A better lower bound on average degree of 4-list-critical graphs

Landon Rabern

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Abstract

This short note proves that every incomplete k-list-critical graph has average degree at least $k-1+\frac{k-3}{k^2-2k+2}$. This improves the best known bound for $k=4,5,6$. The same bound holds for online k-list-critical graphs.

1 Introduction

A graph G is *k-list-critical* if G is not $(k-1)$-choosable, but every proper subgraph of G is $(k-1)$-choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of k-list-critical graphs.

Main Theorem. *Every incomplete k-list-critical graph has average degree at least*

$$k-1+\frac{k-3}{k^2-2k+2}.$$

Main Theorem gives a lower bound of $3+\frac{1}{10}$ for 4-list-critical graphs. This is the first improvement over Gallai's bound of $3+\frac{1}{13}$. The same proof shows that Main Theorem holds for online k-list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

Definition. The *maximum independent cover number* of a graph G is the maximum $\operatorname{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Kernel Magic (Kierstead and R. [6])**.** *Every k-list-critical graph G satisfies*

$$2\|G\| \geq (k-2)|G| + \operatorname{mic}(G) + 1.$$

The previous best bounds in Table 1 for k-list-critical graphs hold for k-Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for k-Alon-Tarsi-critical graphs. Any improvement over Gallai's bound of $3+\frac{1}{13}$ for 4-Alon-Tarsi-critical graphs would be interesting.

	k-Critical G				k-List Critical G			
	Gallai [4]	Kriv [9]	KS [7]	KY [8]	KS [7]	KR [5]	CR [2]	Here
k	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—	**3.1000**
5	4.0909	4.1429	—	4.5000	—	4.0984	4.1000	**4.1176**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	5.1076	**5.1153**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1192**	6.1081
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1167**	7.1000
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**	8.0923
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**	9.0853
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0884**	14.0609
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0733**	19.0469

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [4] proved that in a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees. The same is true for k-list-critical graphs [1, 3]. For a graph T and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of T induced on the vertices of degree $k-1$. When k is defined in the context, put $\beta(T) := \beta_k(T)$.

Lemma 1. *If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k-1$, then*

$$2||T|| \leq (k-2)|T| + 2\beta(T).$$

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$. Plainly, T has more than one block. Let A be an endblock of T and let x be the unique cutvertex of T with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$,

$$2\,\|T\| - 2\,\|A\| \leq (k-2)(|T| + 1 - |A|) + 2\beta(T').$$

Since T is a counterexample, $2\,\|A\| > (k-2)(|A|-1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then A is an odd cycle. In both cases, $d_T(x) = k-1$. Consider $T^* := T - V(A)$. By minimality of $|T|$,

$$2\,\|T\| - 2\,\|A\| - 2 \leq (k-2)(|T| - |A|) + 2\beta(T^*).$$

Since T is a counterexample, $2\,\|A\| + 2 > (k-2)\,|A| + 2(\beta(T) - \beta(T^*))$. In T^*, all of x's neighbors have degree at most $k-2$. But $d_T(x) = k-1$, so some vertex in $\{x\} \cup N(x)$ is in a maximum independent set of degree $k-1$ vertices in T. Hence $\beta(T^*) \leq \beta(T) - 1$, which gives

$$2\,\|A\| > (k-2)\,|A|\,,$$

a contradiction since $k \geq 4$. □

Proof of Main Theorem. Let $G \neq K_k$ be a k-list-critical graph. The theorem is trivially true if $k \leq 3$, so suppose $k \geq 4$. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k-1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. By Lemma 1,

$$2\|\mathcal{L}\| \leq (k-2)|\mathcal{L}| + 2\beta(\mathcal{L})$$

Hence,

$$\begin{aligned}
2\|G\| &= 2\|\mathcal{H}\| + 2\|\mathcal{H}, \mathcal{L}\| + 2\|\mathcal{L}\| \\
&= 2\|\mathcal{H}\| + 2((k-1)|\mathcal{L}| - 2\|\mathcal{L}\|) + 2\|\mathcal{L}\| \\
&= 2\|\mathcal{H}\| + 2(k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \\
&\geq 2\|\mathcal{H}\| + k|\mathcal{L}| - 2\beta(\mathcal{L}),
\end{aligned}$$

which is

$$\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|. \tag{1}$$

Let M be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Then

$$\operatorname{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

Applying Kernel Magic and using (1) gives

$$\begin{aligned}
2\|G\| &\geq (k-2)|G| + M + (k-1)\beta(\mathcal{L}) + 1 \\
&\geq (k-2)|G| + M + (k-1)\left(\|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|\right) + 1 \\
&= (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| - (k-1)\|G\| + 1.
\end{aligned}$$

Hence

$$(k+1)\|G\| \geq (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| + 1 \tag{2}$$

Let $\mathcal{C}$ be the components of $G[\mathcal{H}]$. Then $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C \in \mathcal{C}$. Whence

$$M + (k-1)\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k\frac{|C|}{\chi(C)} + (k-1)\|C\|. \tag{3}$$

If $\mathcal{L} = \emptyset$, then G has average degree at least $k \geq k - 1 + \frac{k-3}{k^2-2k+2}$. So, assume $\mathcal{L} \neq \emptyset$. Then $G[\mathcal{H}]$ is $(k-1)$-colorable by k-list-criticality of G. In particular, $\chi(C) \leq k-1$ for every $C \in \mathcal{C}$. For every $C \in \mathcal{C}$,

$$k\frac{|C|}{\chi(C)} + (k-1)\|C\| \geq \left(k - \frac{1}{2}\right)|C|. \tag{4}$$

To see this, first suppose $C \in \mathcal{C}$ is not a tree. Then $\|C\| \geq |C|$ and hence $k\frac{|C|}{\chi(C)} + (k-1)\|C\| \geq k\frac{|C|}{k-1} + (k-1)|C| \geq (k-\frac{1}{2})|C|$. If C is a tree, then $\chi(C) \leq 2$ and hence

$k\frac{|C|}{\chi(C)} + (k-1)\,\|C\| \geq k\frac{|C|}{2} + (k-1)(|C|-1) \geq (k - \frac{1}{2})\,|C|$ unless $|C| = 1$. This proves (4) since the bound is trivially satisfied when $|C| = 1$.

Now combining (2), (3) and (4) with the basic bound

$$|\mathcal{L}| \geq k\,|G| - 2\,\|G\|\,,$$

gives

$$\begin{aligned}
(k+1)\,\|G\| &\geq (k-2)\,|G| + \left(k - \frac{1}{2}\right)|\mathcal{H}| + \frac{k(k-1)}{2}\,|\mathcal{L}| + 1 \\
&= \left(2k - \frac{5}{2}\right)|G| + \frac{k^2-3k+1}{2}\,|\mathcal{L}| + 1 \\
&\geq \left(2k - \frac{5}{2}\right)|G| + \frac{k^2-3k+1}{2}\,(k\,|G| - 2\,\|G\|) + 1.
\end{aligned}$$

After some algebra, this becomes

$$2\,\|G\| \geq \left(k - 1 + \frac{k-3}{k^2-2k+2}\right)|G| + \frac{2}{k^2-2k+2}.$$

That proves the theorem. □

The right side of equation (4) in the above proof can be improved to $k\,|C|$ unless C is a K_2 where both vertices have degree k in G. If these K_2's could be handled, the average degree bound would improve to $k - 1 + \frac{k-3}{(k-1)^2}$.

Conjecture. *Every incomplete (online) k-list-critical graph has average degree at least*

$$k - 1 + \frac{k-3}{(k-1)^2}.$$

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