graph theory notes*

Haxell’s independent transversal lemma

In 1995, Penny Haxell [5, 4] proved a lemma that gives a necessary condition for the existence of an independent transversal. This lemma is a very powerful tool for many coloring problems. In [2], Haxell gave a simpler proof of her lemma using the technique from Haxell and Szabó [3]. We prove the following variation of the lemma using the same technique (see [6, 1] for the original proof).

Transversal Lemma (Haxell, Aharoni-Berger-Ziv, King). Let $H$ be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose there exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min \{t, |V_i| - t\}$. For any $S \subseteq V(H)$ with $|S| < \min \{|V_1|, \ldots, |V_r|\}$, there is an independent transversal $I$ of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$.

In fact, a more general statement holds. First we need some notation. Write $X := \pi^{-1}(1)$, $Y := \pi^{-1}(2)$. The multigraph with vertex set $J \cup X \cup Y$ and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.

Lemma 1. Let $G$ be a graph and $\pi: V(G) \to [k]$ a proper $k$-coloring of $G$. Suppose that $\pi$ has no $G$-independent transversal, but for every $e \in E(G)$, $\pi$ has a $(G - e)$-independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching $M$ of $G[\pi^{-1}(J)]$ with $xy \in M$ such that:

1. $\bigcup M$ totally dominates $G[\pi^{-1}(J)]$, 

2. the multigraph with vertex set $J$ and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.

Proof. Suppose the lemma is false and choose a counterexample $G$ with $\pi: V(G) \to [k]$ so as to minimize $k$. Let $xy \in E(G)$. By assumption $\pi$ has a $(G - xy)$-independent transversal $T$. Note that we must have $x, y \in T$ lest $T$ be a $G$-independent transversal of $\pi$.

By symmetry we may assume that $\pi(x) = k - 1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k - 1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x, y\}) - E(X, Y)$. Define $\zeta: V(H) \to [k - 1]$ by $\zeta(v) := \min \{\pi(v), k - 1\}$. Note that since $x, y \in T$, we have $|\zeta^{-1}(i)| \geq 1$ for each $i \in [k - 2]$. Put $Z := \zeta^{-1}(k - 1)$. Then $Z \neq \emptyset$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

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Suppose $\zeta$ has an $H$-independent transversal $S$. Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a $G$-independent transversal of $\pi$, a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that $\zeta$ has no $H'$-independent transversal. Now $d(z) \geq 1$ for each $z \in Z$ for otherwise $T - \{x, y\} \cup \{z\}$ would be an $H'$-independent transversal of $\zeta$. Pick $zw \in E(H')$. By minimality of $k$, we have $J \subseteq [k - 1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching $M$ of $H'[\zeta^{-1}(J)]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'[\zeta^{-1}(J)]$,

2. the multigraph with vertex set $J$ and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since $H'$ is a spanning subgraph of $H$, $\bigcup M$ totally dominates $H[\zeta^{-1}(J)]$ and hence $\bigcup M'$ totally dominates $G[\pi^{-1}(J')]$. Moreover, the multigraph in (2) for $M'$ and $J'$ is formed by splitting the vertex $k - 1 \in J$ into two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma.

Proof of Transversal Lemma. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let $V_1', \ldots , V_r'$ be the induced partition of $H'$. Then there is no independent transversal of $V_1', \ldots , V_r'$ and $|V_i'| \geq 1$ for each $i \in [r]$. Create a graph $Q$ by removing edges from $H'$ until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma on $yz$ with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree $T$ with vertex set $J$ and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V_a'$ and $v \in V_b'$. By our condition, for each $uv \in E(V_i, V_j)$, we have $|N_{H}(u) \cup N_{H}(v)| \leq \min \{|V_i|, |V_j|\}$.

Choose a root $c$ of $T$. Traversing $T$ in leaf-first order and for each leaf $a$ with parent $b$ picking $|V_a|$ from min \{|$V_a$|, $V_b$|\} we get that the vertices in $M$ together dominate at most $\sum_{i \in J \setminus \{c\}} |V_i|$ vertices in $H$. Since $|S| < |V_c|$, $M$ cannot totally dominate $\bigcup_{i \in J} V_i'$, a contradiction.

Note that the condition on $S$ can be weakened slightly. Suppose we have ordered the $V_i$ so that $|V_1| \leq |V_2| \leq \cdots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that $V_1 \nsubseteq S$, there is an independent transversal $I$ of $V_1, \ldots , V_r$ with $I \cap S = \emptyset$. The proof is the same except when we choose our root $c$, choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

References


