A better lower bound on average degree of 4-list-critical graphs

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Abstract
This short note proves that every incomplete \(k\)-list-critical graph has average degree at least \(k - 1 + \frac{k - 3}{k^2 - 2k + 2}\). This improves the best known bound for \(k = 4, 5, 6\). The same bound holds for online \(k\)-list-critical graphs.

1 Introduction
A graph \(G\) is \(k\)-list-critical if \(G\) is not \((k - 1)\)-choosable, but every proper subgraph of \(G\) is \((k - 1)\)-choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of \(k\)-list-critical graphs.

Main Theorem. Every incomplete \(k\)-list-critical graph has average degree at least

\[ k - 1 + \frac{k - 3}{k^2 - 2k + 2}. \]

Main Theorem gives a lower bound of \(3 + \frac{1}{10}\) for 4-list-critical graphs. This is the first improvement over Gallai’s bound of \(3 + \frac{1}{13}\). The same proof shows that Main Theorem holds for online \(k\)-list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

Definition. The maximum independent cover number of a graph \(G\) is the maximum \(\text{mic}(G)\) of \(\|I, V(G) \setminus I\|\) over all independent sets \(I\) of \(G\).

Kernel Magic (Kierstead and R. [6]). Every \(k\)-list-critical graph \(G\) satisfies

\[ 2\|G\| \geq (k - 2)|G| + \text{mic}(G) + 1. \]

The previous best bounds in Table 1 for \(k\)-list-critical graphs hold for \(k\)-Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for \(k\)-Alon-Tarsi-critical graphs. Any improvement over Gallai’s bound of \(3 + \frac{1}{13}\) for 4-Alon-Tarsi-critical graphs would be interesting.
Table 1: History of lower bounds on the average degree $d(G)$ of $k$-critical and $k$-list-critical graphs $G$.

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</table>

2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called Gallai trees. Gallai [4] proved that in a $k$-critical graph, the vertices of degree $k - 1$ induce a disjoint union of Gallai trees. The same is true for $k$-list-critical graphs [1, 3]. For a graph $T$ and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of $T$ induced on the vertices of degree $k - 1$. When $k$ is defined in the context, put $\beta(T) := \beta_k(T)$.

**Lemma 1.** If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k - 1$, then

$$2 ||T|| \leq (k - 2)||T|| + 2\beta(T).$$

**Proof.** Suppose the lemma is false and choose a counterexample $T$ minimizing $|T|$. Plainly, $T$ has more than one block. Let $A$ be an endblock of $T$ and let $x$ be the unique cutvertex of $T$ with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$, we have

$$2 ||T|| - 2 ||A|| \leq (k - 2)(|T| + 1 - |A|) + 2\beta(T').$$

Since $T$ is a counterexample, $2 ||A|| > (k - 2)(|A| - 1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then $A$ is an odd cycle. In both cases, $d_T(x) = k - 1$. Consider $T^* := T - V(A)$. By minimality of $|T|$, we get

$$2 ||T|| - 2 ||A|| - 2 \leq (k - 2)(||T|| - |A|) + 2\beta(T^*).$$

Since $T$ is a counterexample, $2 ||A|| + 2 > (k - 2) |A| + 2(\beta(T) - \beta(T^*))$. In $T^*$, all of $x$’s neighbors have degree at most $k - 2$. But $d_T(x) = k - 1$, so some vertex in $\{x\} \cup N(x)$ is in a maximum independent set of degree $k - 1$ vertices in $T$. Hence $\beta(T^*) \leq \beta(T) - 1$, which gives

