A NOTE ON REED'S CONJECTURE

LANDON RABERN

Abstract. In [5], Reed conjectures that every graph satisfies $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$. We prove this holds for graphs with disconnected complement. Combining this fact with a result of Molloy proves the conjecture for graphs satisfying $\chi > \left\lceil \frac{n}{2} \right\rceil$. Generalizing this we prove that the conjecture holds for graphs satisfying $\chi > \frac{n+3-\alpha}{2}$. It follows that the conjecture holds for graphs satisfying $\Delta \geq n + 2 - \left(\alpha + \sqrt{n+5-\alpha}\right)$. In the final section, we show that if G is an even order counterexample to Reed's conjecture, then $\overline{G}$ has a 1-factor.

1. Introduction

Reed's conjecture states that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil. \tag{1}$$

The weaker statement that every graph satisfies

$$\chi \leq \frac{\omega+n}{2}, \tag{2}$$

follows easily by observing that in any optimal coloring the set of vertices in singleton color classes induces a clique.

We prove the following intermediate result.

Key Lemma. $\chi \leq \frac{\omega + \frac{n+\Delta+1}{2}}{2}$

Using this result it is easy to prove Reed's conjecture for graphs with disconnected complement. Combining this fact with a result of Molloy tells us that Reed's conjecture holds for graphs satisfying $\chi > \left\lceil \frac{n}{2} \right\rceil$. Building on this result, we show that the conjecture holds for graphs satisfying $\chi > \frac{n+3-\alpha}{2}$. With a bit more work we determine that the conjecture holds for graphs satisfying $\Delta \geq n + 2 - \left(\alpha + \sqrt{n+5-\alpha}\right)$.

Further analysis allows us to prove some related results.

In all that follows, *graph* will mean finite simple graph with non-empty vertex set. Let $\mathbb{G}$ be the collection of all graphs. Let $R_t \subseteq \mathbb{G}$ be the graphs satisfying $\chi \leq \frac{1}{2}(\omega + \Delta + 1) + t$.

Definition 1. Given graphs A and B, their *join* $A + B$ is the graph with vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B) \cup \{ab \mid a \in V(A), b \in V(B)\}$. Also, if X and Y are collections of graphs, we let $X + Y = \{A + B \mid A \in X, B \in Y\}$.

First a few basic facts about joins.

Lemma 1.1. *Let A and B be graphs. Then*
(a) $|A + B| = |A| + |B|$,
(b) $\omega(A + B) = \omega(A) + \omega(B)$,
(c) $\chi(A + B) = \chi(A) + \chi(B)$,
(d) $\Delta(A + B) = \max\{\Delta(A) + |B|, |A| + \Delta(B)\}$.

Proof. These all follow immediately from the definitions. □

2. Reed's Conjecture For Joins Of Graphs

Proposition 2.1. *Let A and B be graphs. Then $A + B \in R_0$.*

Proof. Applying the Key Lemma to A and B and adding the inequalities yields

$$\chi(A) + \chi(B) \leq \frac{1}{2}\left(\omega(A) + \omega(B) + \frac{\Delta(A) + |B| + |A| + \Delta(B) + 2}{2}\right).$$

Using Lemma 1.1 (b),(c), and (d), this becomes

$$\chi(A + B) \leq \frac{1}{2}\left(\omega(A + B) + \frac{2\Delta(A + B) + 2}{2}\right) = \frac{1}{2}(\omega(A + B) + \Delta(A + B) + 1).$$

Hence $A + B \in R_0$. □

Lemma 2.2. $\mathbb{G} + R_t \subseteq R_t$ *for all* $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. Let $G \in \mathbb{G}$ and $H \in R_t$. Applying (2) to G gives

$$\chi(G) \leq \tfrac{1}{2}(\omega(G) + |G|).$$

Also, since $H \in R_t$,

$$\chi(H) \leq \tfrac{1}{2}(\omega(H) + \Delta(H) + 1) + t.$$

Adding these inequalities and applying Lemma 2 (b) and (c) gives

$$\chi(G + H) \leq \tfrac{1}{2}(\omega(G + H) + |G| + \Delta(H) + 1) + t.$$

Now Lemma 2(d) gives $|G| + \Delta(H) \leq \Delta(G + H)$ and the result follows. □

Remark. We will use Lemma 2.2 in our proof of the Key Lemma, we note that we did not use the Key Lemma in its proof. It should also be noted that Lemma 2.2 just says that R_t is an ideal in the abelian semigroup $(\mathbb{G}, +)$.

