graph theory notes*

Haxell's independent transversal lemma

In 1995, Penny Haxell [5, 4] proved a lemma that gives a necessary condition for the existence of an independent transversal. This lemma is a very powerful tool for many coloring problems. In [2], Haxell gave a simpler proof of her lemma using the technique from Haxell and Szabó [3]. We prove the following variation of the lemma using the same technique (see [6, 1] for the original proof).

Transversal Lemma (Haxell, Aharoni-Berger-Ziv, King). *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose there exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min\{t, |V_i| - t\}$. For any $S \subseteq V(H)$ with $|S| < \min\{|V_1|, \ldots, |V_r|\}$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$.*

In fact, a more general statement holds. First we need some notation. Write $f \colon A \twoheadrightarrow B$ for a surjective function from A to B. Let G be a graph. For a k-coloring $\pi \colon V(G) \twoheadrightarrow [k]$ of G and a subgraph H of G we say that $I := \{x_1, \ldots, x_k\} \subseteq V(H)$ is an H-independent transversal of π if I is an independent set in H and $\pi(x_i) = i$ for all $i \in [k]$.

Lemma 1. *Let G be a graph and $\pi \colon V(G) \twoheadrightarrow [k]$ a proper k-coloring of G. Suppose that π has no G-independent transversal, but for every $e \in E(G)$, π has a $(G - e)$-independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching M of $G[\pi^{-1}(J)]$ with $xy \in M$ such that:*

1. *$\bigcup M$ totally dominates $G[\pi^{-1}(J)]$,*
2. *the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.*

Proof. Suppose the lemma is false and choose a counterexample G with $\pi \colon V(G) \twoheadrightarrow [k]$ so as to minimize k. Let $xy \in E(G)$. By assumption π has a $(G - xy)$-independent transversal T. Note that we must have $x, y \in T$ lest T be a G-independent transversal of π.

By symmetry we may assume that $\pi(x) = k - 1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k-1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x, y\}) - E(X, Y)$. Define $\zeta \colon V(H) \to [k-1]$ by $\zeta(v) := \min\{\pi(v), k-1\}$. Note that since $x, y \in T$, we have $|\zeta^{-1}(i)| \geq 1$ for each $i \in [k-2]$. Put $Z := \zeta^{-1}(k-1)$. Then $Z \neq \emptyset$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

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Suppose ζ has an H-independent transversal S. Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a G-independent transversal of π, a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that ζ has no H'-independent transversal. Now $d(z) \geq 1$ for each $z \in Z$ for otherwise $T - \{x, y\} \cup \{z\}$ would be an H'-independent transversal of ζ. Pick $zw \in E(H')$. By minimality of k, we have $J \subseteq [k-1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching M of $H'[\zeta^{-1}(J)]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'[\zeta^{-1}(J)]$,
2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since H' is a spanning subgraph of H, $\bigcup M$ totally dominates $H[\zeta^{-1}(J)]$ and hence $\bigcup M'$ totally dominates $G[\pi^{-1}(J')]$. Moreover, the multigraph in (2) for M' and J' is formed by splitting the vertex $k-1 \in J$ into two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. □

Proof of Transversal Lemma. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let $V_1', \ldots, V_r'$ be the induced partition of H'. Then there is no independent transversal of $V_1', \ldots, V_r'$ and $|V_i'| \geq 1$ for each $i \in [r]$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 1 on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree T with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V_a'$ and $v \in V_b'$. By our condition, for each $uv \in E(V_i, V_j)$, we have $|N_H(u) \cup N_H(v)| \leq \min\{|V_i|, |V_j|\}$.

Choose a root c of T. Traversing T in leaf-first order and for each leaf a with parent b picking $|V_a|$ from $\min\{|V_a|, |V_b|\}$ we get that the vertices in M together dominate at most $\sum_{i \in J \setminus \{c\}} |V_i|$ vertices in H. Since $|S| < |V_c|$, M cannot totally dominate $\bigcup_{i \in J} V_i'$, a contradiction. □

Note that the condition on S can be weakened slightly. Suppose we have ordered the V_i so that $|V_1| \leq |V_2| \leq \cdots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that $V_1 \not\subseteq S$, there is an independent transversal I of $V_1, \ldots, V_r$ with $I \cap S = \emptyset$. The proof is the same except when we choose our root c, choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

References

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