$$2 ||A|| > (k - 2) |A|,$$

a contradiction since $k \geq 4$. \qed
Proof of Main Theorem. Let $G \neq K_k$ be a $k$-list-critical graph. The theorem is trivially true if $k \leq 3$, so suppose $k \geq 4$. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k - 1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. By Lemma \[1\],
\[
2 \|\mathcal{L}\| \leq (k - 2)|\mathcal{L}| + 2\beta(\mathcal{L})
\]
Hence,
\[
2 \|G\| = 2 \|\mathcal{H}\| + 2 \|\mathcal{H}, \mathcal{L}\| + 2 \|\mathcal{L}\|
= 2 \|\mathcal{H}\| + 2((k - 1)|\mathcal{L}| - 2 \|\mathcal{L}\|) + 2 \|\mathcal{L}\|
= 2 \|\mathcal{H}\| + 2(k - 1)|\mathcal{L}| - 2 \|\mathcal{L}\|
\geq 2 \|\mathcal{H}\| + k|\mathcal{L}| - 2\beta(\mathcal{L}),
\]
which is
\[
\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2} |\mathcal{L}| - \|G\|.
\tag{1}
\]
Let $M$ be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets $I$ of $G$ with $I \subseteq \mathcal{H}$. Then
\[
\text{mic}(G) \geq M + (k - 1)\beta(\mathcal{L}).
\]
Applying Kernel Magic and using (1) gives
\[
2 \|G\| \geq (k - 2)|G| + M + (k - 1)\beta(\mathcal{L}) + 1
\geq (k - 2)|G| + M + (k - 1) \left( \|\mathcal{H}\| + \frac{k}{2} |\mathcal{L}| - \|G\| \right) + 1
\]
\[
= (k - 2)|G| + M + (k - 1)\|\mathcal{H}\| + \frac{k(k - 1)}{2} |\mathcal{L}| - (k - 1)\|G\| + 1.
\]
Hence
\[
(k + 1)\|G\| \geq (k - 2)|G| + M + (k - 1)\|\mathcal{H}\| + \frac{k(k - 1)}{2} |\mathcal{L}| + 1
\tag{2}
\]
Let $\mathcal{C}$ be the components of $G[\mathcal{H}]$. Then $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C \in \mathcal{C}$. Whence
\[
M + (k - 1)\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k \frac{|C|}{\chi(C)} + (k - 1)\|C\|.
\tag{3}
\]
If $\mathcal{L} = \emptyset$, then $G$ has average degree at least $k \geq k - 1 + \frac{k - 3}{k^2 - 2k + 2}$. So, assume $\mathcal{L} \neq \emptyset$. Then $G[\mathcal{H}]$ is $(k - 1)$-colorable by $k$-list-criticality of $G$. In particular, $\chi(C) \leq k - 1$ for every $C \in \mathcal{C}$. For every $C \in \mathcal{C},$
\[
k \frac{|C|}{\chi(C)} + (k - 1)\|C\| \geq \left( k - \frac{1}{2} \right) |C|.
\tag{4}
\]
To see this, first suppose $C \in \mathcal{C}$ is not a tree. Then $\|C\| \geq |C|$ and hence $k \frac{|C|}{\chi(C)} + (k - 1)\|C\| \geq k \frac{|C|}{k - 1} + (k - 1)|C| \geq (k - \frac{1}{2})|C|$. If $C$ is a tree, then $\chi(C) \leq 2$ and hence
\[ k \frac{|C|}{\chi(C)} + (k - 1) \|C\| \geq k \frac{|C|}{2} + (k - 1)(|C| - 1) \geq (k - \frac{1}{2})|C| \] unless \(|C| = 1\). This proves (4) since the bound is trivially satisfied when \(|C| = 1\).

Now combining (2), (3) and (4) with the basic bound

\[ |\mathcal{L}| \geq k |G| - 2 \|G\|, \]

gives

\[ (k + 1) \|G\| \geq (k - 2) |G| + \left( k - \frac{1}{2} \right) |\mathcal{H}| + \frac{k(k - 1)}{2} |\mathcal{L}| + 1 \]

\[ = \left( 2k - \frac{5}{2} \right) |G| + \frac{k^2 - 3k + 1}{2} |\mathcal{L}| + 1 \]

\[ \geq \left( 2k - \frac{5}{2} \right) |G| + \frac{k^2 - 3k + 1}{2} (k |G| - 2 \|G\|) + 1. \]

After some algebra, this becomes

\[ 2 \|G\| \geq \left( k - 1 + \frac{k - 3}{k^2 - 2k + 2} \right) |G| + \frac{2}{k^2 - 2k + 2}. \]

That proves the theorem. \( \square \)

The right side of equation (4) in the above proof can be improved to \( k |C| \) unless \( C \) is a \( K_2 \) where both vertices have degree \( k \) in \( G \). If these \( K_2 \)'s could be handled, the average degree bound would improve to \( k - 1 + \frac{k - 3}{(k - 1)^2} \).

**Conjecture.** Every incomplete (online) \( k \)-list-critical graph has average degree at least

\[ k - 1 + \frac{k - 3}{(k - 1)^2}. \]

**References**