3. Proof Of The Key Lemma

Combining Lemma 2.2 with the following two results of Molloy (see [1]) allows us to prove Reed's conjecture for graphs satisfying $\alpha = 2$.

Lemma 3.1. *Let G be a graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$. Then there exists $X \subseteq V(G)$ such that $\overline{G - X}$ is disconnected and $\chi(G - X) = \chi(G)$.*

Lemma 3.2. *If G is a vertex critical graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$, then $\overline{G}$ is disconnected.*

Proposition 3.3. *If G is a graph with $\alpha(G) \leq 2$, then $G \in R_{\frac{1}{2}}$.*

Proof. Assume this is not the case and let G be a counterexample with the minimum number of vertices, say $|G| = n$. Since $\alpha(G) \leq 2$, we see that $V(G) \smallsetminus N(v) \cup \{v\}$ induces a clique for each $v \in V(G)$. Hence $\omega(G) \geq n - \delta(G) - 1$ which gives

$$\Delta(G) + \omega(G) + 1 \geq n.$$

So, since $G \notin R_{\frac{1}{2}}$, we have $\chi(G) > \left\lceil \frac{n}{2} \right\rceil$.
Now, using minimality of G, we see that G is vertex critical. Thus $\overline{G}$ is disconnected by Lemma 3.2. Hence we have $m \geq 2$ and (non-empty) graphs $C_1, \ldots, C_m$ such that $G = C_1 + \cdots + C_m$. But, for $1 \leq i \leq m$, minimality of G gives $C_i \in R_{\frac{1}{2}}$ since $\alpha(C_i) \leq \alpha(G) \leq 2$ and $|C_i| < n$. Hence $G = C_1 + \cdots + C_m \in R_{\frac{1}{2}}$ by Lemma 2.2. This contradiction completes the proof. □

To prove our Key Lemma, we will also need the following result from [2].

Theorem 3.4. *Let $I_1, \ldots, I_m$ be disjoint independent sets in a graph G. Then*

$$\chi(G) \leq \frac{1}{2}\left(\omega(G) + |G| - \sum_{j=1}^{m} |I_j| + 2m - 1\right). \tag{3}$$

Remark. If we increase the bound in this theorem by $\frac{1}{2}$ we get a corollary of (2).

To apply Theorem 3.4 and Proposition 3.3 we need a definition.

Definition 2. Let G be a graph and r a positive integer. A collection of disjoint independent sets in G each with at least r vertices will be called an r-*greedy* partial coloring of G. A vertex of G is said to be *missed* by a partial coloring just in case it appears in none of the independent sets.

We use the following consequence of Theorem 3.4.

Corollary 3.5. *Let G be a graph which is not complete and C an r-greedy partial coloring of G. Then*

$$\chi(G) \leq \tfrac{1}{2}(\omega(G) + |G| - (r-2)|C| - 1).$$

Combining Corollary 3.5 for $r = 3$ with the following Lemma yields the Key Lemma.

Lemma 3.6. *Let G be a graph and of all 3-greedy partial colorings of G, let C be one that misses the minimum number of vertices. Then*

$$\chi(G) \leq \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}.$$

Proof. The first case to consider is when C misses zero vertices. In this case, C is a proper coloring of G and hence $\chi(G) \leq |C|$. Thus

$$\chi(G) \leq \tfrac{1}{2}(\chi(G) + |C|) \leq \tfrac{1}{2}(\Delta(G) + 1 + |C|) \leq \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}.$$

Otherwise, C misses at least one vertex and by the minimality condition placed on C, each vertex missed by C must be adjacent to at least one vertex in each element of C. Hence $\Delta(G - \cup C) \leq \Delta(G) - |C|$. In addition, $\alpha(G - \cup C) \leq 2$. Thus, applying Proposition 3.3 to $G - \cup C$, yields

$$\begin{aligned}
\chi(G) &\leq |C| + \chi(G - \cup C) \\
&\leq |C| + \tfrac{1}{2}(\omega(G - \cup C) + \Delta(G - \cup C) + 1) + \tfrac{1}{2} \\
&\leq |C| + \tfrac{1}{2}(\omega(G) + \Delta(G) - |C| + 1) + \tfrac{1}{2} \\
&= \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + \tfrac{|C|+1}{2}.
\end{aligned}$$

□

4. Further Details

Proposition 2.1 allows us to prove Reed's conjecture for graphs satisfying $\chi > \lceil \frac{n}{2} \rceil$.

