The combinatorial nullstellensatz and Schauz’s coefficient formula

In [2], Alon and Tarsi introduced a beautiful algebraic technique for proving the existence of list colorings. Alon [1] further developed this technique into the Combinatorial Nullstellensatz. Fix an arbitrary field $\mathbb{F}$. We write $f_{k_1, \ldots, k_n}$ for the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in the polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$.

**Combinatorial Nullstellensatz (Alon).** Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i=1}^n k_i = \deg(f)$. If $f_{k_1, \ldots, k_n} \neq 0$, then for any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$, there exists $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$ with $f(a_1, \ldots, a_n) \neq 0$.

Michalek [5] gave a very short proof of the Combinatorial Nullstellensatz just using long division. Schauz [6] sharpened the Combinatorial Nullstellensatz by proving the following coefficient formula. Versions of this result were also proved by Hefetz [3] and Lasoń [4]. Our presentation is similar to Lasoń’s.

**Coefficient Formula (Schauz).** Suppose $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i=1}^n k_i = \deg(f)$. For any $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$, we have

$$f_{k_1, \ldots, k_n} = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} \frac{f(a_1, \ldots, a_n)}{N(a_1, \ldots, a_n)},$$

where

$$N(a_1, \ldots, a_n) := \prod_{i=1}^n \prod_{b \in A_i \setminus \{a_i\}} (a_i - b).$$

We first give Michalek’s proof of the Combinatorial Nullstellensatz and use this to derive the coefficient formula.

**Proof of Combinatorial Nullstellensatz.** Suppose the result is false and choose $f \in \mathbb{F}[x_1, \ldots, x_n]$ for which it fails minimizing $\deg(f)$. Then $\deg(f) \geq 2$ and we have $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i=1}^n k_i = \deg(f)$ and $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| \geq k_i + 1$ such that $f(a_1, \ldots, a_n) = 0$ for all $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$. By symmetry, we may assume that $k_1 > 0$. Fix $a \in A_1$ and divide $f$ by $x_1 - a$ to get $f = (x_1 - a)Q + R$ where the degree of $x_1$ in $R$ is zero. Then the

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Then it is readily verified that $(a_1, \ldots, a_n) \in (A_1 \setminus \{a\}) \times \cdots \times A_n$ such that $Q(a_1, \ldots, a_n) \neq 0$. Since $0 = f(a_1, \ldots, a_n) = (a_1 - a)Q(a_1, \ldots, a_n) + R(a_1, \ldots, a_n)$ we must have $R(a_1, \ldots, a_n) \neq 0$. But $x_1$ has degree zero in $R$, so $R(a_1, \ldots, a_n) = R(a_1, \ldots, a_n) \neq 0$. Finally, this means that $f(a_1, \ldots, a_n) = (a - a)Q(a_1, \ldots, a_n) + R(a_1, \ldots, a_n) \neq 0$, a contradiction. 

Proof of Coefficient Formula. Let $f \in \mathbb{F}[x_1, \ldots, x_n]$ and $k_1, \ldots, k_n \in \mathbb{N}$ with $\sum_{i \in [n]} k_i = \deg(f)$. Also, let $A_1, \ldots, A_n \subseteq \mathbb{F}$ with $|A_i| = k_i + 1$. For each $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, let $\chi_{(a_1, \ldots, a_n)}$ be the characteristic function of the set $\{(a_1, \ldots, a_n)\}$; that is $\chi_{(a_1, \ldots, a_n)} : A_1 \times \cdots \times A_n \to \mathbb{F}$ with $\chi_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) = 1$ when $(x_1, \ldots, x_n) = (a_1, \ldots, a_n)$ and $\chi_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) = 0$ otherwise. Consider the function

$$F = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} f(a_1, \ldots, a_n)\chi_{(a_1, \ldots, a_n)}.$$ 

Then $F$ agrees with $f$ on all of $A_1 \times \cdots \times A_n$ and hence $f - F$ is zero on $A_1 \times \cdots \times A_n$. We will apply the Combinatorial Nullstellensatz to $f - F$ to conclude that $(f - F)_{k_1, \ldots, k_n} = 0$ and hence $f_{k_1, \ldots, k_n} = F_{k_1, \ldots, k_n}$ where $F_{k_1, \ldots, k_n}$ will turn out to be our desired sum. To apply the Combinatorial Nullstellensatz, we need to represent $F$ as a polynomial, we can do so by representing each $\chi_{(a_1, \ldots, a_n)}$ as a polynomial as follows. For $(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n$, let

$$N(a_1, \ldots, a_n) := \prod_{i \in [n]} \prod_{b \in A_i \setminus \{a_i\}} (a_i - b).$$

Then it is readily verified that

$$\chi_{(a_1, \ldots, a_n)}(x_1, \ldots, x_n) = \frac{\prod_{i \in [n]} \prod_{b \in A_i \setminus \{a_i\}} (x_i - b)}{N(a_1, \ldots, a_n)}.$$ 

Using this to define $F$ we get

$$F(x_1, \ldots, x_n) = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} f(a_1, \ldots, a_n)\frac{\prod_{i \in [n]} \prod_{b \in A_i \setminus \{a_i\}} (x_i - b)}{N(a_1, \ldots, a_n)}.$$

Now $\deg(F) = \sum_{i \in [n]} (|A_i| - 1) = \sum_{i \in [n]} k_i = \deg(f)$. Since $f - F$ is zero on $A_1 \times \cdots \times A_n$, applying the Combinatorial Nullstellensatz to $f - F$ with $k_1, \ldots, k_n$ and sets $A_1, \ldots, A_n$ gives $(f - F)_{k_1, \ldots, k_n} = 0$ and hence

$$f_{k_1, \ldots, k_n} = F_{k_1, \ldots, k_n} = \sum_{(a_1, \ldots, a_n) \in A_1 \times \cdots \times A_n} f(a_1, \ldots, a_n)\frac{1}{N(a_1, \ldots, a_n)}.$$

References


