graph theory notes*

The combinatorial nullstellensatz and Schauz's coefficient formula

In [2], Alon and Tarsi introduced a beautiful algebraic technique for proving the existence of list colorings. Alon [1] further developed this technique into the *Combinatorial Nullstellensatz*. Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1,\ldots,k_n}$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$.

Combinatorial Nullstellensatz (Alon)**.** *Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$. If $f_{k_1,\ldots,k_n} \neq 0$, then for any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$, there exists $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ with $f(a_1, \ldots, a_n) \neq 0$.*

Michałek [5] gave a very short proof of the Combinatorial Nullstellensatz just using long division. Schauz [6] sharpened the Combinatorial Nullstellensatz by proving the following coefficient formula. Versions of this result were also proved by Hefetz [3] and Lasoń [4]. Our presentation is similar to Lasoń's.

Coefficient Formula (Schauz)**.** *Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$. For any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$, we have*

$$f_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)},$$

where

$$N(a_1, \ldots, a_n) := \prod_{i \in [n]} \prod_{b \in A_i \setminus \{a_i\}} (a_i - b).$$

We first give Michałek's proof of the Combinatorial Nullstellensatz and use this to derive the coefficient formula.

Proof of Combinatorial Nullstellensatz. Suppose the result is false and choose $f \in \mathbb{F}[x_1, \ldots, x_n]$ for which it fails minimizing $\deg(f)$. Then $\deg(f) \geq 2$ and we have $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$ and $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$ such that $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. By symmetry, we may assume that $k_1 > 0$. Fix $a \in A_1$ and divide f by $x_1 - a$ to get $f = (x_1 - a)Q + R$ where the degree of x_1 in R is zero. Then the

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coefficient of $x_1^{k_1-1}x_2^{k_2}\cdots x_n^{k_n}$ in Q must be non-zero and $\deg(Q) < \deg(f)$. So, by minimality of $\deg(f)$ there is $(a_1,\ldots,a_n) \in (A_1 \setminus \{a\}) \times \cdots \times A_n$ such that $Q(a_1,\ldots,a_n) \neq 0$. Since $0 = f(a_1,\ldots,a_n) = (a_1 - a)Q(a_1,\ldots,a_n) + R(a_1,\ldots,a_n)$ we must have $R(a_1,\ldots,a_n) \neq 0$. But x_1 has degree zero in R, so $R(a,\ldots,a_n) = R(a_1,\ldots,a_n) \neq 0$. Finally, this means that $f(a,\ldots,a_n) = (a-a)Q(a,\ldots,a_n) + R(a,\ldots,a_n) \neq 0$, a contradiction. □

Proof of Coefficient Formula. Let $f \in \mathbb{F}[x_1,\ldots,x_n]$ and $k_1,\ldots,k_n \in \mathbb{N}$ with $\sum_{i\in[n]} k_i = \deg(f)$. Also, let $A_1,\ldots,A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$. For each $(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n$, let $\chi_{(a_1,\ldots,a_n)}$ be the characteristic function of the set $\{(a_1,\ldots,a_n)\}$; that is $\chi_{(a_1,\ldots,a_n)} \colon A_1 \times \cdots \times A_n \to \mathbb{F}$ with $\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = 1$ when $(x_1,\ldots,x_n) = (a_1,\ldots,a_n)$ and $\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = 0$ otherwise. Consider the function

$$F = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\chi_{(a_1,\ldots,a_n)}.$$

Then F agrees with f on all of $A_1 \times \cdots \times A_n$ and hence $f - F$ is zero on $A_1 \times \cdots \times A_n$. We will apply the Combinatorial Nullstellensatz to $f - F$ to conclude that $(f - F)_{k_1,\ldots,k_n} = 0$ and hence $f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n}$ where $F_{k_1,\ldots,k_n}$ will turn out to be our desired sum. To apply the Combinatorial Nullstellensatz, we need to represent F as a polynomial, we can do so by representing each $\chi_{(a_1,\ldots,a_n)}$ as a polynomial as follows. For $(a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n$, let

$$N(a_1,\ldots,a_n) := \prod_{i\in[n]} \prod_{b\in A_i\setminus\{a_i\}} (a_i - b).$$

Then it is readily verified that

$$\chi_{(a_1,\ldots,a_n)}(x_1,\ldots,x_n) = \frac{\prod_{i\in[n]}\prod_{b\in A_i\setminus\{a_i\}}(x_i - b)}{N(a_1,\ldots,a_n)}.$$

Using this to define F we get

$$F(x_1,\ldots,x_n) = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} f(a_1,\ldots,a_n)\frac{\prod_{i\in[n]}\prod_{b\in A_i\setminus\{a_i\}}(x_i - b)}{N(a_1,\ldots,a_n)}.$$

Now $\deg(F) = \sum_{i\in[n]}(|A_i| - 1) = \sum_{i\in[n]} k_i = \deg(f)$. Since $f - F$ is zero on $A_1 \times \cdots \times A_n$, applying the Combinatorial Nullstellensatz to $f - F$ with $k_1,\ldots,k_n$ and sets $A_1,\ldots,A_n$ gives $(f - F)_{k_1,\ldots,k_n} = 0$ and hence

$$f_{k_1,\ldots,k_n} = F_{k_1,\ldots,k_n} = \sum_{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n} \frac{f(a_1,\ldots,a_n)}{N(a_1,\ldots,a_n)}.$$

□

References

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