Corollary 4.1. *If G is a graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$, then $G \in R_0$.*

Proof. Let G be a graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$. Then, by Lemma 3.1, we have $X \subseteq V(G)$ such that $\overline{G - X}$ is disconnected and $\chi(G - X) = \chi(G)$. Since $\overline{G - X}$ is disconnected, there exist graphs A and B such that $G - X = A + B$. Hence, by Proposition 2.1,

$$\chi(G) = \chi(G - X) \leq \tfrac{1}{2}(\omega(G - X) + \Delta(G - X) + 1) \leq \tfrac{1}{2}(\omega(G) + \Delta(G) + 1).$$

Whence $G \in R_0$. □

We can generalize the above corollary by passing from R_0 to $R_{\frac{1}{2}}$.

Corollary 4.2. *If G is a graph with $\chi(G) > \frac{|G|+3-\alpha(G)}{2}$, then $G \in R_{\frac{1}{2}}$.*

Proof. Let G be a graph with $\chi(G) > \frac{|G|+3-\alpha(G)}{2}$ and I an independent set in G with $\alpha(G)$ vertices. Put $H = G \setminus I$. Then

$$\begin{aligned}\chi(H) &\geq \chi(G) - 1 \\ &> \frac{|G| + 3 - \alpha(G)}{2} - 1 \\ &= \frac{|G| + 1 - \alpha(G)}{2} \\ &= \frac{|H| + 1}{2}.\end{aligned}$$

Hence, by Corollary 4.1, we have

$$\chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2}.$$

But I is a maximal independent set and hence each vertex of H is adjacent to at least one vertex in I. In particular, $\Delta(H) \leq \Delta(G) - 1$. Whence

$$\chi(G) \leq \chi(H) + 1 \leq \frac{\omega(H) + \Delta(G) - 1 + 1}{2} + 1 \leq \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{1}{2}.$$

The theorem follows. □

Remark. After noting that the conditions on the graph order in both Corollary 4.1 and Corollary 4.2 are sufficient to force the existence of a doubly critical edge in a minimum counterexample to Reed's conjecture, it is natural to ask if similar results can be proven under the weaker condition that the graph contain a doubly critical edge. This is indeed the case as is shown in [3] and [4]. The first paper generalizes the method of proof using singleton color classes that was sketched in the introduction as justification of (2). It is proven that a graph with more than $\frac{\omega}{2}$ singleton color classes satisfies Reed's conjecture. The second paper proves Reed's conjecture for claw-free graphs containing a doubly critical edge and shows that general graphs containing a doubly critical edge satisfy $\chi \leq \frac{1}{3}\omega + \frac{2}{3}(\Delta + 1)$.

Corollary 4.3. *Let G be a graph and $t \geq \frac{1}{2}$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \omega(G) - \alpha(G) + 2$.*

Proof. This is an immediate consequence of the previous corollary. □

Lemma 4.4. *Let G be a graph with $\alpha(G) \leq 2$. Then $\omega(G)^2 + \omega(G) \geq |G|$.*

Proof. Let K be a maximal clique in G. Then each vertex of $G - K$ is non-adjacent to at least one vertex in K and hence some vertex $v \in K$ is non-adjacent to at least $\frac{|G-K|}{|K|}$ vertices. Since $\alpha(G) \leq 2$, the vertices non-adjacent to v form a clique. Whence $\omega(G) \geq \frac{|G-K|}{|K|} = \frac{|G|-\omega(G)}{\omega(G)}$, which yields

$$\omega(G)^2 + \omega(G) \geq |G|.$$

□

Proposition 4.5. *If G is a graph with* $\Delta(G) \geq |G|+2-\left(\alpha(G) + \sqrt{|G|+5-\alpha(G)}\right)$, *then* $G \in R_{\frac{1}{2}}$.

