega(G) = |K| + 1$ as well. The statements about $\overline{G}$ follow in a similar manner.

(b) Assume each vertex of K is adjacent in G to at least one vertex of I. Then, in $\overline{G}$, each vertex of $\overline{K}$ is nonadjacent to at least one vertex of $\overline{I}$. Hence $\omega(\overline{G}) = |I|$. The other equalities follow from (a).

(c) We have $G/[I]$ complete if and only if $N_{G/[I]}([I]) = K$. This happens if and only if each vertex of K is adjacent to at least one vertex of I.

□

Lemma 4.5. *Let*

$$G = H_0 \to H_1 \to \ldots \to H_{r-1} \to H_r = K_t$$

be a sequence where each term is obtained from the previous one by associating two non-adjacent vertices. If $\chi(H_{r-1}) = \chi(H_r)$, then $\omega(H_{r-1}) = \omega(H_r)$.

Proof. Since H_r is complete, H_{r-1} is an independent set of size 2 strongly attached to a clique; hence, by Lemma 4.4(a), $\omega(H_{r-1}) = \chi(H_{r-1}) = \chi(H_r) = \omega(H_r)$. □

Theorem 4.6. *Let $I_1, \ldots, I_m$ be disjoint independent sets in a graph G. Then*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + |G| - \sum_{j=1}^{m} |I_j| + 2m - 1\right). \tag{2}$$

Proof. Associate I_1 through I_m in turn to yield a sequence

$$G = H_0 \to H_1 \to \ldots \to H_{m-1} \to H_m = B, \tag{3}$$

and let $A = H_{m-1}$, so that B is obtained from A by associating I_m to a single vertex.
We distinguish two cases.
Case 1: B is complete, so that $B = K_{\chi(B)}$. Then, by Lemma 4.4(c), A consists of I_m strongly attached to a clique. By Corollary 3.3 and Lemma 4.4(b),

$$\chi(A) \leq |A| - \chi(\overline{A}) + 1 = |A| - |I_m| + 1,$$

so that, since $\chi(A) = \omega(A)$ by Lemma 4.4(a),

$$2\chi(A) \leq \omega(A) + |A| - |I_m| + 1. \tag{4}$$

Since $\omega(G \setminus \{v\}) \geq \omega(G) - 1$ for all $v \in V(G)$, Lemma 4.2 tells us that associating an independent set to a single point increases ω by at most one. Hence

$$\omega(A) \leq \omega(G) + m - 1. \tag{5}$$

Also, $|G| - |A| = \sum_{j=1}^{m-1}(|I_j| - 1) = \sum_{j=1}^{m}|I_j| - |I_m| - m + 1$, so that

$$|A| - |I_m| = |G| - \sum_{j=1}^{m}|I_j| + m - 1. \tag{6}$$

Since $\chi(G) \leq \chi(A)$ by the first inequality of Lemma 1.4(i), substituting (5) and (6) into (4) gives

$$\begin{aligned} 2\chi(G) \leq 2\chi(A) &\leq \omega(G) + m - 1 + |G| - \sum_{j=1}^{m}|I_j| + m - 1 + 1 \\ &= \omega(G) + |G| - \sum_{j=1}^{m}|I_j| + 2m - 1, \end{aligned}$$

which is (2).

Case 2: B is not complete. Consider the sequence

$$B \to \ldots \to C \to K_{\chi(B)}, \tag{7}$$

where each term is obtained from the previous one by associating two non-adjacent vertices. Then, by the first inequality in Lemma 1.4(i),

$$\chi(B) \leq \chi(C) \leq \chi(K_{\chi(B)}) = \chi(B).$$

Hence $\chi(C) = \chi(B) = \chi(K_{\chi(B)})$ and we may apply Lemma 4.5 to conclude

$$\omega(C) = \omega(K_{\chi(B)}) = \chi(B). \tag{8}$$

In addition, it is clear that

$$|C| = \chi(B) + 1. \tag{9}$$

Applying Lemma 4.2 as in (5), but this time to a combination of sequences (3) and (7) between G and C, gives

$$\omega(C) \leq \omega(G) + m + |B| - |C|, \tag{10}$$

and $|G| - |B| = \sum_{j=1}^{m} |I_j| - m$, so that, by (8), (9) and (10),

$$\begin{aligned} 2\chi(B) = \omega(C) + |C| - 1 &\leq \omega(G) + m + |B| - 1 \\ &= \omega(G) + m + |G| - \sum_{j=1}^{m} |I_j| + m - 1. \end{aligned}$$

Since $\chi(G) \leq \chi(B)$ by the first inequality of Lemma 1.4(i), the theorem follows.

□

Since the vertex-set of an induced subgraph H of G can be partitioned into $\chi(H)$ independent sets, the following is an equivalent formulation of Theorem 4.6.

Theorem 4.7. *Let G be a graph. Then, for any induced subgraph H of G,*

$$\chi(G) \leq \chi(H) + \tfrac{1}{2}(\omega(G) + |G| - |H| - 1).$$

Corollary 4.8. *Let G be a graph. Then*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \tau(G)}{2} \right\rceil.$$

Proof. Apply Theorem 4.6 to a single independent set with $\omega(\overline{G})$ elements to get

$$\chi(G) \leq \frac{1}{2}(\omega(G) + |G| - \omega(\overline{G}) + 1). \tag{11}$$

Since $S \subseteq V(G)$ is a vertex cover if and only if $V(G) \setminus S$ is an independent set,

$$\tau(G) + \omega(\overline{G}) = |G|.$$

The result follows. □

Note that this is a generalization of the Nordhaus-Gaddum upper bound since replacing G by $\overline{G}$ in (11) and adding the two inequalities yields $\chi(G) + \chi(\overline{G}) \leq |G| + 1$.

Conjecture 4.9 (Reed [4]). *Let G be a graph. Then*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil .$$

Corollary 4.8 establishes this for all graphs G with $\tau(G) \leq \Delta(G) + 1$; equivalently, for all graphs with $\delta(\overline{G}) \leq \omega(\overline{G})$. In particular, if $\delta(\overline{G}) \leq 2$ then either $\delta(\overline{G}) \leq 2 \leq \omega(\overline{G})$ or $\omega(\overline{G}) = 1$ and hence G is complete. Thus Reed's conjecture holds for any graph G with $\Delta(G) \geq |G| - 3$.

Corollary 4.10. *Let G be a triangle-free graph. Then*

$$\chi(G) \leq 2 + \tfrac{1}{2}\delta(\overline{G}).$$

Proof. Since G is triangle-free, $\omega(\overline{G}) \geq \Delta(G)$. It follows from (11) that

$$\chi(G) \leq \frac{1}{2}(\omega(G) + |G| - \Delta(G) + 1) = \frac{1}{2}(\omega(G) + \delta(\overline{G}) + 2) \leq \frac{1}{2}(4 + \delta(\overline{G})),$$

which is the required result.

□

References

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