Proof. We prove the contrapositive. Let G be a graph with n vertices, maximal degree Δ, clique number ω, and independence number α such that $G \notin R_{\frac{1}{2}}$. Let I be a maximal independent set in G. Let S be a maximal collection of disjoint 3-vertex indepedent sets of $G - I$. Since $\alpha(G - (\cup S) \cup I) \leq 2$, we may apply Lemma 4.4 to get $\omega^2 + \omega \geq |G - (\cup S) \cup I| = n - \alpha - 3|S|$. Hence

$$|S| \geq \frac{n - \alpha - (\omega^2 + \omega)}{3}. \tag{4}$$

Now, combining the fact that $G \notin R_{\frac{1}{2}}$ with Corollary 3.5 , we have $n-\alpha-|S|+1 > \Delta + 2$. Putting this together with (4) we have

$$n - \alpha - \Delta - 1 > |S| \geq \frac{n - \alpha - (\omega^2 + \omega)}{3}.$$

Which implies that

$$\Delta < \frac{2n + \omega^2 + \omega - 2\alpha - 3}{3}. \tag{5}$$

By Corollary 4.3, $\omega \leq n - \Delta - \alpha$. Plugging this into (5) and doing a little algebra, we find that $\Delta < n - \alpha + 2 - \sqrt{n - \alpha + 5}$. This completes the proof of the contrapositive. □

Corollary 4.6. *If G is a graph with* $\Delta(G) \geq |G| - \left(1 + \sqrt{|G|+2}\right)$, *then* $G \in R_{\frac{1}{2}}$.

Proof. Let G be a graph with $\Delta(G) \geq |G| - \left(1 + \sqrt{|G|+2}\right)$. If $\alpha(G) \leq 2$, then $G \in R_{\frac{1}{2}}$ by Proposition 3.3. Otherwise, $\alpha(G) \geq 3$ and $G \in R_{\frac{1}{2}}$ by Proposition 4.5. □

5. Graphs Without Perfect Matchings In Their Complements

The bound on Δ in Corollary 4.3 only works for $t \geq \frac{1}{2}$. In this section we need bounds on Δ that work for $t = 0$ as well. Using a single independent set of maximal order in Theorem 3.4, we deduce the following.

Corollary 5.1. *Let G be a graph. Then*

$$\chi(G) \leq \tfrac{1}{2}(\omega(G) + |G| - \alpha(G) + 1).$$

Corollary 5.2. *Let G be a graph and $t \in \frac{1}{2}\mathbb{Z}$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \alpha(G)$.*

Proof. Assume $G \notin R_t$. Applying Corollary 5.1 gives

$$\tfrac{1}{2}(\omega(G) + \Delta(G) + 1) + t < \chi(G) \leq \tfrac{1}{2}(\omega(G) + |G| - \alpha(G) + 1).$$

The corollary follows. □

Corollary 5.3. *Let G be a graph and $t \geq 0$. If $G \notin R_t$, then $\Delta(G) + 1 \leq |G| - 2t - \omega(G)$.*

Proof. This is an immediate consequence of the Corollary 4.1. □

Note that for $t \geq \frac{1}{2}$ in Corollary 5.2, we must have $\alpha(G) \geq 3$ by Propostion 3.3, so the lemma gives $\Delta(G) + 1 \leq |G| - 2t - 3$.

Lemma 5.4. *If $k \geq 2$ and $G_1, \ldots, G_k$ are graphs with $\Delta(G_i) + 1 \leq |G_i| - 3$ for each i, then*

$$G_1 + \cdots + G_k \in R_{2-k}.$$

Proof. Assume this is not the case and let $G_1, \ldots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{2-(k-1)} = R_{3-k}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. Since $G_k \notin R_{t-\frac{1}{2}}$, using Corollary 5.2 for $t \geq 1$ and the fact that $\Delta(G_k) + 1 \leq |G_k| - 3$ for $t \leq \frac{1}{2}$, we find that $\Delta(G_k) + 1 \leq |G_k| - 2t - 2$. We have,

$$\begin{aligned}
\chi(D + G_k) &= \chi(D) + \chi(G_k) \\
&\leq \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + \Delta(G_k) + 2) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) + 3 - k + t \\
&\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) + 3 - k + t \\
&\leq \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + \tfrac{1}{2}(-2t - 3 + 1) + 3 - k + t \\
&= \tfrac{1}{2}(\omega(D + G_k) + \Delta(D + G_k) + 1) + 2 - k.
\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{2-k}$, contradicting our assumption. □

The hypotheses of this lemma can be weakened, but we do not use the following stronger lemma in what follows.

Lemma 5.5. *If $k \geq 2$ and $G_1, \ldots, G_k$ are graphs, which are not 5-cycles, with $\Delta(G_i) + 1 \leq |G_i| - 2$ for each i, then*

$$G_1 + \cdots + G_k \in R_{2-k}.$$

Proof. Similar to Lemma 5.4. Graphs with $\Delta(G_i) + 1 \leq |G_i| - 3$ only matter if $G_i \in R_{\frac{1}{2}} \setminus R_0$. Corollary 5.3 shows that such graphs have $\omega(G_i) \leq 2$ and Lemma 5.2 shows they have $\alpha(G) \leq 2$. Thus they have order less than 6 and we see that the only one that breaks the lemma is C_5. □

Definition 3. The *matching number* of a graph G, denoted $\nu(G)$ is the number of edges in a maximum matching of G.

Proposition 5.6. *Let G be a graph. If $\nu(\overline{G}) < \left\lfloor \frac{|G|}{2} \right\rfloor$, then $G \in R_0$.*

Proof. Assume $\nu(\overline{G}) < \left\lfloor \frac{|G|}{2} \right\rfloor$. Then, by Tutte's Theorem, we have $X \subseteq V(G)$ such that $\overline{G - X}$ has at least m odd components; where $m \geq |X| + 2$ if $|G|$ is even and $m \geq |X| + 3$ if $|G|$ is odd. Hence, we have graphs $G_1, \ldots, G_m$ such that $G - X = G_1 + \cdots + G_m$. Note that by picking one vertex from each component we induce a clique. Hence $\omega(G) \geq m$. To get a contradiction, assume $G \notin R_0$. First assume there is some G_i for which $\Delta(G_i) + 1 \geq |G_i| - 2$, then

$$\Delta(G)+1 \geq \Delta(G-X)+1 \geq |G_1|+\cdots+|G_{i-1}|+\Delta(G_i)+|G_{i+1}|+\cdots+|G_m|+1 \geq |G|-|X|-2.$$

Since $G \notin R_0$,

$$\chi(G) > \tfrac{1}{2}(\omega(G) + \Delta(G) + 1) \geq \tfrac{1}{2}(m + (|G| - |X| - 2)) \geq \left\lceil \frac{|G|}{2} \right\rceil.$$

Hence $G \in R_0$ by Corollary 4.1! Thus, we may assume $\Delta(G_i)+1 \leq |G_i|-3$ for each i. Now Lemma 5.4 yields $G - X \in R_{-|X|}$. Whence $G \in R_0$. This contradiction completes the proof. □

Corollary 5.7. *Let G be an even order graph. If $G \notin R_0$, then $\overline{G}$ has a 1-factor.*

Definition 4. A graph is called *matching covered* if every edge participates in a perfect matching.

Corollary 5.8. *Let G be an even order graph with $G \notin R_1$. Then $\overline{G}$ is matching covered.*

Lemma 5.4 can be generalized.

Lemma 5.9. *Let $m \in \mathbb{N}$. Let $k \geq 2$ and $G_1, \ldots, G_k$ be graphs such that $|G_i| < r(m, m) \Rightarrow G_i \in R_{\frac{1}{2}}$. If $\Delta(G_i) + 1 \leq |G_i| - m$ for each i, then*

$$G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}.$$

Proof. Assume this is not the case and let $G_1, \ldots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{(m-1)(1-\frac{k-1}{2})}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. We would like to have $\Delta(G_k)+1 \leq |G_k|-2t-(m-1)$. If $t \leq \frac{1}{2}$, then we are all good since $\Delta(G_k) + 1 \leq |G_k| - m$. So, to get a contradiction, assume $t \geq 1$ and $\Delta(G_k)+1 > |G_k|-2t-(m-1)$. Then, by Corollary 5.3, $\omega(G_k) \leq (m-1)$.

Also, by Lemma 5.2, $\alpha(G_k) \le (m-1)$. Hence $|G_k| < r(m,m)$ contradicting the fact that $G_k \notin R_{\frac{1}{2}}$. Hence we do indeed have $\Delta(G_k)+1 \le |G_k| - 2t - (m-1)$.
We have,

$$\begin{aligned}
\chi(D+G_k) &= \chi(D) + \chi(G_k) \\
&\le \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D) + \Delta(G_k) + 2) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
&\le \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
&\le \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + \tfrac{1}{2}(-2t - 4 + 1) + (m-1)(1 - \tfrac{k-1}{2}) + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + (m-1)(1 - \tfrac{k}{2}).
\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}$, contradicting our assumption. □

We can do a bit better than Lemma 5.9 in the following special case.

Lemma 5.10. *If $k \ge 2$ and $G_1, \ldots, G_k$ are non-complete graphs, then*

$$G_1 + \cdots + G_k \in R_{1-\frac{k}{2}}.$$

Proof. Assume this is not the case and let $G_1, \ldots, G_k$ constitute a counterexample with the smallest k. Then, by Proposition 2.1, $k > 2$. Set $D = G_1 + \cdots + G_{k-1}$. Note that $D \in R_{1-\frac{(k-1)}{2}}$ by the minimality of k. Let $t \in \frac{1}{2}\mathbb{Z}$ be minimal such that $G_k \in R_t$. Since $G_k \notin R_{t-\frac{1}{2}}$, if $t \ge \frac{1}{2}$, then, by Corollary 5.3, $\Delta(G_k) + 1 \le |G_k| - 2t - 1$. If $t \le 0$ and $\Delta(G_k)+1 > |G_k|-2t-1$, then $t = 0$ and $\Delta(G_k)+1 = |G_k|$; however, Corollary 5.1 shows that the only such graphs are complete graphs which we have excluded. Whence $\Delta(G_k) + 1 \le |G_k| - 2t - 1$. We have,

$$\begin{aligned}
\chi(D+G_k) &= \chi(D) + \chi(G_k) \\
&\le \tfrac{1}{2}(\omega(D) + \omega(G_k) + \Delta(D) + \Delta(G_k) + 2) + 1 - \tfrac{(k-1)}{2} + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D) + \Delta(G_k) + 2) + 1 - \tfrac{(k-1)}{2} + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D) + |G_k| + 1 + \Delta(G_k) - |G_k| + 1) + 1 - \tfrac{(k-1)}{2} + t \\
&\le \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + \tfrac{1}{2}(\Delta(G_k) - |G_k| + 1) + 1 - \tfrac{(k-1)}{2} + t \\
&\le \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + \tfrac{1}{2}(-2t - 2 + 1) + 1 - \tfrac{(k-1)}{2} + t \\
&= \tfrac{1}{2}(\omega(D+G_k) + \Delta(D+G_k) + 1) + 1 - \tfrac{k}{2}.
\end{aligned}$$

Hence $G_1 + \cdots + G_k \in R_{1-\frac{k}{2}}$, contradicting our assumption. □

Combining Reed's conjecture with Lemma 5.9 proves the following conjecture.

Conjecture 5.11. *Let $m \in \mathbb{N}$. If $k \geq 2$ and $G_1, \ldots, G_k$ are graphs with $\Delta(G_i) + 1 \leq |G_i| - m$ for each i, then*

$$G_1 + \cdots + G_k \in R_{(m-1)(1-\frac{k}{2})}.$$

References

[1] Michael Molloy. Chromatic neighborhood sets. *J. Graph Theory.*, 31:303-311, 1999.

[2] Landon Rabern. On Graph Associations. *SIAM J. Discrete Math.*, 20(2):529–535, 2006.

[3] Landon Rabern. Coloring and The Lonely Graph. *arXiv:0707.1069*, 2007.

[4] Landon Rabern. Coloring Graphs Containing a Doubly Critical Edge. *Submitted to Journal of Graph Theory*, 2007.

[5] Bruce Reed. ω, Δ, and χ. *J. Graph Theory.*, 27:177-212, 1997.